MA-207 Differential Equations II

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Let us write down some classical ODE's.

- (Euler equation) $\alpha x^2 y'' + \beta x y' + \gamma y = 0$
- (Bessel equation) $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$. We will next look at this case more closely.
- (Laguerre equation) $x y'' + (1 - x) y' + \lambda y = 0$

Gamma function

Define for all $p \geq 1$, the Gamma function

$$\Gamma(p) := \int_0^\infty t^{p-1} e^{-t} dt$$

There is a problem if $p < 1$, since t^{p-1} is unbounded near 0.
For $p > 1$, there is no problem because e^{-t} is rapidly decreasing.

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

For any real number $p \geq 1$,

$$\Gamma(p+1) = \lim_{x \to \infty} \int_0^x t^p e^{-t} dt = p \left(\lim_{x \to \infty} \int_0^x t^{p-1} e^{-t} dt \right) = p\Gamma(p)$$

$$\Gamma(p+1) = p\,\Gamma(p) \implies \Gamma(p) = \frac{\Gamma(p+1)}{p} \qquad (*)$$

We use $(*)$ to extend the gamma function to all real numbers except non-positive integers $0, -1, -2, \ldots$

Note $0 < p < 1 \implies 1 < p+1 < 2$, hence $\Gamma(p+1)$ is defined. We use $(*)$ to define $\Gamma(p)$.

Next, $-1 < p < 0 \implies 0 < p+1 < 1$. Since $\Gamma(p+1)$ is defined above; use $(*)$ to define $\Gamma(p)$. Proceed like this

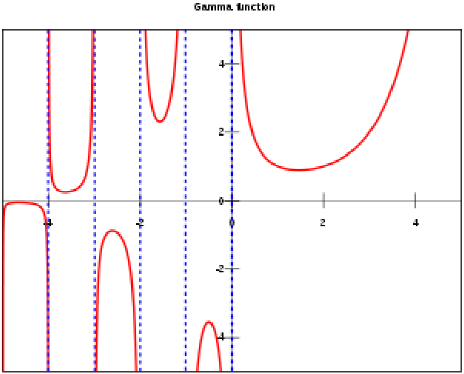
For example, $\Gamma(-\frac{5}{2}) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})} = \frac{\Gamma(\frac{1}{2})(= \sqrt{\pi})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}$

Further

$$\lim_{p\to 0}\Gamma(p)=\lim_{p\to 0}\frac{\Gamma(p+1)}{p}=\pm\infty$$

according as $p\to 0$ from right or left.

The graph of Gamma function is shown below.



Gamma function

Though the gamma function is now defined for all real numbers (except the non positive integers),
the integral representation is valid only for $p > 0$.

It is useful to rewrite

$$\frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)}$$

This holds for all p if we impose the natural condition that the reciprocal of Γ evaluated at a non positive integer is 0.

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty t^{-1/2} e^{-t} dt \\ &= 2\int_0^\infty e^{-s^2} ds \qquad \text{(use the substitution } t = s^2\text{)} \\ &= \sqrt{\pi}\end{aligned}$$

Gamma function

By translating,

$$
\begin{aligned}
\Gamma(1/2) & & &= \sqrt{\pi} &&\approx 1.772 \\
\Gamma(-1/2) &= \frac{\Gamma(1/2)}{-1/2} & &= -2\sqrt{\pi} &&\approx -3.545 \\
\Gamma(-3/2) &= \frac{\Gamma(-1/2)}{-3/2} & &= \frac{4}{3}\sqrt{\pi} &&\approx 2.363 \\
\Gamma(3/2) &= \frac{1}{2}\Gamma(1/2) & &= \frac{1}{2}\sqrt{\pi} &&\approx 0.886 \\
\Gamma(5/2) &= \frac{3}{2}\Gamma(3/2) & &= \frac{3}{4}\sqrt{\pi} &&\approx 1.329 \\
\Gamma(7/2) &= \frac{5}{2}\Gamma(5/2) & &= \frac{15}{8}\sqrt{\pi} &&\approx 3.323
\end{aligned}
$$

Bessel equation

Bessel equation is the second-order linear ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad p \geq 0 \qquad\qquad (*)$$

Its solutions are called Bessel functions.

Bessel functions have applications in physics and engineering.

Since $x = 0$ is a regular singular point of $(*)$, we get a Frobenius solution, called Bessel function of first kind.

The second linearly independent solution of $(*)$ is called Bessel function of second kind.

For Frobenius solution, put $y(x) = x^r \sum_{n=0}^{\infty} a_n(r)x^n \quad a_0 = 1$.

Bessel equation: First solution

The indicial equation, that is, coefficient of x^r, for Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ is

$$I(r) = r(r-1) + r - p^2 = r^2 - p^2 = 0$$

The roots are $r_1 = p$ and $r_2 = -p$.
For recurrence relations, equating coefficient of x^{n+r} to 0 (for $n \geq 1$) we get

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \qquad n \geq 2$$
$$((r+1)^2 - p^2)a_1(r) = 0 \implies a_1(r) = 0$$

So all odd terms $a_{2n+1}(r) = 0$.

$$a_{2n}(r) = \frac{-1}{(r+2n)^2 - p^2}\, a_{2n-2}$$
$$= \frac{(-1)^n}{((r+2)^2 - p^2)((r+4)^2 - p^2)\ldots((r+2n)^2 - p^2)}$$

For Frobenius solution, set $r = p$ the larger root.

$$a_{2n}(p) = \frac{(-1)^n}{((p+2)^2 - p^2)((p+4)^2 - p^2)\dots((p+2n)^2 - p^2)}$$
$$= \frac{(-1)^n}{(2(2p+2))(4(2p+4))\dots(2n(2p+2n))}$$
$$= \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)}$$

The solution $$y_1(x) = x^p \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n}$$

converges on $(0, \infty)$.

Multiply $y_1(x)$ by $\frac{1}{2^p \Gamma(1+p)}$ to get to the convenient form

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n} \quad x > 0.$$

This is called the Bessel function of first kind of order p.

$$J_p(x) := \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} \quad x > 0.$$

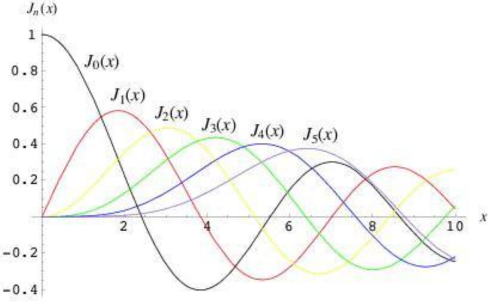
The Bessel function of order 0 is

$$J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!n!} \left(\frac{x}{2}\right)^{2n}$$
$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!3!}\left(\frac{x}{2}\right)^6 + \dots$$

The Bessel function of order 1 is

$$J_1(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$
$$= \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots$$

Both $J_0(x)$ and $J_1(x)$ have a damped oscillatory behavior having an infinite number of zeros.



Further, they satisfy derivative identities similar to $\cos x$ and $\sin x$.

$$J_0'(x) = -J_1(x) \qquad [xJ_1(x)]' = xJ_0(x)$$

Second independent solution of Bessel equation

Recall $r_1 = p$ and $r_2 = -p$ are roots of indicial equation.
So that $r_1 - r_2 = 2p$.
The analysis to get a second independent solution of the Bessel equation splits into the following cases

- $2p$ is not an integer
- $2p$ is an odd positive integer
- $2p$ is an even positive integer
- $p = 0$

Second independent solution of Bessel equation

Case 1: $2p$ is not an integer

Solving the recursion

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \quad n \geq 2 \qquad a_1(r) = 0.$$

for $r = -p$, we obtain

$$y_2(x) = x^{-p} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n! (1-p) \dots (n-p)} x^{2n}$$

Multiplying by $\dfrac{1}{2^{-p}\Gamma(1-p)}$ (Caution: This should be a nonzero real number!)

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n \geq 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n} \qquad x > 0.$$

This is a second solution of the Bessel equation linearly independent of $J_p(x)$.
It is unbounded near $x = 0$.

Recall that the second solution is given by

$$y_2(x) = \sum_{n \geq 0} A_n'(-p)x^{n-p} + \sum_{n \geq 0} A_n(-p)x^{n-p}\log x$$

where

$$A_n(r) := (r+p)a_n(r)$$

$2p$ is an odd positive integer, that is, $p = \frac{2l+1}{2}$ for some $l > 0$.

We have seen that $A_{2n+1}(r) = (r+p)a_{2n+1}(r) = 0$

$$a_{2n}(r) = \frac{(-1)^n}{\prod_{i=1}^{n}((r+2i)^2 - p^2)}.$$

Since the polynomial $\prod_{i=1}^{n}((r+2i)^2-p^2)$ evaluated at $r=-p$, is $\prod_{i=1}^{n} 4i(i-p) \neq 0$,

the function $a_{2n}(r)$ is analytic in a neighborhood of $-p$.

Thus, $A_{2n}(-p)=0$ and $A'_{2n}(-p)=a_{2n}(-p)$.
Thus, in this case we obtain that the second solution is

$$y_2(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)}\, x^{2n-p}$$

Multiplying by $\dfrac{1}{2^{-p}\Gamma(1-p)}$ (Caution: This should be a nonzero real number!)

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n} \quad x>0.$$

In this case p is a positive integer. As before, $A_{2n+1}(r) = 0$.
The polynomial $\prod_{i=1}^{n}((r+2i)^2 - p^2)$ evaluated at $r = -p$, is $\prod_{i=1}^{n} 4i(i-p)$,
Thus, if $n < p$, then $a_{2n}(r)$ is analytic in a neighborhood of $-p$.
Thus, if $n < p$, then $A_{2n}(-p) = 0$ and

$$A'_{2n}(-p) = a_{2n}(-p) = \frac{(-1)^n}{2^{2n}n!(1-p)\dots(n-p)} = \frac{1}{2^{2n}n!(p-n)!}$$

If $n \geq p$, then

$$\begin{aligned} A_{2n}(-p) &= \frac{2(-1)^n}{2^{2n}n!(1-p)\dots(-1)\cdot 1\cdot 2\cdots(n-p)} \\ &= \frac{-2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!} \end{aligned}$$

Define

$$f(r) := \left(\prod_{i=1}^{p-1}((r+2i)^2-p^2)\right)(r+3p)\left(\prod_{i=p+1}^{n}((r+2i)^2-p^2)\right) \quad (*)$$

Then

$$A_{2n}(r)f(r) = (-1)^n$$

Differentiating the above and setting $r = -p$ we get

$$A'_{2n}(-p)f(-p) + A_{2n}(-p)f'(-p) = 0$$

Taking log and differentiating $(*)$ we get

$$f'(-p) = f(-p)\Big(\frac{1}{2p} + \sum_{i\in\{1,2,\dots,n\}\backslash p} \frac{1}{2i} + \frac{1}{2(i-p)}\Big)$$

$$= f(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big),$$

where

$$H_0 = 0, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Thus,

$$A'_{2n}(-p) = -A_{2n}(-p)\left(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right)$$
$$= \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\left(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right)$$

Thus, we get

$$y_2(x) = \sum_{n=0}^{p-1} \frac{1}{2^{2n}n!(p-n)!}x^{2n-p} +$$
$$\sum_{n\geq p} \frac{(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\Big(H_n - H_{p-1} + H_{n-p}\Big)x^{2n-p} +$$
$$-\sum_{n\geq p} \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}x^{2n-p}\log x$$

is a second solution.

Second independent solution of Bessel equation

Case 3: $p = 0$ (Repeated root case)

The indicial equation has a repeated root $r_1 = r_2 = 0$,

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2\dots(r+2n)^2} \qquad a_{2n+1}(r) = 0$$

Differentiating $a_{2n}(r)$ with respect to r, we get

$$a_{2n}(r)' = -2a_{2n}(r)\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2n}\right)$$

$$a'_{2n}(0) = \frac{(-1)^{n-1}H_n}{2^{2n}(n!)^2}, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

By theorem stated earlier, the second solution is

$$y_2(x) = J_0(x)\ln x - \sum_{n\geq 1}\frac{(-1)^n H_n}{2^{2n}(n!)^2}x^{2n} \qquad x > 0$$

where $y_1(x) = J_0(x) = \sum_{n\geq 0}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}$ is Frobenius solution.

For real p, define

$$J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

1. The above is a well defined power series once we know that the Gamma function never vanishes.
2. If $p \notin \{0, 1, 2, \ldots\}$ $J_p(x)$ and $J_{-p}(x)$ are the two independent solutions of the Bessel equation.
3. If $p \in \{0, 1, 2, \ldots\}$ then $J_{-p}(x) = (-1)^p J_p(x)$. Thus, in this case the second solution is not $J_{-p}(x)$.

Bessel identities

1. $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$
2. $$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

   The above two can be obtained by formally differentiating the power series.

3. $$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$
4. $$J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

   These follow from (1) and (2). Expand LHS and divide by $x^{\pm p}$;

5. $$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$$
6. $$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

Add and subtract (3) and (4) to get (5) and (6).

Problem: Let $p > 0$. Show that between any two consecutive positive zeros of $J_p(x)$, there exists precisely one zero of $J_{p-1}(x)$ and precisely one zero of $J_{p+1}(x)$

Solution: Let $0 < c < d$ be two consecutive zeros of $J_p(x)$.

So $x^p J_p(x)$ vanishes at c and d. By Rolle's theorem,

$$[x^p J_p(x)]'(b) = 0 \quad \text{for some} \quad b \in (c, d)$$

As

$$[x^p J_p(x)]' = x^p J_{p-1}(x)$$

we get $J_{p-1}(b) = 0$.

Repeating the above argument with the identity $[x^{-p} J_p(x)]' = -x^{-p} J_{p+1}(x)$, we get that $J_{p+1}(x)$ has a root in (c, d).

Thus, we have proved that both $J_{p-1}(x)$ and $J_{p+1}(x)$ have at least one root in (c, d).

If $J_{p-1}(x)$ had two roots in (c, d), then from above, we conclude that $J_p(x)$ would have a root in (c, d). However, this contradicts the assumption that c and d are consecutive roots. Thus, J_{p-1} has exactly one root in (c, d).

Similarly, $J_{p+1}(x)$ has exactly one root in (c, d).

Problem: Find a and c so that $J_2(x) - J_0(x) = aJ_c''(x)$.

Solution: Using $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$ for $p = 1$, we get

$J_0(x) - J_2(x) = 2J_1'(x)$

Now using $[x^{-p}J_p(x)]' = -x^{-p}J_{p+1}$ for $p = 0$, we get

$J_0'(x) = -J_1(x)$.

Therefore,

$J_2(x) - J_0(x) = -2J_1'(x) = 2J_0''(x)$.

Hence $a = 2$ and $c = 0$. □

Consequences of Bessel identities

We can use

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x$$

to see that $J_p(x)$ are elementary functions for $p \in \mathbb{Z} + \frac{1}{2}$.

For example,

- $$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

- $$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$$

Consequences of Bessel identities

- $J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$

  $= \sqrt{\frac{2}{\pi x}} \left(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x \right)$

These functions are called spherical Bessel functions as they arise in solving wave equations in spherical coordinates.

An interesting theorem

An algebraic function is any function $y = f(x)$ that satisfies an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

for some n, where each $P_i(x)$ is a polynomial.

Take algebraic functions, trigonometric functions (example $\sin x,\ \cos x,\ \tan x$),inverse trigonometric functions (example $\sin^{-1} x,\ \cos^{-1} x,\ \tan^{-1} x$),exponential and logarithmic functions, (example $e^{x^2},\ \log(x^2 + x + 1)$) and all other functions that can be constructed from these functions by adding, subtracting, multiplying, dividing and composition of functions. Any function which can be constructed like this will be called an elementary function.Thus

$$y = tan\left[\frac{xe^{1/x^2} + \tan^{-1}(1 + x^2) + \sqrt{x^2 + 3}}{\sin x \cos 2x - \sqrt{\log x} + x^{3/2}}\right]^{1/3}$$

is an elementary function.

An interesting theorem

Theorem (Liouville)

$J_{m+\frac{1}{2}}(x)$'s are the only Bessel functions which are elementary functions.

Qualitative properties of solutions

It is rarely possible to solve 2nd order linear ODE

$$y'' + P(x)y' + Q(x)y = 0$$

in terms of familiar elementary functions.

Then how do we understand the nature and properties of solutions.

It is surprising that we can obtain quite a bit of information about the solution from the ODE itself.

Theorem (Sturm separation theorem)

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

P, Q continuous on (a, b). Then
(1) $y_1(x)$ and $y_2(x)$ have no common zero in (a, b).
(2) Between any two successive zeros of $y_1(x)$, there is exactly one zero of $y_2(x)$ and vice versa.

Proof of (1). Consider the Wronskian

$$W(x) := W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

It satisfies the differential equation $W' = -P(x)W$ and so is given by

$$W(x) = C\exp\Big(\int_{a_0}^{x} -P(t)dt\Big) \qquad a_0 \in (a, b)$$

Qualitative properties of solutions

In particular, since y_1 and y_2 are linearly independent, the Wronskian is nonzero and so it never vanishes. This proves (1).

Proof of (2). Let x_1 and x_2 be successive zeros of $y_1(x)$.

First let us show y_2 has a zero in (x_1, x_2).
The Wronskian $W(x)$ has the same sign in the interval (a, b) as it never vanishes. Thus, $W(x_1)$ and $W(x_2)$ have the same sign.

$$0 \neq W(x_1) = -y_1'(x_1)y_2(x_1) \qquad 0 \neq W(x_2) = -y_1'(x_2)y_2(x_2)$$

We conclude that $y_1'(x_1)$ and $y_1'(x_2)$ are nonzero.

It follows that $y_1'(x_1)$ and $y_1'(x_2)$ have opposite signs since x_1 and x_2 are consecutive zeros of y_1.

It follows that $y_2(x_1)$ and $y_2(x_2)$ have opposite signs. Thus, $y_2(x)$ has a zero in (x_1, x_2).

Qualitative properties of solutions

If $y_2(x)$ had two zeros in the interval $x_1 < \alpha < \beta < x_2$, then by the same reasoning, y_1 will have a zero in (α, β), which contradicts the assumption that x_1 and x_2 are successive zeros of y_1. This completes the proof of the theorem.

As a consequence, if y_1 and y_2 are linearly independent solution of $y'' + P(x)y' + Q(x)y = 0$, P, Q continuous on (a, b) then the number of zeros of y_1 and y_2 on (a, b) differ by at most 1.

In particular, either both have finite number of zeros or both have infinite number of zeros in (a, b).

For further discussion, we need that any ODE in the "standard" form $y'' + P(x)y' + Q(x)y = 0$ can be written in the "normal" form $u'' + q(x)u = 0$.

Define $v(x) := \exp\Big(\int_{a_0}^{x} -\frac{1}{2} P(t)dt \Big)$ and set $u(x) = \frac{y(x)}{v(x)}$.

One easily checks that $u(x)$ satisfies the differential equation

$$u'' + q(x)u = 0 \qquad q(x) := Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$$

It is clear that the zeros of u are the same as those of y.

Also note that we need $P(x)$ to be once differentiable.

Qualitative properties of solutions

Theorem

Let $q(x)$ be continuous on the interval (α, β). Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on finite interval $[a, b] \subset (\alpha, \beta)$. Then $u(x)$ has at most finite number of zeros in $[a, b]$.
Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.

Proof. Assume $u(x)$ has infinitely many zeros in $[a, b]$. Then $\exists\, x_0 \in [a, b]$ and a sequence of zeros $x_n \neq x_0$ such that $x_n \to x_0$ as $n \to \infty$.
$u(x_0) = \lim_{x_n \to x_0} u(x_n) = 0$ (u is continuous) and

$$u'(x_0) = \lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0\,.$$

Since $u(x_0)$ and $u'(x_0)$ are both 0, it follows that the Wronskian at x_0 is 0. But this is a contradiction as the Wronskian at x_0 is nonzero. □

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$. If $q(x) < 0$ in (a, b) and continuous then $u(x)$ has atmost one zero in (a, b).

Proof. Assume $u(x_0) = 0$. Then $u'(x_0) \neq 0$, since Wronskian $W(x_0) \neq 0$.

Assume x_1 is next zero of $u(x)$ after x_0.

If necessary, multiply by -1 and assume that $u'(x_0) > 0$.

Then $u(x) > 0$ on (x_0, x_1).

Since $u''(x) = -q(x)u(x) > 0$ on (x_0, x_1), $u'(x)$ is an increasing function on (x_0, x_1).

By Rolle's theorem u' has a zero in (x_0, x_1).

But this is not possible as u' is increasing on (x_0, x_1). $\square$

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ Let $q(x)$ be continuous and $q(x) > 0$ for all $x > x_0 > 0$.

If $\int_{x_0}^{\infty} q(x)\,dx = \infty$,

then $u(x)$ has infinitely many zeros on $(0, \infty)$.

Proof. Assume $u(x)$ has only finitely many zeros on $(0, \infty)$.

Then there is $x_1 > x_0$ such that $u(x) \neq 0$ for $x \geq x_1$. Assume $u(x) > 0$ for $x \geq x_1$.

Then $u''(x) = -q(x)u(x) < 0$ for $x \geq x_1$. Hence $u'(x)$ is decreasing for $x \geq x_1$.

If we show that $u'(x_2) < 0$ for some $x_2 > x_1$, then we get for $x > x_2$

$$
\begin{aligned}
u(x) = \int_{x_2}^{x} u'(t)dt + u(x_2) &\leq \int_{x_2}^{x} u'(x_2)dt + u(x_2) \\
&\leq u'(x_2)(x - x_2) + u(x_2)
\end{aligned}
$$

Thus if x is sufficiently large, then $u(x) < 0$, a contradiction.

To show that $u'(x) < 0$ for some $x > x_1$. Put

$$v(x) = -\frac{u'(x)}{u(x)}, \quad \text{for} \quad x \geq x_1$$

$$v' = \frac{-u''u + u'^2}{u^2} = \frac{q(x)u^2 + u'^2}{u^2} = q(x) + v(x)^2$$

Integrating we get

$$v(x) - v(x_1) = \int_{x_1}^{x} q(x)\,dx + \int_{x_1}^{x} v(x)^2\,dx$$

$\int_{x_0}^{\infty} q(x)\,dx = \infty \implies v(x) > 0$ for large x.

Thus, $u'(x) = -u(x)v(x)$ and this shows that $u'(x) < 0$ for x large.

Theorem

In Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ *Substituting* $u(x) = \sqrt{x}y(x)$, *we get*

$$u'' + \left[1 + \frac{1-4p^2}{4x^2}\right]u = 0$$

$q(x) = 1 + \dfrac{1-4p^2}{4x^2}$ *is continuous and* $q(x) > 0$ *for* $x > x_0 > 0$. *Further,*

$$\int_{x_0}^{\infty}\left(1 + \frac{1-4p^2}{4x^2}\right)dx = \infty$$

By previous theorem, $u(x)$, *hence any Bessel function has infinitely many zeros on* $(0, \infty)$.

Corollary

Let $Z^{(p)}$ be the set of zeros of Bessel function $J_p(x)$ on $(0, \infty)$. Since $Z^{(p)}$ is an infinite set, it is not bounded.

We will conside the following question.

Write $Z^{(p)} = \{x_1, x_2, \ldots\}$ as increasing sequence $x_n < x_{n+1}$.

Question. What is the limit of $x_{n+1} - x_n$ as $n \to \infty$?

We will need the Sturm comparison theorem.

Qualitative properties of solutions

Theorem (Sturm Comparison theorem)

Let $y(x)$ be a non-trivial solutions of

$$y'' + q(x)y = 0$$

and $z(x)$ be a non-trivial solutions of

$$z'' + r(x)z = 0$$

where $q(x) > r(x) > 0$ are continuous.
Then $y(x)$ vanishes at least once between any two consecutive zeros of $z(x)$.

Compare $y'' + 4y = 0$ and $z'' + z = 0$.

Here $(q(x) =) \, 4 > (r(x) =) \, 1 > 0$

Zeros of $y(x)$ are $\pi/2$ apart and that of $z(x)$ are π apart.

Proof of Sturm Comparison theorem.

Let $x_1 < x_2$ be consecutive zeros of $z(x)$.

Assume $y(x)$ has no zero in (x_1, x_2).

We may assume $z(x) > 0$ and $y(x) > 0$ on (x_1, x_2). Hence $z'(x_1) > 0$ and $z'(x_2) < 0$.

Consider the function $W(x) = y(x)z'(x) - y'(x)z(x)$

$$W'(x) = yz'' - y''z = y(-rz) - (-qy)z = (q - r)yz > 0$$

on (x_1, x_2).

Integrating from x_1 to x_2, we get

$$W(x_2) - W(x_1) > 0 \implies W(x_2) > W(x_1)$$

But $W(x_1) = y(x_1)z'(x_1) > 0$ and $W(x_2) = y(x_2)z'(x_2) < 0$, a contradiction. □

Qualitative properties of solutions

Theorem

Substituting $u(x) = \sqrt{x}y(x)$ *in Bessel equation, we get Bessel equation in normal form* $(p \geq 0)$

$$u'' + q(x)u = 0, \qquad q(x) = 1 + \frac{1 - 4p^2}{4x^2}$$

- $p < 1/2 \implies q(x) > 1$
- $p = 1/2 \implies q(x) = 1$ *(Well known, hence, uninteresting)*
- $p > 1/2 \implies q(x) < 1$

Use $z'' + z = 0$ *and Sturm comparison theorem.*

Let $y_p(x)$ *be a non-trivial solution of Bessel equation. Then we get* ...

Qualitative properties of solutions

Theorem

- $\boxed{p < 1/2} \implies$ *Between any two roots of* $\alpha \cos x + \beta \sin x$ *there is a root of* $y_p(x)$.
- $\boxed{p = 1/2} \implies x_2 - x_1 = \pi$
- $\boxed{p > 1/2} \implies$ *Between any two roots of* $y_p(x)$ *there is a root of* $\alpha \cos x + \beta \sin x$.

We can say more than the above.

Claim: Suppose $p < 1/2$ and $a < b < c$ are consecutive roots of $u(x)$. Then $b - a < c - b$. That is, the difference between the successive roots keeps increasing.

To see this, consider the function $f := u(x - b + a)$ defined on the interval (b, ∞).

It is a trivial check that f satisfies the differential equation

$$f'' + r(x)f = 0 \qquad r(x) := q(x - b + a)$$

Since $p < 1/2$ the function q is strictly decreasing. Thus, on (b, ∞) we have $r(x) > q(x) > 0$.

Applying Sturm's comparison theorem we get that there is a $b < x_0 < c$ such that $f(x_0) = u(x_0 - b + a) = 0$.

Clearly,

- $b < x_0 \implies a < x_0 - b + a$
- $a < b \implies x_0 - b + a < x_0$

Thus,

$$a < x_0 - b + a < x_0 < c$$

However, $a < b < c$ are successive roots of $u(x)$. This forces that

$$x_0 - b + a = b \qquad \text{that is} \qquad x_0 = 2b - a$$

As $x_0 < c$ we get that $2b - a < c$, that is, $b - a < c - b$.

Next we claim that the difference between any two successive roots of u is strictly less than π.

If not, then let $a < b$ be successive roots such that $b - a \geq \pi$

Since u has infinitely many roots, and their difference is strictly increasing, we may assume that $b - a > \pi$.

But now we can choose $\alpha, \beta \in \mathbb{R}$ such that $\alpha \cos x + \beta \sin x$ has two roots in (a, b), which contradicts Sturm's comparison theorem.

Thus, we have proved that if $\{x_n\}$ are the roots of u in increasing order, then the difference $x_{n+1} - x_n$ is strictly increasing and bounded above by π.

Next let us show that these differences converge to π. If not, then $(x_{n+1} - x_n) \to \gamma < \pi$. Choose $1 < \delta$, sufficiently close to 1 such that $\gamma < \frac{\pi}{\delta} < \pi$.

The function $q(x)$ is decreasing to 1. Therefore, there is a $x_0 \in \mathbb{R}$, sufficiently large, such that $q(x_0) < \delta^2$. Apply Sturm's comparison on the interval (x_0, ∞) to the differential equations $u'' + q(x)u = 0$ and $z'' + \delta^2 z = 0$.

Thus, between any two roots of u there is a root of z. Let a and b be two consecutive roots of u such that $x_0 < a < b$. Since $b - a < \gamma < \frac{\pi}{\delta}$, find a' and b' such that $x_0 < a' < a < b < b'$ and $b' - a' = \frac{\pi}{\delta}$.

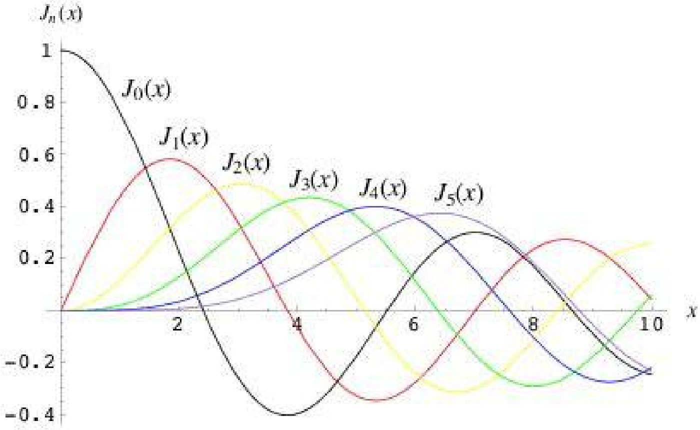
Find α and β such that the function $\alpha \cos \delta\, x + \beta \sin \delta\, x$ vanishes at a'. This function is a solution to the ODE $z'' + \delta^2 z = 0$. The next root of this function is at $a' + \frac{\pi}{\delta} = b'$. Thus, we get a contradiction to Sturm's theorem which says that there is a root of this function in the interval (a, b).

Thus, we have proved

Theorem

If $p < 1/2$ *then the sequence of differences of roots of* u*,* $x_{n+1} - x_n$ *is increasing and tends to* π*.*

Similarly, we can prove that if $p > 1/2$ *then the sequence of difference of roots of* u *is decreasing and tends to* π*.*



$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
$J_0(x)$
$J_1(x)$
$J_2(x)$
$J_3(x)$
$J_4(x)$
$J_5(x)$
2
4
6
8
10
x

Expansion in terms of Bessel functions

The first few zeroes of Bessel functions are tabulated below.

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Question. Why are we concerned with zeros of Bessel function $J_p(x)$?

It is often required in mathematical physics to expand a given function in terms of Bessel functions.

Expansion in terms of Bessel functions

Definition

For a scalar a, the scaled Bessel functions $J_p(ax)$ are solutions of

$$x^2y'' + xy' + (a^2x^2 - p^2)y = 0$$

known as scaled Bessel equation.

Simplest and most useful expansions are of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_{p,n}x) = a_1 J_p(\lambda_{p,1}x) + a_2 J_p(\lambda_{p,2}x) + \dots$$

where $f(x)$ is defined on, (say) $[0, 1]$, and $\lambda_{p,n}$'s are zeros of Bessel function $J_p(x)$, $p \geq 0$.

Qn. How to compute the coefficients a_n?

Orthogonality

Define an inner product on functions on $[0, 1]$ by

$$\langle f, g\rangle := \int_0^1 xf(x)g(x)\,dx$$

This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by x and the interval of integration is from 0 to 1.

We call a function on $[0, 1]$ square integrable with respect to this inner product if

$$\int_0^1 xf(x)^2 dx < \infty$$

The multiplying factor x is called a weight function.

Orthogonality

Fix $p \geq 0$. Let $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ denote the set of zeros of $J_p(x)$ on $(0, \infty)$.

Theorem

The set of scaled Bessel functions

$$\{J_p(\lambda_{p,1}x), J_p(\lambda_{p,2}x), \ldots\}$$

form an orthogonal family w.r.t. above inner product, i.e. $\langle J_p(\lambda_{p,k}x), J_p(\lambda_{p,l}x)\rangle :=$

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)\,dx = \begin{cases} \frac{1}{2}[J_{p+1}(\lambda_{p,k})]^2 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Proof of orthogonality of scaled Bessel functions

If a, b are positive scalars, then $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfies

$$u'' + \frac{1}{x}u' + \left(a^2 - \frac{p^2}{x^2}\right)u = 0$$

$$v'' + \frac{1}{x}v' + \left(b^2 - \frac{p^2}{x^2}\right)v = 0$$

Multiply by v and u resp. and subtract, we get

$$(vu'' - uv'') + \frac{1}{x}(vu' - uv') + (a^2 - b^2)uv = 0$$

$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

$$(x(u'v - v'u))' = (b^2 - a^2)xuv$$

Orthogonality

$$(b^2 - a^2) \int_0^1 xuv\, dx = [x(u'v - v'u)]\Big|_0^1 = (u'v - v'u)(1)$$

$$(b^2 - a^2) \int_0^1 x J_p(ax) J_p(bx)\, dx = J_p'(a) J_p(b) - J_p'(b) J_p(a)$$

So if $a = \lambda_{p,k}$ and $b = \lambda_{p,l}$ are **distinct** (RHS is 0 since these are roots of $J_p(x)$), then

$$\int_0^1 x J_p(\lambda_{p,k} x) J_p(\lambda_{p,l} x)\, dx = 0$$

To compute the norm of $J_p(\lambda_{p,k} x)$, consider

$$\begin{aligned} 2x^2 u' \left[u'' + \frac{1}{x} u' + (a^2 - \frac{p^2}{x^2}) u \right] &= 0 \\ = [x^2 u'^2 + (a^2 x^2 - p^2) u^2]' - 2a^2 x u^2 \end{aligned}$$

Orthogonality

Integrate on $[0,1]$,

$$2a^2 \int_0^1 xu^2\,dx = [x^2u'^2 + (a^2x^2 - p^2)u^2]\Big|_0^1$$

Since $p \geq 0$, $(pu(0))^2 = (pJ_p(0))^2 = 0$.

Thus, $\big(x^2u'^2 + (a^2x^2 - p^2)u^2\big)(0) = 0$.

Further, $u'(1) = aJ_p'(a)$, so we get

$$\big(x^2u'^2 + (a^2x^2 - p^2)u^2\big)(1) = a^2J_p'(a)^2 + (a^2 - p^2)J_p(a)^2$$

Put $a = \lambda_{p,k}$ to get

$$2\lambda_{p,k}^2 \int_0^1 xJ_p(\lambda_{p,k}x)^2\,dx = \lambda_{p,k}^2 J_p'(\lambda_{p,k})^2$$

Thus,

$$\int_0^1 xJ_p(\lambda_{p,k}x)^2\,dx = \frac{1}{2}J_p'(\lambda_{p,k})^2 = \frac{1}{2}J_{p+1}(\lambda_{p,k})^2$$

for last equality, use $x = \lambda_{p,k}$ in $J_p'(x) - \dfrac{p}{x}J_p(x) = J_{p+1}(x)$

Expansion in terms of Bessel functions

Theorem

Fix $p \geq 0$ *and* $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ *: zeros of* $J_p(x)$ *on* $(0, \infty)$*. Any square-integrable function* $f(x)$ *on* $[0, 1]$ *can be expanded in a series of scaled Bessel functions* $J_p(\lambda_{p,n} x)$ *as*

$$f(x) = \sum_{n \geq 1} c_n J_p(\lambda_{p,n} x)$$

where

$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x\, f(x)\, J_p(\lambda_{p,n} x)\, dx$$

This is Fourier-Bessel series of $f(x)$ *for parameter* p*.*

Example. Let us compute the Fourier-Bessel series (for $p=0$) of $f(x)=1$ in the interval $0 \le x \le 1$.

Use $\frac{d}{dx}\big(x^p J_p(x)\big) = x^p J_{p-1}(x)$ for $p=1$.

$$\int_0^1 x\, J_0(\lambda_{0,n}x)\, dx = \frac{1}{\lambda_{0,n}} x\, J_1(\lambda_{0,n}x)\Big|_0^1 = \frac{J_1(\lambda_{0,n})}{\lambda_{0,n}}$$

$$c_n = \frac{2}{[J_1(\lambda_{0,n})]^2} \int_0^1 x\, f(x)\, J_0(\lambda_{0,n}x)\, dx = \frac{2}{\lambda_{0,n} J_1(\lambda_{0,n})}$$

Thus, the Fourier-Bessel series of $f(x)$ is

$$\sum_{n\ge 1} \frac{2}{\lambda_{0,n} J_1(\lambda_{0,n})} J_0(\lambda_{0,n}x)$$

By next theorem, this converges to 1 for $0 < x < 1$.

Expansion in terms of Bessel functions

Convergence in norm

Fourier-Bessel series converges to $f(x)$ in norm, i.e.

$$\|f(x) - \sum_{n=1}^{m} c_n J_p(\lambda_{p,n} x)\| \text{ converges to } 0 \text{ as } m \to \infty$$

For pointwise convergence, we have

Bessel expansion theorem

Assume f and f' have at most a finite number of jump discontinuities in $[0,1]$, then the Bessel series converges for $0 < x < 1$ to

$$\frac{f(x_-) + f(x_+)}{2}$$

At $x = 1$, the series always converges to 0 for all f,
at $x = 0$, if $p = 0$ then it converges to $f(0_+)$.
at $x = 0$, if $p > 0$ then it converges to 0.