MA-207 Differential Equations II

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The boundary condition in problem 5 is called periodic.

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Problems $1 - 5$ are called eigenvalue problems. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions.

Eigen Value problems $y'' + \lambda y = 0$

Theorem

1. *Problems* $1 - 5$ *have no negative eigenvalues.*
2. $\lambda = 0$ *is an eigenvalue of Problems* 2 *and* 5 *with associated eigenfunctions* $y_0 = 1$.
3. $\lambda = 0$ *is not an eigenvalue of Problems* $1, 3$ *and* 4.

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Suppose $\lambda < 0$ is an eigenvalue. Let us write $\lambda = -a^2$.

Rewrite the differential equation as $y'' = a^2 y$. The general solution to this is $y(x) = Ce^{ax} + De^{-ax}$. In problem 1 we have the condition $y(0) = y(L) = 0$. This forces that $C + D = 0$ and $Ce^{aL} + De^{-aL} = 0$. One checks easily that this forces $C = D = 0$.

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In problem 2 we have the condition that $y'(0) = y'(L) = 0$. This gives $aC - aD = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Since $a \neq 0$, this forces $C = D = 0$.

continued . . ..

In problem 3 we have the conditions $y(0) = y'(L) = 0$. This gives $C + D = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Again this forces $C = D = 0$.

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In problem 5, we have $y(-L) = y(L)$, that is, $-aL + b = aL + b$. This forces that $a = 0$. Thus, in this case too $y(x) = \text{const}$. □

EVP 1 $\quad y'' + \lambda y = 0; \qquad y(0) = 0, \;\; y(L) = 0$

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \;\; y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \;\; n = 1, 2, \ldots,$$

with associated eigenfunctions

$$y_n(x) = \sin\frac{n\pi x}{L}, \;\; n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof.

Any eigen value must be positive (by previous theorem).

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If $y(x)$ is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

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$$\implies \lambda_n = \frac{n^2\pi^2}{L^2}$$

is an eigenvalue with an associated eigenfunction

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The eigenvalue problem

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and infinitely many positive eigenvalues

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Proof. Similar to the proof of Problem 1, hence is left as an exercise.

EVP 3 $\quad y'' + \lambda y = 0; \qquad y(0) = 0, \;\; y'(L) = 0$

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \ \ y'(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}$$

with associated eigenfunctions

$$y_n = \sin\frac{(2n+1)\pi x}{2L}, \quad n = 0, 1, 2, \ldots .$$

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Note that in EVP 5 every positive eigenvalue has two dimensional space of associated eigen functions.

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We will study Fourier series w.r.t. different orthogonal systems.

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Show directly that eigenfunctions of (1-4) are orthogonal on $[0, L]$ and of (5) is orthogonal on $[-L, L]$.

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From now on, we will always be working with functions in some inner product space of the type $L^2[a, b]$. In such a space, the norm of a function is defined to be $\|f\| := \langle f, f\rangle^{1/2}$.

Theorem

Let $f \in L^2[-L, L]$. *Consider the series*

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The above series converges to f in the L^2-norm, that is,

$$\lim_{N \longrightarrow \infty} \left\| f - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\| = 0$$

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Therefore, at every point x of continuity of f, the Fourier series converges to $f(x)$.

If we re-define $f(x)$ at every point of discontinuity x as $\frac{1}{2}[f(x+) + f(x-)]$ then the Fourier series represents the function everywhere. Thus two functions can have same Fourier series.

Example

Let us find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

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Example (continued ...)

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Example (continued ...)

Thus, the Fourier series of $f(x)$ is

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2} \cos \frac{n\pi x}{2} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1 + 3\cos n\pi}{n} \sin \frac{n\pi x}{2}$$

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To summarize,

$$F(x) = \begin{cases} 5/4, & x = \pm 2 \\ -x, & -2 < x < 0 \\ 1/4, & x = 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

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The the coefficient of ϕ_n in the expansion of f is given by

$$a_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} .$$

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In this situation, we will loosely write

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Next we shall use the above to find normed basis for some other vector spaces. We will see that, just as in the case of basis, a normed basis need not be unique.

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Restricting this expansion to $[0, L]$ we get the required expansion of f.

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The functions $\{\sin \frac{n\pi x}{L}\}_{n \geq 1}$ are the eigen functions for EVP 1. Thus, we have proved

Theorem

The eigen functions for EVP 1 form a normed basis for $L^2[0,1]$.

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Since $\sin\frac{n\pi x}{L}$ is an odd function and $\tilde{f}$ is even, it follows $\tilde{f}(x)\sin\frac{n\pi x}{L}$ is an odd function. Thus, $b_n = 0$.

This proves that

$$\tilde{f}(x) = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{L}$$

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Eigen functions of EVP 3 as normed basis for $L^2[0,1]$

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

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\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
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To solve this we will use the Fourier expansion in terms of EVP 1. This will be done in the coming lectures.

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In some situations, the term in the box will become 0, making the above computation easy.

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Example. Find the Fourier expansion of

$$f(x) = x^2(3L - 2x) \quad \text{on} \quad [0, L]$$

in terms of eigen functions for EVP 2.

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$$f'(x) = 6Lx - 6x^2 \implies f'(0) = f'(L) = 0.$$

Note $f'''(x) = -12$. Thus, using derivative transfer we get

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Examples of Fourier series

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Thus $a_{2n} = 0$ and $a_{2n-1} = \dfrac{-48L^3}{(2n-1)^4\pi^4}$.

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Therefore, the required Fourier expansion of $f(x)$ on $[0, L]$ is

$$c\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$$

with $c = \frac{96L^3}{\pi^3}$.

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$$-9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\Big]$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L + \frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

Examples of Fourier series

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Therefore, the required Fourier expansion of $f(x)$ on $[0, L]$ is

$$\frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$$

□