MA-207 Differential Equations II

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Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0
2. $x_0 \in \mathbb{R}$ is called regular singular point if x_0 is not an ordinary point and both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.
   If x_0 is regular singular then there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that
   $$p(x) = \frac{b(x)}{(x - x_0)} \qquad q(x) = \frac{c(x)}{(x - x_0)^2}$$
3. If $x_0 \in \mathbb{R}$ is not ordinary or regular singular, then we call it irregular singular.

Thus,

- The best possible situation is when x_0 is an ordinary point.
- If x_0 is not ordinary, then the next best possible situation is when it is regular singular. We will next learn how to deal with this situation.
- Finally, we have the situation when x_0 is not regular singular, that is, it is irregular singular. We will not deal with this case in this course.

Example

$x = 0$ is an irregular singular point of $x^3 y'' + xy' + y = 0$

Let us write the ODE in standard form

$$y'' + \frac{1}{x^2} y' + \frac{1}{x^3} y = 0$$

Then

$$p(x) = \frac{1}{x^2} \qquad q(x) = \frac{1}{x^3}$$

Clearly,

$$xp(x) = \frac{1}{x} \qquad x^2 q(x) = \frac{1}{x}$$

are not analytic at 0. Thus, $x = 0$ is an irregular singular point.

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

All $x \neq 0$ are ordinary points.

Assume $\boxed{x > 0}$

Note that $y = x^r$ solves the equation iff

$$r(r-1) + b_0r + c_0 = 0$$

$$\iff r^2 + (b_0 - 1)r + c_0 = 0$$

Let r_1 and r_2 denote the roots of this quadratic equation.

Solutions in the regular singular case

Example (continues ...)

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

  are two independent solutions.

- If the roots $r_1 = r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad (\log x)x^{r_1}$$

  are two independent solutions.

- If the roots are complex (written as $a \pm ib$), then

$$x^a \cos(b \log x) \quad \text{and} \quad x^a \sin(b \log x)$$

  are two independent solutions.

This example motivates us to look for solutions which involve x^r.

First solution in regular singular case

Consider $x^2y'' + xb(x)y' + c(x)y = 0$ with

$$b(x) = \sum_{i\geq 0} b_i x^i \qquad c(x) = \sum_{i\geq 0} c_i x^i$$

analytic functions in a small neighborhood of 0.

$x = 0$ is a regular singular point.

Define the indicial equation

$$I(r) := r(r-1) + b_0 r + c_0$$

Look for solution of the type

$$y(x) = \sum_{n\geq 0} a_n x^{n+r}$$

by substituting this into the differential equation and setting the coefficient of x^{n+r} to 0.

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i$$

   We need this to be 0

Let r_1 and r_2 be roots of $I(r) = 0$. Assume r_1 and r_2 are real and $r_1 \geq r_2$.

Define $a_0 = 1$.

Set $r = r_1$ in the above equation and define a_n, for $n \geq 1$, inductively by the equation

$$a_n(r_1) = -\frac{\sum_{i=0}^{n-1} b_{n-i}(i+r_1)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i}{I(n+r_1)}$$

Since $I(n+r_1) \neq 0$ for $n \geq 1$, $a_n(r_1)$ is a well defined real number.

Thus,

$$y_1(x) = \sum_{n \geq 0} a_n(r_1) x^{n+r_1}$$

is a possible solution to the above differential equation.

First solution in regular singular case

Theorem

Consider the ODE $x^2y'' + xb(x)\,y' + c(x)\,y = 0 \qquad (*)$

where $b(x)$ *and* $c(x)$ *are analytic at* 0. *Then* $x = 0$ *is a regular singular point of ODE.*

Then $(*)$ *has a solution of the form*

$$y(x) = x^r \sum_{n\geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \qquad (**)$$

The solution $(**)$ *is called Frobenius solution or fractional power series solution.*

The power series $\sum_{n\geq 0} a_n x^n$ *converges on* $(-\rho, \rho)$, *where* ρ *is the minimum of the radius of convergence of* $b(x)$ *and* $c(x)$. *We will consider the solution* $y(x)$ *in the open interval* $(0, \rho)$.

Second solution in regular singular case

The analysis now breaks into the following three cases

- $r_1 - r_2 \notin \mathbb{Z}$
- $r_1 = r_2$
- $0 \neq r_1 - r_2 \in \mathbb{Z}$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Thus, as before, the second solution is given by

$$y_2(x) = \sum_{n \geq 0} a_n(r_2) x^{n+r_2}$$

Example

Consider the ODE $\quad x^2 y'' - \frac{x}{2} y' + \frac{(1+x)}{2} y = 0$

Observe that $x = 0$ is a regular singular point.

$$\begin{aligned} I(r) &= r(r-1) - \tfrac{1}{2} r + \tfrac{1}{2} \\ &= (2r(r-1) - r + 1)/2 \\ &= (2r^2 - 3r + 1)/2 \\ &= (r-1)(2r-1)/2 \end{aligned}$$

Roots of $I(r) = 0$ are $\boxed{r_1 = 1}$ and $\boxed{r_2 = 1/2}$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

Their difference $r_1 - r_2 = 1/2$ is not an integer.

The equation defining a_n, for $n \geq 1$, is

$$I(n+r)a_n + \frac{1}{2}a_{n-1} = 0$$

Thus,

$$a_n(r) = -\frac{a_{n-1}(r)}{(n+r-1)(2n+2r-1)}$$

Thus,

$$a_n(r_1) = a_n(1) = -\frac{a_{n-1}}{n(2n+1)}$$

$$= (-1)^n \frac{1}{n!((2n+1)(2n-1)\dots 3}$$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

$$y_1(x) = x\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n+1)(2n-1)\dots 3}\right)$$

$$a_n(r_2) = -\frac{a_{n-1}}{n(2n-1)}$$

$$= (-1)^n \frac{1}{n!(2n-1)(2n-3)\dots 1}$$

$$y_2(x) = x^{1/2}\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n-1)(2n-3)\dots 1}\right)$$

Since $|a_n|$ are smaller that $\frac{1}{n!}$, it is clear that both solutions converge on $(0, \infty)$.

Second solution: $r_1 = r_2$

Consider the differential operator

$$L := x^2 \frac{d^2}{dx^2} + xb(x)\frac{d}{dx} + c(x)$$

Consider the function of two variables

$$\psi(r,x) := \sum_{n \geq 0} a_n(r) x^{n+r}$$

Then one checks easily that

$$L\psi(r,x) = \sum_{n \geq 0} E(n) x^{n+r}$$

where

$$E(0) := I(r)a_0, \qquad \text{and for } n \geq 1$$

$$E(n) := I(n+r)a_n(r) + \sum_{i=0}^{n-1}(i+r)b_{n-i}a_i(r) + \sum_{i=0}^{n-1} c_{n-i}a_i(r)$$

Second solution: $r_1 = r_2$

$I(r)$ is the indicial equation, given by $r(r-1) + b_0 r + c_0$. The roots are $r_1 \geq r_2$.

Setting $a_0(r) = 1$ and $E(n) = 0$ allows us to inductively define functions $a_n(r)$.

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator of $a_n(r)$ is $\prod_{i=1}^{n} I(i+r)$.

The functions $a_n(r)$ are analytic at r_1. They are analytic at r_2 if $r_1 - r_2 \notin \mathbb{Z}$.

In particular, if we put $r = r_1$, then it gives a solution since

$$L\psi(r_1, x) = I(r_1)x^{r_1} = 0$$

Explicitly this solution is

$$y_1(x) = x^{r_1} \sum_{n \geq 0} a_n(r_1) x^n$$

If $r_1 - r_2 \notin \mathbb{Z}$ then the second solution is given by

$$y_2(x) = x^{r_2} \sum_{n \geq 0} a_n(r_2) x^n$$

Now let us consider the case when I has repeated roots

Since I has repeated roots $r_1 = r_2$, it follows that, for every $n \geq 1$, the polynomial $\prod_{i=1}^{n} I(i + r)$ does not vanish at $r = r_1$

Consequently, it is clear that the $a_n(r)$ are analytic in a small neighborhood around $r = r_1 = r_2$.

Second solution: $r_1 = r_2$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L\psi(r,x) = I(r)x^r$. Clearly the operators L and $\frac{d}{dr}$ commute with each other, and so we get

$$\begin{aligned}
\frac{d}{dr}L\psi(r,x) &= L\frac{d}{dr}\psi(r,x) \\
&= L\sum_{n\geq 0}\left(a_n'(r)x^{n+r} + a_n(r)x^{n+r}\log x\right) = \frac{d}{dr}I(r)x^r \\
&= I'(r)x^r + I(r)x^r\log x
\end{aligned}$$

Thus, if we plug in $r = r_1 = r_2$ in the above, then we get

$$L\Big(\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + a_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Second solution: $r_1 = r_2$

Theorem (Second solution: $r_1 = r_2$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} a_n(r_2)x^{n+r_2}\log x$$

Example

Consider the ODE

$$x^2 y'' + 3xy' + (1 - 2x)y = 0$$

This has a regular singularity at $x = 0$.

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

Let us find the Frobenius solution directly by putting

$$y = x^r \sum_{n \geq 0} a_n(r) x^n \qquad a_0 = 1$$

$$y' = \sum_{n \geq 0} (n + r) a_n(r) x^{n+r-1}$$

$$y'' = \sum_{n \geq 0}^{\infty} (n + r)(n + r - 1) a_n(r) x^{n+r-2}$$

Second solution: $r_1 = r_2$

Example (continues ...)

$$x^2y(x,r)'' + 3xy(x,r)' + (1-2x)y(x,r)$$
$$= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 3(n+r) + 1\right] a_n(r) x^{n+r}$$
$$- \sum_{n=0}^{\infty} 2a_n(r) x^{n+r+1}$$

Recursion relations for $n \geq 1$ are

$$a_n(r) = \frac{2a_{n-1}(r)}{(n+r)(n+r-1) + 3(n+r) + 1}$$
$$= \frac{2a_{n-1}(r)}{(n+r+1)^2}$$
$$= \frac{2^n}{[(n+r+1)(n+r)\dots(r+2)]^2}\, a_0$$

Example (continues ...)

Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2} x^n$$

The power series converges on $(0, \infty)$.

The second solution is

$$y_2(x) = y_1(x) \log x + x^{-1} \sum_{n \geq 1} a_n'(-1) x^n$$

where

$$a_n(r) = \frac{2^n}{[(n+r+1)(n+r)\ldots(r+2)]^2}$$

$$a_n'(r) = \frac{-2.2^n \, [(n+r+1)(n+r)\ldots(r+2)]'}{[(n+r+1)(n+r)\ldots(r+2)]^3}$$

Example (continued)

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

Putting $r = -1$, we get

$$a_n'(-1) = -\frac{2^{n+1}H_n}{(n!)^2}$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

(These are the partial sums of the harmonic series.)

So the second solution is

$$y_2(x) = y_1(x)\log x - \frac{1}{x}\sum_{n\geq 1}\frac{2^{n+1}H_n}{(n!)^2}\,x^n$$

It is clear that this series converges on $(0, \infty)$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Define

$$N := r_1 - r_2$$

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator is exactly $\prod_{i=1}^{n} I(i+r)$.

The polynomial $\prod_{i=1}^{n} I(i+r)$ evaluated at r_2 vanishes iff $n \geq N$. For $n \geq N$ it vanishes to order exactly 1.

Thus, if we define

$$A_n(r) := a_n(r)(r - r_2)$$

then it is clear that for every $n \geq 0$, the function $A_n(r)$ is analytic in a neighborhood of r_2.

In particular, $A_n(r_2)$ and $A'_n(r_2)$ are well defined real numbers.

Multiplying the equation $L\psi(r,x) = I(r)x^r$ with $r - r_2$ we get

$$(r - r_2)L\psi(r,x) = L(r - r_2)\psi(r,x) = (r - r_2)I(r)x^r$$

Note that

$$(r - r_2)\psi(r,x) = \sum_{n \geq 0} A_n(r)x^{n+r}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L(r-r_2)\psi(r,x) = (r-r_2)I(r)x^r$ to get

$$\begin{aligned}\frac{d}{dr}L(r-r_2)\psi(r,x) &= L\frac{d}{dr}(r-r_2)\psi(r,x)\\ &= \frac{d}{dr}(r-r_2)I(r)x^r\\ &= I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r\log x\end{aligned}$$

Thus we get

$$\begin{aligned}L\frac{d}{dr}\Big(\sum_{n\geq 0}A_n(r)x^{n+r}\Big) &= L\frac{d}{dr}\Big(\sum_{n\geq 0}A_n(r)x^{n+r}\Big)\\ &= L\Big(\sum_{n\geq 0}A_n'(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big)\\ &= I(r)x^r + (r-r_2)I'(r)x^r +\\ &\quad (r-r_2)I(r)x^r\log x\end{aligned}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

If we set $r = r_2$ into the equation

$$L\Big(\sum_{n\geq 0} A_n'(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big) = I(r)x^r + (r-r_2)I'(r)x^r + (r-r_2)I(r)x^r\log x$$

we get the second solution

$$L\Big(\sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + A_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Theorem (Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} A_n(r_2)x^{n+r_2}\log x$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example

Consider the ODE $\qquad xy'' - (4+x)y' + 2y = 0 \qquad (*)$

Multiplying $(*)$ with x, we get $x = 0$ is a regular singular point.

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots differing by a positive integer.

Put $y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n, \quad a_0(r) = 1$, into the ODE to get

$$x \sum_{n \geq 0} (n+r)(n+r-1) a_n(r) x^{n+r-2}$$

$$-(4+x) \sum_{n \geq 0} (n+r) a_n(r) x^{n+r-1} + 2 \sum_{n \geq 0} a_n(r) x^{n+r} = 0$$

the coefficient of x^{n+r-1} for $n \geq 1$ gives

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) \\ +2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

$$a_n(r) = \frac{(n+r-3)}{(n+r)(n+r-5)}\, a_{n-1}$$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)}\, a_0$$

For the first solution, set $r = r_1 = 5$ (and $a_0 = 1$), we get

$$a_n(5) = \frac{(n+2)\dots(3)}{(n+5)\dots 6(n)\dots 1}$$

$$= \frac{(n+2)!/2}{(n!)(n+5)!/5!}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Example (continues ...)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus

$$y_1(x) = \sum_{n \geq 0} \frac{60}{n!(n+5)(n+4)(n+3)} x^{n+5}$$

Recall $N = r_1 - r_2 = 5 - 0$ is integer, so the second solution is

$$y_2(x) = \sum_{n \geq 0} A_n'(r_2) x^{n+r_2} + \sum_{n \geq 0} A_n(r_2) x^{n+r_2} \log x$$

where, for $n \geq 0$

$$A_n(r) = (r - r_2) a_n(r)$$

Since $r_2 = 0$, the above becomes

$$A_n(r) = r a_n(r)$$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Thus, $A_n(0) = 0$ for all $n \geq 0$; and $A'_n(0) = a_n(0)$ for all $n \geq 0$.

$a_1(0) = \frac{1}{2}$; $a_2(0) = \frac{1}{12}$;
It is easily checked that for $n \geq 3$

$$a_n(r) = \frac{(n + r - 3)(n + r - 4)}{n!12}$$

Thus, $a_3(0) = a_4(0) = 0$.

Example

Therefore a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{n\geq 5} \frac{(n-3)(n-4)}{n!12} x^n$$

$$= 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{k\geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

Since

$$\sum_{k\geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

is a multiple of $y_1(x)$,
we get that a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12}.$$

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

The larger root always yields a fractional power series solution.

In the first case, the smaller root also yields a fractional power series solution.

In the second and third cases, the second solution may involve a log term.