MA-207 Differential Equations II

Ronnie Sebastian

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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We will develop Fourier series representation of functions that will be used to solve PDE considered later.

Consider the following Boundary Value Problems (BVP), where $\lambda \in \mathbb{R}$ and $L > 0$.

1. Problem 1. $y'' + \lambda y = 0 \qquad y(0) = 0, \qquad y(L) = 0.$
2. Problem 2. $y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y'(L) = 0.$
3. Problem 3. $y'' + \lambda y = 0 \qquad y(0) = 0, \qquad y'(L) = 0.$
4. Problem 4. $y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y(L) = 0.$
5. Problem 5. $y'' + \lambda y = 0 \quad y(-L) = y(L),\ y'(-L) = y'(L).$

The boundary condition in problem 5 is called periodic.

Eigen Value problems $y'' + \lambda y = 0$

Questions. Let us fix any one of the above problems, say Problem 1.

1. For what values of λ does problem 1 have a non-trivial solutions?
2. If it has solutions, what are the solutions?

Any λ for which problem 1 has a non-trivial solution is called an eigenvalue of problem 1.

Given an eigenvalue, if we take the space of all functions which satisfy the problem with respect to this eigenvalue, then it is a trivial check that this space forms a vector space.

Nonzero solutions for an eigenvalue λ are called λ-eigenfunction, or eigenfunction associated with λ.

Problems $1 - 5$ are called eigenvalue problems. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions.

Eigen Value problems $y'' + \lambda y = 0$

Theorem

1. *Problems* $1 - 5$ *have no negative eigenvalues.*
2. $\lambda = 0$ *is an eigenvalue of Problems* 2 *and* 5 *with associated eigenfunctions* $y_0 = 1$.
3. $\lambda = 0$ *is not an eigenvalue of Problems* $1, 3$ *and* 4.

Proof.

Let us prove the first two; the third is left as an exercise.

Suppose $\lambda < 0$ is an eigenvalue. Let us write $\lambda = -a^2$.

Rewrite the differential equation as $y'' = a^2 y$. The general solution to this is $y(x) = Ce^{ax} + De^{-ax}$. In problem 1 we have the condition $y(0) = y(L) = 0$. This forces that $C + D = 0$ and $Ce^{aL} + De^{-aL} = 0$. One checks easily that this forces $C = D = 0$.

In problem 2 we have the condition that $y'(0) = y'(L) = 0$. This gives $aC - aD = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Since $a \neq 0$, this forces $C = D = 0$.

continued . . . .

In problem 3 we have the conditions $y(0) = y'(L) = 0$. This gives $C + D = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Again this forces $C = D = 0$.

Similarly, do the other problems.

Now consider the second statement in the theorem. If $\lambda = 0$, the clearly, the solution has to be of the form $y(x) = ax + b$.

In problem 2 we have $y'(0) = y'(L) = 0$, and so $a = 0$. Thus, $y(x) = \text{constant}$ is the solution in this case.

In problem 5, we have $y(-L) = y(L)$, that is, $-aL + b = aL + b$. This forces that $a = 0$. Thus, in this case too $y(x) = \text{const.}$ □

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \;\; y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, \;\; n = 1, 2, \ldots,$$

with associated eigenfunctions

$$y_n(x) = \sin \frac{n \pi x}{L}, \;\; n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof.

Any eigen value must be positive (by previous theorem).

If $y(x)$ is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = 0 \implies c_1 = 0$$

$$\implies y(x) = c_2 \sin \sqrt{\lambda} x \quad \text{with} \quad c_2 \neq 0$$

$$y(L) = 0 \implies \sin \sqrt{\lambda} L = 0 \implies \sqrt{\lambda} L = n\pi$$

$$\implies \lambda_n = \frac{n^2 \pi^2}{L^2}$$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin \frac{n\pi x}{L}$$

□

EVP 2 $\quad y'' + \lambda y = 0; \quad y'(0) = 0, \ y'(L) = 0$

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y'(0) = 0, \ \ y'(L) = 0$$

has an eigenvalue $\lambda_0 = 0$ *with eigenfunction* $y_0 = 1$

and infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L} \quad n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof. Similar to the proof of Problem 1, hence is left as an exercise.

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \;\; y'(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}$$

with associated eigenfunctions

$$y_n = \sin\frac{(2n+1)\pi x}{2L}, \quad n = 0, 1, 2, \ldots.$$

There are no other eigenvalues.

Proof.

Any eigen value must be positive (by previous theorem).

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = 0 \implies c_1 = 0$$

$$\implies y(x) = c_2 \sin \sqrt{\lambda} x \quad \text{with} \quad c_2 \neq 0$$

$$y'(L) = 0 \implies \sqrt{\lambda} \cos \sqrt{\lambda} L = 0 \implies \sqrt{\lambda} L = \tfrac{2n+1}{2}\pi$$

$$\implies \lambda_n = \frac{(2n+1)^2 \pi^2}{4L^2}$$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin \frac{(2n+1)\pi x}{2L}$$

□

EVP 4 and EVP 5

Left as exercises.

Note that in EVP 5 every positive eigenvalue has two dimensional space of associated eigen functions.

Orthogonality

Definition

We say two integrable functions f and g are orthogonal on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\,dx = 0$$

More generally, we say functions $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ (finite or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0 \quad \text{whenever} \quad i \neq j$$

We have already seen orthogonality of Legendre function.
We will study Fourier series w.r.t. different orthogonal systems.

Consider the eigenfunctions

1. $\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \sin\frac{n\pi x}{L}, \ldots$
2. $1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \ldots$
3. $\sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, \ldots, \sin\frac{(2n-1)\pi x}{2L}, \ldots$
4. $\cos\frac{\pi x}{2L}, \cos\frac{3\pi x}{2L}, \ldots, \cos\frac{(2n-1)\pi x}{2L}, \ldots$
5. $1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}, \ldots$

Show directly that eigenfunctions of (1-4) are orthogonal on $[0, L]$ and of (5) is orthogonal on $[-L, L]$.

The space of square integral functions

We will study series expansions in terms of eigenfunctions. It is used to solve PDEs.

For this we consider the vector space of functions on $[a, b]$ and define an inner product on it

$$\langle f, g\rangle := \int_a^b f(x)g(x)dx$$

Denote by $L^2[a, b]$ the subspace of those functions satisfying $\langle f, f\rangle < \infty$.

To say this is a subspace, one needs to check that if $f, g \in L^2[a, b]$ then $f + g \in L^2[a, b]$. We shall assume this fact.

From now on, we will always be working with functions in some inner product space of the type $L^2[a, b]$. In such a space, the norm of a function is defined to be $\|f\| := \langle f, f\rangle^{1/2}$.

Theorem

Let $f \in L^2[-L, L]$. Consider the series

$$F_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

which is called the Fourier series of f on $[-L, L]$. Here

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx \qquad \text{and for } n > 0$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

The above series converges to f in the L^2-norm, that is,

$$\lim_{N \longrightarrow \infty} \left\| f - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\| = 0$$

The above means that the function defined by the series $F_f(x)$ is equal to $f(x)$ in a reasonably strong sense.

We started with a function $f(x)$ and cooked up a sequence of numbers a_n's and b_n's. Using these we cooked up a function $F_f(x)$. Although this function is "equal" to $f(x)$ in a reasonably strong sense, as we remarked above, it may not be equal to $f(x)$ pointwise. We now make some remarks about when $F_f(x)$ will be equal to $f(x)$.

Definition

A function f on $[a, b]$ is said to be piecewise smooth if

1. f has atmost finitely many points of discontinuity.
2. f' exists and has atmost finitely many points of discontinuity.
3. $f(x_0+) := \lim_{x \to x_0^+} f(x)$ and $f'(x_0+) := \lim_{x \to x_0^+} f'(x)$ exists if $a \leq x_0 < b$.
4. $f(x_0-) = \lim_{x \to x_0^-} f(x)$ and $f'(x_0-) = \lim_{x \to x_0^-} f'(x)$ exists if $a < x_0 \leq b$.

Pointwise convergence of Fourier series

Theorem

Let $f(x)$ be a piecewise smooth function on $[-L, L]$. Then the Fourier series

$$F_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

of f converges to

$$F_f(x) = \begin{cases} \frac{1}{2}[f((-L)+) + f(L-)] & x = -L, L \\ \frac{1}{2}[f(x+) + f(x-)] & x \in (-L, L) \end{cases}$$

Therefore, at every point x of continuity of f, the Fourier series converges to $f(x)$.

If we re-define $f(x)$ at every point of discontinuity x as $\frac{1}{2}[f(x+) + f(x-)]$ then the Fourier series represents the function everywhere. Thus two functions can have same Fourier series.

Example

Let us find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$.

$$a_0 = \frac{1}{4} \int_{-2}^{2} f(x)\, dx = \frac{1}{4} \left[\int_{-2}^{0} (-x)\, dx + \int_{0}^{2} \frac{1}{2}\, dx \right] = \frac{3}{4}$$

If $n \geq 1$, then

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2}\, dx \\ &= \frac{1}{2} \left[\int_{-2}^{0} (-x) \cos \frac{n\pi x}{2}\, dx + \int_{0}^{2} \frac{1}{2} \cos \frac{n\pi x}{2}\, dx \right] \end{aligned}$$

Example (continued ...)

$$= \frac{1}{2}\left[-x\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \int_{-2}^{0}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx + 0\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}\left(-\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{0}$$

$$= \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2}\frac{1}{2}\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2n\pi}(1 + 3\cos n\pi)$$

Example (continued ...)

Thus, the Fourier series of $f(x)$ is

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2} \cos \frac{n\pi x}{2} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1 + 3\cos n\pi}{n} \sin \frac{n\pi x}{2}$$

$\square$

Let us compute $F(x)$ at discontinuous points.

Example (continued ...)

$$F(-2) = F(2) = \frac{1}{2}\left(f(-2+) + f(2-)\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$$

$$F(0) = \frac{1}{2}\left(f(0-) + f(0+)\right) = \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$$

To summarize,

$$F(x) = \begin{cases} 5/4, & x = \pm 2 \\ -x, & -2 < x < 0 \\ 1/4, & x = 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

Recall

Let V be a vector space along with an inner product. For example, $L^2[-1,1]$ or $L^2[0,1]$ with the standard inner products.

Suppose we have an orthogonal set $\{\phi_1, \phi_2, \ldots\}$ which has the following property. For every function f we have a series $\sum_{i \geq 1} a_i \phi_i$ which converges to f, that is,

$$\lim_{n \to \infty} \|f - \sum_{i=1}^{n} a_i \phi_i\| = 0\,.$$

Then we say that set $\{\phi_1, \phi_2, \ldots\}$ is a normed basis for V.

Note that this is different from the notion of basis, where we need that every vector should be written as a finite linear combination of the basis vectors.

The the coefficient of ϕ_n in the expansion of f is given by

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}\,.$$

In this situation, we will loosely write

$$f = \sum_{n \geq 1} a_n \phi_n \, .$$

The theorem on existence of Fourier series for elements in $L^2[-1,1]$ can be rephrased as saying that the set

$$\{1\} \cup \{\cos \frac{n\pi x}{L}\} \cup \{\sin \frac{n\pi x}{L}\}$$

is a normed basis for $L^2[-1,1]$. Notice that these functions are precisely the eigen functions to EVP 5.

Next we shall use the above to find normed basis for some other vector spaces. We will see that, just as in the case of basis, a normed basis need not be unique.

Eigen functions of EVP 1 as normed basis for $L^2[0,1]$

Let f be an element of $L^2[0,L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n\geq 1} b_n \sin\frac{n\pi x}{L}\,.$$

To see this, let us first extend f to $[-L,L]$ by defining $f(x) = -f(-x)$ for $x \in [-L,0)$. Denote the extension by $\tilde{f}$.

Then we know that $\tilde{f}$ has a Fourier expansion

$$\tilde{f}(x) = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L}\tilde{f}(x)dx \qquad a_n = \frac{1}{L}\int_{-L}^{L}\tilde{f}(x)\cos\frac{n\pi x}{L}dx \qquad n > 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L}\tilde{f}(x)\sin\frac{n\pi x}{L}dx$$

Eigen functions of EVP 1 as normed basis for $L^2[0,1]$

Now note that by the way $\tilde{f}$ has been defined, it is an odd function. Thus, $a_0 = 0$.

Since $\cos \frac{n\pi x}{L}$ is an even function and $\tilde{f}$ is odd, it follows $\tilde{f}(x) \cos \frac{n\pi x}{L}$ is an odd function. Thus, $a_n = 0$.

This proves that

$$\tilde{f}(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L} \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

The functions $\{\sin \frac{n\pi x}{L}\}_{n \geq 1}$ are the eigen functions for EVP 1. Thus, we have proved

Theorem

The eigen functions for EVP 1 form a normed basis for $L^2[0,1]$.

Eigen functions of EVP 2 as normed basis for $L^2[0,1]$

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = a_0 + \sum_{n\geq 1} a_n \cos \frac{n\pi x}{L}$$

To see this, let us first extend f to $[-L, L]$ by defining $f(x) = f(-x)$ for $x \in [-L, 0)$. Denote the extension by $\tilde{f}$. Then we know that $\tilde{f}$ has a Fourier expansion

$$\tilde{f}(x) = a_0 + \sum_{n\geq 1} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \tilde{f}(x)dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\frac{n\pi x}{L}dx \qquad n > 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\frac{n\pi x}{L}dx$$

Now note that by the way $\tilde{f}$ has been defined, it is an even function.

Since $\sin\frac{n\pi x}{L}$ is an odd function and $\tilde{f}$ is even, it follows $\tilde{f}(x)\sin\frac{n\pi x}{L}$ is an odd function. Thus, $b_n = 0$.

This proves that

$$\tilde{f}(x) = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Restricting this expansion to $[0,L]$ we get the required expansion of f.

The functions $\{\cos\frac{n\pi x}{L}\}_{n\geq 1}$ are the eigen functions for EVP 2. Thus, we have proved

Theorem

The eigen functions for EVP 2 form a normed basis for $L^2[0,1]$.

Eigen functions of EVP 3 as normed basis for $L^2[0,1]$

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n\geq 1} a_n \sin \frac{(2n-1)\pi x}{2L}$$

Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = f(2L - x)$ for $x \in (L, 2L)$.

From what we saw earlier, we can expand f_1 as a series in the eigen functions for EVP 1, that is,

$$f_1(x) = \sum_{n\geq 1} b_n \sin \frac{n\pi x}{2L}$$

We claim that $b_{2n} = 0$. This is easily checked using the definition:

$$b_n = \frac{2}{2L} \int_0^{2L} f_1(x) \sin \frac{n\pi x}{2L}\, dx$$

$$= \frac{1}{L} \int_0^{L} f(x) \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \int_L^{2L} f(2L - x) \sin \frac{n\pi x}{2L} dx$$

Let us rewrite the second integral in the sum

$$\int_L^{2L} f(2L-x)\sin\frac{n\pi x}{2L}dx$$

$$(x' = 2L - x), \qquad = \int_L^0 f(x')\sin(n\pi - \frac{n\pi x'}{2L})(-dx')$$

$$= \int_0^L (-1)^{n+1} f(x)\sin\frac{n\pi x}{2L}\,dx$$

Putting this back we get

$$b_n = \frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}dx + \frac{1}{L}\int_0^L (-1)^{n+1} f(x)\sin\frac{n\pi x}{2L}dx$$

So $b_{2n} = 0$.

Eigen functions of EVP 3 as normed basis for $L^2[0,1]$

Thus

$$f_1(x) = \sum_{n\geq 1} b_{2n-1} \sin \frac{(2n-1)\pi x}{2L}$$

$$b_{2n-1} = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

The functions $\{\sin \frac{(2n-1)\pi x}{2L}\}_{n\geq 1}$ are the eigen functions for EVP 3. Thus, we have proved

Theorem

The eigen functions for EVP 3 form a normed basis for $L^2[0,1]$.

Eigen functions of EVP 4 as normed basis for $L^2[0,1]$

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n\geq 1} a_n \cos\frac{(2n-1)\pi x}{2L}$$

Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = -f(2L - x)$ for $x \in (L, 2L)$.

From what we saw earlier, we can expand f_1 as a series in the eigen functions for EVP 2, that is,

$$f_1(x) = b_0 + \sum_{n\geq 1} b_n \cos\frac{n\pi x}{2L}$$

We claim that $b_{2n} = 0$. This is easily checked using the definition:

$$b_0 = \frac{1}{2L}\int_0^{2L} f_1(x)\, dx = 0$$

$$b_n = \frac{2}{2L}\int_0^{2L} f_1(x)\cos\frac{n\pi x}{2L}\,dx$$

$$= \frac{1}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{2L}dx + \frac{1}{L}\int_L^{2L} -f(2L-x)\cos\frac{n\pi x}{2L}dx$$

Let us rewrite the second integral in the sum

$$\int_L^{2L} -f(2L-x)\cos\frac{n\pi x}{2L}dx$$

$$(x' = 2L - x), \qquad = -\int_L^{0} f(x')\cos(n\pi - \frac{n\pi x'}{2L})(-dx')$$

$$= \int_0^{L} (-1)^{n+1} f(x)\cos\frac{n\pi x}{2L}\,dx$$

Putting this back we get

$$b_n = \frac{1}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{2L}dx + \frac{1}{L}\int_0^{L} (-1)^{n+1} f(x)\cos\frac{n\pi x}{2L}dx$$

So $b_{2n} = 0$.

Thus

$$f_1(x) = \sum_{n\geq 1} b_{2n-1} \cos \frac{(2n-1)\pi x}{2L}$$

.

$$b_{2n-1} = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

The functions $\{\cos \frac{(2n-1)\pi x}{2L}\}_{n\geq 1}$ are the eigen functions for EVP 4. Thus, we have proved

Theorem

The eigen functions for EVP 4 form a normed basis for $L^2[0,1]$.

What is the point of all this?

Ok! So we have seen five theorems now.

Each of these says that there is a certain normed linear space V, and that eigen functions of a certain problem form a normed basis for V.

Also note that the four of the above theorems are easy consequences of the main one.

Ok! So what! What is so interesting about this? Who cares?

The power of these theorems will be revealed when we use them to solve PDE's. Recall the PDE we saw in the first lecture.

$$\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0\\
&u(0,t) = 0 && t \geq 0\\
&u(L,t) = 0, && t \geq 0\\
&u(x,0) = x(L-x), && 0 \leq x \leq L
\end{aligned}$$

To solve this we will use the Fourier expansion in terms of EVP 1. This will be done in the coming lectures.

An observation: derivative transfer

Often we need to find Fourier expansion of polynomial functions in terms of the eigenfunctions of Problems 1-4 satisfying the boundary conditions.

We can use "derivative transfer principle" to find Fourier coefficients. Recall what it means to integrate by parts.

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}\,.$$

Thus, integrating this we get

$$\int_a^b f\frac{dg}{dx}\,dx = -\int_a^b g\frac{df}{dx}\,dx + \boxed{\text{f(b)g(b)-f(a)g(a)}}\,.$$

In some situations, the term in the box will become 0, making the above computation easy.

An observation: derivative transfer

Consider the situation where $f(x)$ is a polynomial with $f(0) = 0 = f(L)$. We get Fourier series on $[0, L]$ in terms of eigen functions for EVP 1.

$$f(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{n\pi} \int_0^L f'(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{-2}{L} \left(\frac{L}{n\pi} \right)^2 \int_0^L f''(x) \sin \frac{n\pi x}{L} \, dx$$

An observation: derivative transfer

Consider the situation where $f(x)$ is a polynomial with $f'(0) = 0 = f'(L)$. We get Fourier series on $[0, L]$ in terms of eigen functions for EVP 2.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad 0 \le x \le L$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{-2}{n\pi}\int_0^L f'(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-2L}{n^2\pi^2}\int_0^L f''(x)\cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3\int_0^L f'''(x)\sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

An observation: derivative transfer

Consider the situation where $f(x)$ is a polynomial with $f(0) = 0 = f'(L)$. We get Fourier series on $[0, L]$ in terms of eigen functions for EVP 3.

$$f(x) = \sum_{n \geq 1} c_n \sin \frac{(2n-1)\pi x}{2L} \, dx$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{4}{(2n-1)\pi} \int_0^L f'(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{-2}{L} \left(\frac{2L}{(2n-1)\pi} \right)^2 \int_0^L f''(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

An observation: derivative transfer

Consider the situation where $f(x)$ is a polynomial with $f'(0) = 0 = f(L)$. We get Fourier series on $[0, L]$ in terms of eigen functions for EVP 4.

$$f(x) = \sum_{n\geq 1} d_n \cos \frac{(2n-1)\pi x}{2L}\, dx$$

$$d_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-4}{(2n-1)\pi}\int_0^L f'(x)\sin\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2\int_0^L f''(x)\cos\frac{(2n-1)\pi x}{2L}\, dx$$

Find the Fourier expansion of

$f(x) = x(x^2 - 3Lx + 2L^2)$ on $[0, L]$ in terms of the eigen functions for EVP 1.

Soln. Note $f(0) = 0 = f(L)$, $f''(x) = 6(x - L)$. Thus, using derivative transfer we see the Fourier coefficient is

$$b_n = \frac{-2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^L f''(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{-12L}{n^2\pi^2} \int_0^L (x - L) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{12L^2}{n^3\pi^3} \left[(x - L) \cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\, dx \right]$$

$$= \frac{12L^2}{n^3\pi^3} \left[L - \frac{L}{n\pi} \sin\frac{n\pi x}{L}\Big|_0^L \right] = \frac{12L^3}{n^3\pi^3}$$

Therefore, the required Fourier expansion of $f(x)$ on $[0, L]$ is

$$\frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L} \qquad \square$$

Example. Find the Fourier expansion of

$$f(x) = x^2(3L - 2x) \text{ on } [0, L]$$

in terms of eigen functions for EVP 2.

Soln.

$$a_0 = \frac{1}{L}\int_0^L (3Lx^2 - 2x^3)\,dx$$

$$= \frac{1}{L}\left(Lx^3 - \frac{x^4}{2}\right)_0^L$$

$$= \frac{L^3}{2}$$

$$f'(x) = 6Lx - 6x^2 \implies f'(0) = f'(L) = 0.$$

Note $f'''(x) = -12$. Thus, using derivative transfer we get

$$a_n = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L f'''(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-24}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{24}{L}\left(\frac{L}{n\pi}\right)^4 \cos\frac{n\pi x}{L}\Big|_0^L = \frac{24L^3}{n^4\pi^4}\,\left[(-1)^n - 1\right]$$

Thus $a_{2n} = 0$ and $a_{2n-1} = \dfrac{-48L^3}{(2n-1)^4\pi^4}$.

Thus, the required Fourier expansion of $f(x)$ on $[0, L]$ is

$$\frac{L^3}{2} - \frac{48L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{L}$$

Example Find the Fourier expansion of

$$f(x) = x(2x^2 - 9Lx + 12L^2) \quad \text{on} \quad [0, L]$$

in terms of the eigen functions for EVP 3.

Soln. We again use derivative transfer. Since $f(0) = 0 = f'(L)$ and $f''(x) = 6(2x - 3L)$, we get

$$c_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x) \sin\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-48L}{(2n-1)^2\pi^2} \int_0^L (2x - 3L) \sin\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\Big[(2x - 3L)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-2\int_0^L \cos\frac{(2n-1)\pi x}{2L}\, dx\Big]$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\left[3L - \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

$$= \frac{96L^3}{(2n-1)^3\pi^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]$$

Therefore, the required Fourier expansion of $f(x)$ on $[0, L]$ is

$$c\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$$

with $c = \dfrac{96L^3}{\pi^3}$.

Example. Find the mixed Fourier cosine expansion of $f(x) = 3x^3 - 4Lx^2 + L^3$ on $[0, L]$ in terms of the eigen functions of EVP 4.

Soln. As $f'(0) = 0 = f(L)$, we can use derivative transfer. As $f''(x) = 2(9x - 4L)$, we get

$$d_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x) \cos\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-16L}{(2n-1)^2\pi^2} \int_0^L (9x - 4L) \cos\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\Big[(9x - 4L) \sin\frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-9 \int_0^L \sin\frac{(2n-1)\pi x}{2L}\, dx\Big]$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L + \frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

$$= \frac{32L^3}{(2n-1)^3\pi^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]$$

Therefore, the required Fourier expansion of $f(x)$ on $[0, L]$ is

$$\frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$$

□