# MA-207 Differential Equations II

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September 7, 2021

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- (Laguerre equation) $x y'' + (1 - x) y' + \lambda y = 0$

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For example, $\Gamma(-\frac{5}{2}) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})} = \frac{\Gamma(\frac{1}{2})(=\sqrt{\pi})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}$

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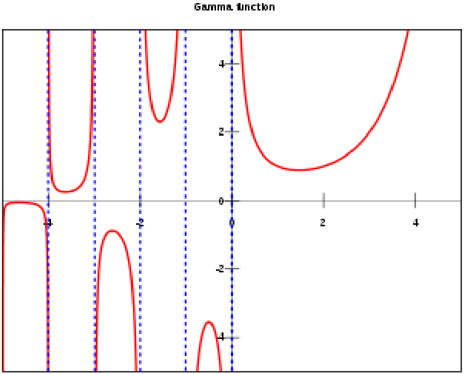
The graph of Gamma function is shown below.



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It is useful to rewrite

$$\frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)}$$

This holds for all $p$ if we impose the natural condition that the reciprocal of $\Gamma$ evaluated at a non positive integer is $0$.

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$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty t^{-1/2} e^{-t} dt \\ &= 2\int_0^\infty e^{-s^2} ds \qquad \text{(use the substitution } t = s^2\text{)} \\ &= \sqrt{\pi}\end{aligned}$$

By translating,

$$\begin{aligned}
\Gamma(1/2) & & &= \sqrt{\pi} &&\approx 1.772 \\
\Gamma(-1/2) &= \frac{\Gamma(1/2)}{-1/2} & &= -2\sqrt{\pi} &&\approx -3.545 \\
\Gamma(-3/2) &= \frac{\Gamma(-1/2)}{-3/2} & &= \frac{4}{3}\sqrt{\pi} &&\approx 2.363 \\
\Gamma(3/2) &= \frac{1}{2}\Gamma(1/2) & &= \frac{1}{2}\sqrt{\pi} &&\approx 0.886 \\
\Gamma(5/2) &= \frac{3}{2}\Gamma(3/2) & &= \frac{3}{4}\sqrt{\pi} &&\approx 1.329 \\
\Gamma(7/2) &= \frac{5}{2}\Gamma(5/2) & &= \frac{15}{8}\sqrt{\pi} &&\approx 3.323
\end{aligned}$$

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The second linearly independent solution of $(*)$ is called Bessel function of second kind.

For Frobenius solution, put $y(x) = x^r \sum_{n=0}^{\infty} a_n(r)x^n \quad a_0 = 1.$

The indicial equation, that is, coefficient of $x^r$, for Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ is

$$I(r) = r(r-1) + r - p^2 = r^2 - p^2 = 0$$

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For recurrence relations, equating coefficient of $x^{n+r}$ to 0 (for $n \geq 1$) we get

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$$a_{2n}(r) = \frac{-1}{(r+2n)^2 - p^2}\, a_{2n-2}$$

$$= \frac{(-1)^n}{((r+2)^2 - p^2)((r+4)^2 - p^2)\ldots((r+2n)^2 - p^2)}$$

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\begin{aligned}
a_{2n}(p) &= \frac{(-1)^n}{((p+2)^2 - p^2)((p+4)^2 - p^2)\dots((p+2n)^2 - p^2)} \\
&= \frac{(-1)^n}{(2(2p+2))(4(2p+4))\dots(2n(2p+2n))} \\
&= \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)}
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The solution $$y_1(x) = x^p \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n}$$

converges on $(0, \infty)$.

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Multiply $y_1(x)$ by $\dfrac{1}{2^p \Gamma(1+p)}$ to get to the convenient form

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n \geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n} \quad x > 0.$$

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The Bessel function of order $0$ is

$$\begin{aligned} J_0(x) &= \sum_{n\geq 0} \frac{(-1)^n}{n!n!} \left(\frac{x}{2}\right)^{2n} \\ &= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!3!}\left(\frac{x}{2}\right)^6 + \dots \end{aligned}$$

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The Bessel function of order $1$ is

$$\begin{aligned} J_1(x) &= \sum_{n\geq 0} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} \\ &= \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots \end{aligned}$$

Both $J_0(x)$ and $J_1(x)$ have a damped oscillatory behavior having an infinite number of zeros.

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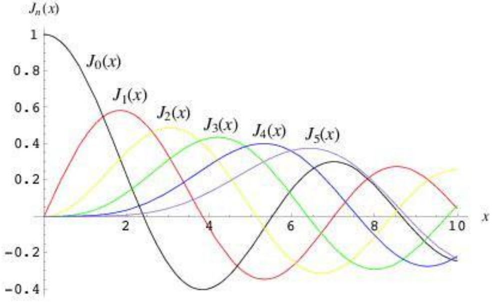

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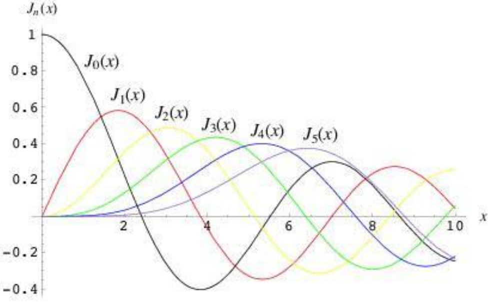


Further, they satisfy derivative identities similar to $\cos x$ and $\sin x$.

$$J_0'(x) = -J_1(x) \qquad\qquad [xJ_1(x)]' = xJ_0(x)$$

# Second independent solution of Bessel equation

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The analysis to get a second independent solution of the Bessel equation splits into the following cases

- $2p$ is not an integer
- $2p$ is an odd positive integer
- $2p$ is an even positive integer
- $p = 0$

Solving the recursion

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \quad n \geq 2 \qquad a_1(r) = 0.$$

for $r = -p$, we obtain

$$y_2(x) = x^{-p} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)} x^{2n}$$

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It is unbounded near $x = 0$.

# Second independent solution of Bessel equation

**Case 1**: $2p$ is not an integer

Recall that the second solution is given by

$$y_2(x) = \sum_{n\geq 0} A_n'(-p)x^{n-p} + \sum_{n\geq 0} A_n(-p)x^{n-p}\log x$$

where

$$A_n(r) := (r+p)a_n(r)$$

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**Case 2(a)**: $2p$ is an odd positive integer

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$$a_{2n}(r) = \frac{(-1)^n}{\prod_{i=1}^{n}((r+2i)^2 - p^2)}.$$

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Thus, $A_{2n}(-p)=0$ and $A'_{2n}(-p)=a_{2n}(-p)$.
Thus, in this case we obtain that the second solution is

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Thus, if $n < p$, then $a_{2n}(r)$ is analytic in a neighborhood of $-p$.
Thus, if $n < p$, then $A_{2n}(-p) = 0$ and

$$A'_{2n}(-p) = a_{2n}(-p) = \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)} = \frac{1}{2^{2n} n!(p-n)!}$$

If $n \geq p$, then

$$\begin{aligned} A_{2n}(-p) &= \frac{2(-1)^n}{2^{2n} n!(1-p)\dots(-1)\cdot 1\cdot 2\cdots(n-p)} \\ &= \frac{-2(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!} \end{aligned}$$

Define

$$f(r) := \left(\prod_{i=1}^{p-1}((r+2i)^2 - p^2)\right)(r+3p)\left(\prod_{i=p+1}^{n}((r+2i)^2 - p^2)\right) \quad (*)$$

Then

$$A_{2n}(r)f(r) = (-1)^n$$

Differentiating the above and setting $r = -p$ we get

$$A'_{2n}(-p)f(-p) + A_{2n}(-p)f'(-p) = 0$$

Taking log and differentiating $(*)$ we get

$$f'(-p) = f(-p)\Big(\frac{1}{2p} + \sum_{i\in\{1,2,\dots,n\}\setminus p} \frac{1}{2i} + \frac{1}{2(i-p)}\Big)$$

$$= f(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big),$$

where

$$H_0 = 0, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Thus,

$$A'_{2n}(-p) = -A_{2n}(-p)\left(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right)$$
$$= \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\left(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right)$$

Thus, we get

$$y_2(x) = \sum_{n=0}^{p-1} \frac{1}{2^{2n}n!(p-n)!}x^{2n-p} +$$
$$\sum_{n\geq p} \frac{(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\left(H_n - H_{p-1} + H_{n-p}\right)x^{2n-p} +$$
$$-\sum_{n\geq p} \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}x^{2n-p}\log x$$

is a second solution.

The indicial equation has a repeated root $r_1 = r_2 = 0$,

# Second independent solution of Bessel equation

**Case 3**: $p = 0$ (Repeated root case)

The indicial equation has a repeated root $r_1 = r_2 = 0$,

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2 \dots (r+2n)^2} \qquad a_{2n+1}(r) = 0$$

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Differentiating $a_{2n}(r)$ with respect to $r$, we get

$$a_{2n}(r)' = -2a_{2n}(r)\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2n}\right)$$

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By theorem stated earlier, the second solution is

$$y_2(x) = J_0(x)\ln x - \sum_{n\geq 1} \frac{(-1)^n H_n}{2^{2n}(n!)^2} x^{2n} \qquad x > 0$$

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where $y_1(x) = J_0(x) = \sum_{n\geq 0}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}$ is Frobenius solution.

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$$J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

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1. The above is a well defined power series once we know that the Gamma function never vanishes.
2. If $p \notin \{0, 1, 2, \dots\}$ $J_p(x)$ and $J_{-p}(x)$ are the two independent solutions of the Bessel equation.
3. If $p \in \{0, 1, 2, \dots\}$ then $J_{-p}(x) = (-1)^p J_p(x)$. Thus, in this case the second solution is not $J_{-p}(x)$.

# Bessel identities

1. $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$
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Add and subtract $(3)$ and $(4)$ to get $(5)$ and $(6)$.

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**Problem**: Let $p > 0$. Show that between any two consecutive positive zeros of $J_p(x)$, there exists precisely one zero of $J_{p-1}(x)$ and precisely one zero of $J_{p+1}(x)$

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Repeating the above argument with the identity $[x^{-p} J_p(x)]' = -x^{-p} J_{p+1}(x)$, we get that $J_{p+1}(x)$ has a root in $(c, d)$.

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Therefore,

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Hence $a = 2$ and $c = 0$.

# Consequences of Bessel identities

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For example,

- $$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$
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These functions are called spherical Bessel functions as they arise in solving wave equations in spherical coordinates.

# An interesting theorem

An algebraic function is any function $y = f(x)$ that satisfies an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

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for some $n$, where each $P_i(x)$ is a polynomial.

Take algebraic functions, trigonometric functions (example $\sin x,\ \cos x,\ \tan x$),inverse trigonometric functions (example $\sin^{-1} x,\ \cos^{-1} x,\ \tan^{-1} x$),exponential and logarithmic functions, (example $e^{x^2},\ \log(x^2 + x + 1)$) and all other functions that can be constructed from these functions by adding, subtracting, multiplying, dividing and composition of functions. Any function which can be constructed like this will be called an elementary function.Thus

$$y = tan\left[\frac{xe^{1/x^2} + \tan^{-1}(1 + x^2) + \sqrt{x^2 + 3}}{\sin x \cos 2x - \sqrt{\log x} + x^{3/2}}\right]^{1/3}$$

is an elementary function.

# An interesting theorem

**Theorem (Liouville)**

$J_{m+\frac{1}{2}}(x)$*'s are the only Bessel functions which are elementary functions.*

It is rarely possible to solve 2nd order linear ODE

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Then how do we understand the nature and properties of solutions.

It is surprising that we can obtain quite a bit of information about the solution from the ODE itself.

**Theorem (Sturm separation theorem)**

*If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of*

$$y'' + P(x)y' + Q(x)y = 0$$

*$P, Q$ continuous on $(a, b)$. Then*
*(1) $y_1(x)$ and $y_2(x)$ have no common zero in $(a, b)$.*
*(2) Between any two successive zeros of $y_1(x)$, there is exactly one zero of $y_2(x)$ and vice versa.*

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It satisfies the differential equation $W' = -P(x)W$ and so is given by

$$W(x) = C\exp\Big(\int_{a_0}^{x} -P(t)dt\Big) \qquad a_0 \in (a,b)$$

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First let us show $y_2$ has a zero in $(x_1, x_2)$.
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We conclude that $y_1'(x_1)$ and $y_1'(x_2)$ are nonzero.

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If $y_2(x)$ had two zeros in the interval $x_1 < \alpha < \beta < x_2$, then by the same reasoning, $y_1$ will have a zero in $(\alpha, \beta)$, which contradicts the assumption that $x_1$ and $x_2$ are successive zeros of $y_1$. This completes the proof of the theorem.

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As a consequence, if $y_1$ and $y_2$ are linearly independent solution of $y'' + P(x)y' + Q(x)y = 0$, $P, Q$ continuous on $(a, b)$ then the number of zeros of $y_1$ and $y_2$ on $(a, b)$ differ by at most $1$.

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In particular, either both have finite number of zeros or both have infinite number of zeros in $(a, b)$.

# Qualitative properties of solutions

For further discussion, we need that any ODE in the "standard" form $y'' + P(x)y' + Q(x)y = 0$ can be written in the "normal" form $u'' + q(x)u = 0$.

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Also note that we need $P(x)$ to be once differentiable.

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*Let $q(x)$ be continuous on the interval $(\alpha, \beta)$. Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on finite interval $[a, b] \subset (\alpha, \beta)$. Then $u(x)$ has at most finite number of zeros in $[a, b]$.*
*Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.*

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Since $u(x_0)$ and $u'(x_0)$ are both 0, it follows that the Wronskian at $x_0$ is 0. But this is a contradiction as the Wronskian at $x_0$ is nonzero.

**Theorem**

*Let* $u(x)$ *be a non-trivial solution of* $u'' + q(x)u = 0$. *If* $q(x) < 0$ *in* $(a, b)$ *and continuous then* $u(x)$ *has atmost one zero in* $(a, b)$.

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But this is not possible as $u'$ is increasing on $(x_0, x_1)$. $\square$

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**Theorem**

*Let* $u(x)$ *be a non-trivial solution of* $u'' + q(x)u = 0$ *Let* $q(x)$ *be continuous and* $q(x) > 0$ *for all* $x > x_0 > 0$.

*If* $\int_{x_0}^{\infty} q(x)\,dx = \infty$,

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**Proof.** Assume $u(x)$ has only finitely many zeros on $(0, \infty)$. Then there is $x_1 > x_0$ such that $u(x) \neq 0$ for $x \geq x_1$. Assume $u(x) > 0$ for $x \geq x_1$.

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Then $u''(x) = -q(x)u(x) < 0$ for $x \geq x_1$. Hence $u'(x)$ is decreasing for $x \geq x_1$.
If we show that $u'(x_2) < 0$ for some $x_2 > x_1$, then we get for $x > x_2$

$$u(x) = \int_{x_2}^{x} u'(t)dt + u(x_2) \leq \int_{x_2}^{x} u'(x_2)dt + u(x_2)$$
$$\leq u'(x_2)(x - x_2) + u(x_2)$$

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Thus, $u'(x) = -u(x)v(x)$ and this shows that $u'(x) < 0$ for $x$ large.

**Theorem**

*In Bessel equation* $x^2y'' + xy' + (x^2 - p^2)y = 0$ *Substituting* $u(x) = \sqrt{x}y(x)$, *we get*

$$u'' + \left[1 + \frac{1 - 4p^2}{4x^2}\right] u = 0$$

$q(x) = 1 + \dfrac{1 - 4p^2}{4x^2}$ *is continuous and* $q(x) > 0$ *for* $x > x_0 > 0$. *Further,*

$$\int_{x_0}^{\infty} \left(1 + \frac{1 - 4p^2}{4x^2}\right) dx = \infty$$

*By previous theorem,* $u(x)$, *hence any Bessel function has infinitely many zeros on* $(0, \infty)$.

**Corollary**

*Let* $Z^{(p)}$ *be the set of zeros of Bessel function* $J_p(x)$ *on* $(0, \infty)$.

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Write $Z^{(p)} = \{x_1, x_2, \ldots\}$ as increasing sequence $x_n < x_{n+1}$.

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Question. What is the limit of $x_{n+1} - x_n$ as $n \to \infty$?

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We will need the Sturm comparison theorem.

**Theorem (Sturm Comparison theorem)**

*Let* $y(x)$ *be a non-trivial solutions of*

$$y'' + q(x)y = 0$$

*and* $z(x)$ *be a non-trivial solutions of*

$$z'' + r(x)z = 0$$

*where* $q(x) > r(x) > 0$ *are continuous.*
*Then* $y(x)$ *vanishes at least once between any two consecutive zeros of* $z(x)$*.*

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*Then $y(x)$ vanishes at least once between any two consecutive zeros of $z(x)$.*

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Integrating from $x_1$ to $x_2$, we get

$$W(x_2) - W(x_1) > 0 \implies W(x_2) > W(x_1)$$

But $W(x_1) = y(x_1)z'(x_1) > 0$ and $W(x_2) = y(x_2)z'(x_2) < 0$, a contradiction. □

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To see this, consider the function $f := u(x - b + a)$ defined on the interval $(b, \infty)$.

It is a trivial check that $f$ satisfies the differential equation

$$f'' + r(x)f = 0 \qquad\qquad r(x) := q(x - b + a)$$

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Next we claim that the difference between any two successive roots of $u$ is strictly less than $\pi$.

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Thus, we have proved that if $\{x_n\}$ are the roots of $u$ in increasing order, then the difference $x_{n+1} - x_n$ is strictly increasing and bounded above by $\pi$.

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Next let us show that these differences converge to $\pi$. If not, then $(x_{n+1} - x_n) \to \gamma < \pi$. Choose $1 < \delta$, sufficiently close to 1 such that $\gamma < \frac{\pi}{\delta} < \pi$.

The function $q(x)$ is decreasing to 1. Therefore, there is a $x_0 \in \mathbb{R}$, sufficiently large, such that $q(x_0) < \delta^2$. Apply Sturm's comparison on the interval $(x_0, \infty)$ to the differential equations
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Thus, between any two roots of $u$ there is a root of $z$. Let $a$ and $b$ be two consecutive roots of $u$ such that $x_0 < a < b$. Since $b - a < \gamma < \frac{\pi}{\delta}$, find $a'$ and $b'$ such that $x_0 < a' < a < b < b'$ and $b' - a' = \frac{\pi}{\delta}$.

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Find $\alpha$ and $\beta$ such that the function $\alpha \cos \delta\, x + \beta \sin \delta\, x$ vanishes at $a'$. This function is a solution to the ODE $z'' + \delta^2 z = 0$. The next root of this function is at $a' + \frac{\pi}{\delta} = b'$. Thus, we get a contradiction to Sturm's theorem which says that there is a root of this function in the interval $(a, b)$.

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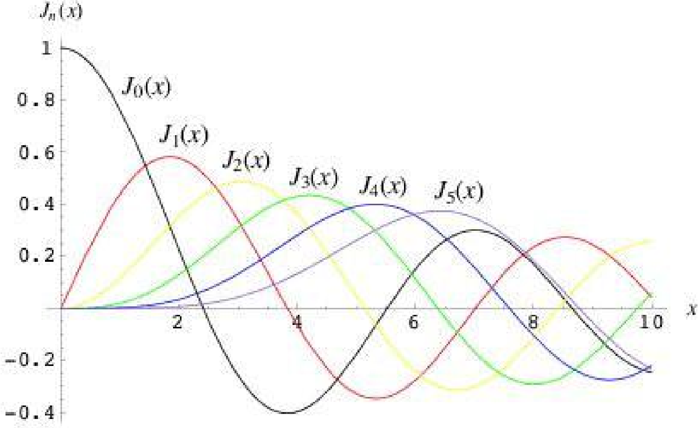
*Similarly, we can prove that if* $p > 1/2$ *then the sequence of difference of roots of* $u$ *is decreasing and tends to* $\pi$*.*



$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
2
4
6
8
10
x
$J_0(x)$
$J_1(x)$
$J_2(x)$
$J_3(x)$
$J_4(x)$
$J_5(x)$

The first few zeroes of Bessel functions are tabulated below.

| | $J_0(x)$ | $J_1(x)$ | $J_2(x)$ | $J_3(x)$ | $J_4(x)$ | $J_5(x)$ |
|---|---|---|---|---|---|---|
| 1 | 2.4048 | 3.8317 | 5.1356 | 6.3802 | 7.5883 | 8.7715 |
| 2 | 5.5201 | 7.0156 | 8.4172 | 9.7610 | 11.0647 | 12.3386 |
| 3 | 8.6537 | 10.1735 | 11.6198 | 13.0152 | 14.3725 | 15.7002 |
| 4 | 11.7915 | 13.3237 | 14.7960 | 16.2235 | 17.6160 | 18.9801 |
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# Expansion in terms of Bessel functions

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Question. Why are we concerned with zeros of Bessel function $J_p(x)$?

It is often required in mathematical physics to expand a given function in terms of Bessel functions.

**Definition**

For a scalar $a$, the scaled Bessel functions $J_p(ax)$ are solutions of

$$x^2y'' + xy' + (a^2x^2 - p^2)y = 0$$

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Simplest and most useful expansions are of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_{p,n}x) = a_1 J_p(\lambda_{p,1}x) + a_2 J_p(\lambda_{p,2}x) + \dots$$

where $f(x)$ is defined on, (say) $[0, 1]$, and $\lambda_{p,n}$'s are zeros of Bessel function $J_p(x)$, $p \geq 0$.

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**Qn.** How to compute the coefficients $a_n$?

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This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by $x$ and the interval of integration is from $0$ to $1$.

We call a function on $[0, 1]$ square integrable with respect to this inner product if

$$\int_0^1 x f(x)^2 dx < \infty$$

The multiplying factor $x$ is called a weight function.

# Orthogonality

Fix $p \geq 0$. Let $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \dots\}$ denote the set of zeros of $J_p(x)$ on $(0, \infty)$.

## Theorem

*The set of scaled Bessel functions*

$$\{J_p(\lambda_{p,1}x), J_p(\lambda_{p,2}x), \dots\}$$

*form an orthogonal family w.r.t. above inner product, i.e.* $\langle J_p(\lambda_{p,k}x), J_p(\lambda_{p,l}x)\rangle :=$

$$\int_0^1 x J_p(\lambda_{p,k}x) J_p(\lambda_{p,l}x)\, dx = \begin{cases} \frac{1}{2}[J_{p+1}(\lambda_{p,k})]^2 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Proof of orthogonality of scaled Bessel functions

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If $a, b$ are positive scalars, then $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfies

$$u'' + \frac{1}{x}u' + \left(a^2 - \frac{p^2}{x^2}\right)u = 0$$

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Multiply by $v$ and $u$ resp. and subtract, we get

$$(vu'' - uv'') + \frac{1}{x}(vu' - uv') + (a^2 - b^2)uv = 0$$

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$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

$$(x(u'v - v'u))' = (b^2 - a^2)xuv$$

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So if $a = \lambda_{p,k}$ and $b = \lambda_{p,l}$ are **distinct** (RHS is 0 since these are roots of $J_p(x)$), then

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)\,dx = 0$$

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To compute the norm of $J_p(\lambda_{p,k}x)$, consider

$$\begin{aligned} 2x^2u'\left[u'' + \frac{1}{x}u' + (a^2 - \frac{p^2}{x^2})u\right] &= 0 \\ = [x^2u'^2 + (a^2x^2 - p^2)u^2]' - 2a^2xu^2 \end{aligned}$$

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Integrate on $[0, 1]$,

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Since $p \geq 0$, $(pu(0))^2 = (pJ_p(0))^2 = 0$.

Thus, $\big(x^2u'^2 + (a^2x^2 - p^2)u^2\big)(0) = 0$.

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$$\int_0^1 xJ_p(\lambda_{p,k}x)^2\,dx = \frac{1}{2}J_p'(\lambda_{p,k})^2 = \frac{1}{2}J_{p+1}(\lambda_{p,k})^2$$

for last equality, use $x = \lambda_{p,k}$ in $J_p'(x) - \dfrac{p}{x}J_p(x) = J_{p+1}(x)$

# Expansion in terms of Bessel functions

**Theorem**

*Fix* $p \geq 0$ *and* $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ *: zeros of* $J_p(x)$ *on* $(0, \infty)$*. Any square-integrable function* $f(x)$ *on* $[0, 1]$ *can be expanded in a series of scaled Bessel functions* $J_p(\lambda_{p,n} x)$ *as*

$$f(x) = \sum_{n \geq 1} c_n J_p(\lambda_{p,n} x)$$

*where*

$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x\, f(x)\, J_p(\lambda_{p,n} x)\, dx$$

*This is Fourier-Bessel series of* $f(x)$ *for parameter* $p$*.*

# Expansion in terms of Bessel functions

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$$\int_0^1 x\, J_0(\lambda_{0,n} x)\, dx = \frac{1}{\lambda_{0,n}} x\, J_1(\lambda_{0,n} x)\Big|_0^1 = \frac{J_1(\lambda_{0,n})}{\lambda_{0,n}}$$

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By next theorem, this converges to $1$ for $0 < x < 1$.

# Expansion in terms of Bessel functions

Convergence in norm

Fourier-Bessel series converges to $f(x)$ in norm, i.e.

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at $x = 0$, if $p = 0$ then it converges to $f(0_+)$.
at $x = 0$, if $p > 0$ then it converges to 0.