MA-207 Differential Equations II

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Now we will start the study of Partial Differential Equations.

In the rest of this course u will denote either a function of two variables or a function of three variables.

A partial differential equation (PDE) is an equation involving u and the partial derivatives of u. Given such an equation, our aim will be to find a function which satisfies this equation.

The order of the PDE is the order of the highest partial derivative of u in the equation.

Examples of some famous PDEs.

1. $u_t - k^2(u_{xx} + u_{yy}) = 0$ two dimensional Heat equation, order 2. Here u is a function of three variables.
2. $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$ two dimensional wave equation, order 2. Here u is a function of three variables.
3. $u_{xx} + u_{yy} = 0$ two dimensional Laplace equation, order 2. Here u is a function of two variables.
4. $u_{tt} + u_{xxxx}$ Beam equation, order 4. Here u is a function of two variables.

Partial Differential Equations: Some Basics

Let $\mathscr{S}$ denote a space of functions. For example, it could denote the space of smooth functions in two variables, or the space of smooth functions in three variables.

A differential operator is a map $D : \mathscr{S} \to \mathscr{S}$.

For example, we could take

$$Du = u(x, y)^2 + 2 \sin x (u_x)^2 + (u_{yy})^3 \,.$$

Definition

A differential operator is said to be linear if it satisfies the condition

$$D(u + v) = D(u) + D(v) \,.$$

Heat equation, Wave equation, Laplace equation and Beam equation are linear PDEs. The example just before the definition is clearly not linear.

Linear Differential Operators

The general form of first order linear differential operator in two variables x, y is

$$L(u) = A(x,y)u_x + B(x,y)u_y + C(x,y)u$$

The general form of first order linear differential operator in three variables x, y, z is

$$L(u) = Au_x + Bu_y + Cu_z + Du$$

where coefficients A, B, C, D and f are functions of x, y and z.
The general form of second order linear PDE in two variables x, y is

$$L(u) = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$$

where coefficients A, B, C, D, E, F and f are functions of x and y.

Classification of second order linear PDE

Consider the linear differential operator L on functions in two variables.

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

where $A, \dots, F$ are functions of x and y.
To the operator L we associate the discriminant $\mathbb{D}(x, y)$ given by

$$\mathbb{D}(x, y) = A(x, y)C(x, y) - B^2(x, y)$$

The operator L is said to be

- elliptic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) > 0$,
- parabolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) = 0$.
- hyperbolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) < 0$,

- Two dimensional Laplace operator $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is elliptic in $\mathbb{R}^2$, since $\mathbb{D} = 1$.
- One dimensional Heat operator (there are two variables, t and x) $H = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$ is parabolic in $\mathbb{R}^2$, since $\mathbb{D} = 0$.
- One dimensional Wave operator (there are two variables, t and x) $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$ is hyperbolic in $\mathbb{R}^2$, since $\mathbb{D} = -1$.

When the coefficients of an operator L are not constant, the type may vary from point to point. Consider the Tricomi operator

$$T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$$

The discriminant $\mathbb{D} = x$. Hence T is elliptic in the half-plane $x > 0$, parabolic on the y-axis and hyperbolic in the half-plane $x < 0$.

Solving PDE's: A few observations

Finally we begin our analysis of how to solve PDE's.

Given a general PDE there is no hope of solving it. However, some of the important PDE's that occur in nature are linear, and these can be solved.

Given a linear differential operator L, our aim will be to solve the equation $Lu = f$ with some boundary conditions. Let us make some observations which will help us breakdown this question into simpler pieces.

Definition

Let L be a linear differential operator. The PDE $Lu = 0$ is called homogeneous and the PDE $Lu = f$, $(f \neq 0)$ is non-homogeneous.

Solving PDE's: A few observations

Principle 1. If $u_1, \ldots, u_N$ are solutions of $Lu = 0$ and $c_1, \ldots, c_N$ are constants, then $\sum_{i=1}^{N} c_i u_i$ is also a solution of $Lu = 0$.

In general, space of solutions of $Lu = 0$ contains infinitely many independent solutions and we may need to use infinite linear combinations of them.

Principle 2. Let L be a differentiable operator of order n. Assume

1. $u_1, u_2, \ldots$ are infinitely many solutions of $Lu = 0$.
2. the series $w = \sum_{i \geq 1} c_i u_i$ with $c_1, c_2, \ldots$ constants, converges to a function, which is differentiable n times;
3. term by term partial differentiation is valid for the series, that is, $Dw = \sum_{i \geq 1} c_i D u_i$, D is any partial differentiation of order $\leq$ order of L.

Then w is again a solution of $Lu = 0$.

Principle 3 for non-homogeneous PDE.
If u_i is a solution of $Lu = f_i$, then

$$w = \sum_{i=1}^{N} c_i u_i \, ,$$

with constants c_i, is a solution of $Lu = \sum_{i=1}^{N} c_i f_i$.

One-dimensional heat equation

The one dimensional heat equation is the PDE

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0\,.$$

Here k is a positive constant.

Imagine a rod of length L whose ends are maintained at a fixed temperature. We may think of x as the space variable and t is the time variable. The function $u(x,t)$ is supposed to give the temperature of the rod at point x and time t.

We can ask to solve this differential equation with various boundary conditions.

Initial-boundary value problem is

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 && t \geq 0 \\ u(L,t) &= 0, && t \geq 0 \\ u(x,0) &= f(x), && 0 \leq x \leq L \end{aligned}$$

We now introduce the method of separation of variables. Let us assume that there is a solution of the form $\boxed{v(x,t) = A(x)B(t)}$ Putting in the initial conditions we get

$$v(0,t) = A(0)B(t) = 0 \quad \text{and} \quad v(L,t) = A(L)B(t) = 0$$

As we don't want B to be identically zero, we get

$$A(0) = 0 \quad \text{and} \quad A(L) = 0.$$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

We also have $v_t = k^2 v_{xx}$. Putting $u(x,t) = A(x)B(t)$ into this we get

$$A(x)B'(t) = k^2 A''(x)B(t)\,.$$

We may rewrite this as

$$\frac{B'(t)}{B(t)} = k^2 \frac{A''(x)}{A(x)}\,.$$

The LHS is a function of t and the RHS is a function of x. The only way both can be equal is if both are equal to the same constant, which we denote by $-\lambda$.

We need to solve eigenvalue problem

1. $A''(x) + \lambda A(x) = 0, \quad A(0) = 0, \quad A(L) = 0, \qquad (*)$
2. $B'(t) = -k^2 \lambda B(t)$

The second problem clearly has solution $B(t) = exp(-k^2 \lambda t)$.

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

We already saw that the first problem has non-trvial solutions only when $\lambda > 0$.
The eigenvalues of $(*)$ are

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$A_n(x) = \sin\frac{n\pi x}{L}, \quad n \geq 1.$$

We get infinitely many solutions for IBVP, one for each $n \geq 1$

$$\begin{aligned} v_n(x,t) &= B_n(t)A_n(x) \\ &= exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L} \end{aligned}$$

Note $\qquad v_n(x,0) = \sin\frac{n\pi x}{L}$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\sin\frac{n\pi x}{L}$$

satisfies the IBVP

$$\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0\\
&u(0,t) = 0 && t \geq 0\\
&u(L,t) = 0 && t \geq 0\\
&u(x,0) = \sin\frac{n\pi x}{L} && 0 \leq x \leq L
\end{aligned}$$

More generally, if $\alpha_1, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m} \alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\sin\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=1}^{m} \alpha_n \sin\frac{n\pi x}{L}.$$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Let us consider the formal series

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} \, t \right) \sin \frac{n\pi x}{L}$$

Heuristically, $u(0,t) = u(L,t) = 0$ and the above series satisfies the equation $u_t = k^2 u_{xx}$. Moreover, setting $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

Thus, for this series to be a solution to our IBVP we would like to have

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L} \qquad 0 \leq x \leq L$$

Is it possible that f has such an expansion?

Given f on $[0, L]$, it has a Fourier series

$$f(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L}$$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Definition

The formal solution of IBVP

$$
\begin{aligned}
u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
u(0,t) &= 0 && t \geq 0 \\
u(L,t) &= 0 && t \geq 0 \\
u(x,0) &= f(x) && 0 \leq x \leq L
\end{aligned}
$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

where

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$ is the Fourier series of f on $[0, L]$,

that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx.$$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Because of negative exponential in $u(x,t)$, the series in $u(x,t)$ converges for all $t > 0$.

Thus, the series $u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right) \sin\frac{n\pi x}{L}$ is a candidate for the solution we are looking for.

However, we need the following conditions too

1. The function u is once differentiable in the variable t and twice differentiable in the variable x.
2. The derivative can be computed by differentiating inside the summation.

If these two conditions are satisfied then $u(x,t)$ is an actual solution of the IBVP.

Both these conditions are satisfied if $f(x)$ is continuous and piecewise smooth on $[0, L]$. Hence we get the next result.

Theorem

Let $f(x)$ *be continuous and piecewise smooth on* $[0, L]$*. Assume* $f(0) = f(L) = 0$*. Let*
$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$ *with* $\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$ *be the Fourier series of* f *on* $[0, L]$*. Then the IBVP*

$$
\begin{aligned}
u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
u(0,t) &= 0 && t \geq 0 \\
u(L,t) &= 0 && t \geq 0 \\
u(x,0) &= f(x) && 0 \leq x \leq L
\end{aligned}
$$

has a solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

Here u_t *and* u_{xx} *can be obtained by term-wise differentiation for* $t > 0$*.*

Example

Let $f(x) = x(x^2 - 3Lx + 2L^2)$. Solve IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 && t > 0 \\ u(L,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \leq x \leq L \end{aligned}$$

The Fourier sine expansion of $f(x)$ is

$$f(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

Therefore, the solution of IBVP is

$$u(x,t) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} exp\left(\frac{-n^2\pi^2k^2}{L^2} t \right) \sin \frac{n\pi x}{L}.$$

Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

Initial-boundary value problem is

$$\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
&u_x(0,t) = 0 && t > 0 \\
&u_x(L,t) = 0, && t > 0 \\
&u(x,0) = f(x), && 0 \le x \le L
\end{aligned}$$

We again use the method of separation of variables. Let us assume that there is a solution of the form $\boxed{v(x,t) = A(x)B(t)}$ Putting in the initial conditions we get

$$v_x(0,t) = A'(0)B(t) = 0 \quad \text{and} \quad v_x(L,t) = A'(L)B(t) = 0$$

As we don't want B to be identically zero, we get

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We also have $v_t = k^2 v_{xx}$. Putting $u(x,t) = A(x)B(t)$ into this we get

$$A(x)B'(t) = k^2 A''(x)B(t)\,.$$

We may rewrite this as

$$\frac{B'(t)}{B(t)} = k^2 \frac{A''(x)}{A(x)}\,.$$

The LHS is a function of t and the RHS is a function of x. The only way both can be equal is if both are equal to the same constant, which we denote by $-\lambda$.

We need to solve eigenvalue problem

1. $A''(x) + \lambda A(x) = 0, \quad A'(0) = 0, \quad A'(L) = 0, \qquad (*)$
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The second problem clearly has solution $B(t) = exp(-k^2 \lambda t)$.

Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

The eigenvalues of $(*)$ are

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$A_n(x) = \cos\frac{n\pi x}{L}, \;\; n \geq 0.$$

We get infinitely many solutions for IBVP, one for each $n \geq 0$

$$\begin{aligned} v_n(x,t) &= B_n(t)A_n(x) \\ &= exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\cos\frac{n\pi x}{L} \end{aligned}$$

Note $\qquad v_n(x,0) = \cos\frac{n\pi x}{L}$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\cos\frac{n\pi x}{L}$$

Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

satisfies the IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 && t > 0 \\ u_x(L,t) &= 0 && t > 0 \\ u(x,0) &= \cos\frac{n\pi x}{L} && 0 \le x \le L \end{aligned}$$

More generally, if $\alpha_0, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=0}^{m} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\cos\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=0}^{m} \alpha_n \cos\frac{n\pi x}{L}.$$

Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

Let us consider the formal series

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} \, t \right) \cos \frac{n\pi x}{L}$$

Heuristically, $u_x(0,t) = u_x(L,t) = 0$ and the above series satisfies the equation $u_t = k^2 u_{xx}$. Moreover, setting $t = 0$ we get

$$u(x,0) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$$

Thus, for this series to be a solution to our IBVP we would like to have

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L} \qquad 0 \le x \le L$$

Is it possible that f has such an expansion?

Given f on $[0, L]$, it has a Fourier cosine series

$$f(x) = \sum_{n\ge 0} a_n \cos \frac{n\pi x}{L}$$

Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

Definition

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 & t > 0 \\ u_x(L,t) &= 0 & t > 0 \\ u(x,0) &= f(x) & 0 \le x \le L \end{aligned}$$

is

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \cos\frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0,L]$ i.e.

$$\alpha_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

Because of negative exponential in $u(x,t)$, the series in $u(x,t)$ converges for all $t > 0$.

Thus, the series $u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right) \cos\frac{n\pi x}{L}$ is a candidate for the solution we are looking for.

However, we need the following conditions too

1. The function u is once differentiable in the variable t and twice differentiable in the variable x.
2. The derivative can be computed by differentiating inside the summation.

If these two conditions are satisfied then $u(x,t)$ is an actual solution of the IBVP.

Both these conditions are satisfied if $f(x)$ is continuous and piecewise smooth on $[0, L]$. Hence we get the next result.

Theorem

$f(x)$ is continuous, piecewise smooth on $[0,L]$; $f'(0) = f'(L) = 0$.

Let $S(x) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L}$ be the Fourier series of f on $[0,L]$.

Then the IBVP

$$
\begin{aligned}
u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
u_x(0,t) &= 0 && t > 0 \\
u_x(L,t) &= 0 && t > 0 \\
u(x,0) &= f(x) && 0 \le x \le L
\end{aligned}
$$

has a solution

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\cos\frac{n\pi x}{L}$$

Here u_t and u_{xx} can be obtained by term-wise differentiation for $t > 0$.

Example

Let $f(x) = x$ on $[0, L]$. Solve IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 & t > 0 \\ u_x(L,t) &= 0 & t > 0 \\ u(x,0) &= f(x) & 0 \le x \le L \end{aligned}$$

The Fourier cosine expansion of $f(x)$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

Therefore, the solution of IBVP is

$$u(x,t) =$$

$$\frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} exp\left(\frac{-(2n-1)^2 \pi^2 k^2}{L^2} t \right) \cos \frac{(2n-1)n\pi x}{L}.$$

Non homogeneous heat equation: Dirichlet boundary cond.

Let us now consider the following PDE

$$\begin{aligned} & u_t - k^2 u_{xx} = F(x,t) \qquad 0 < x < L, \quad t > 0 \\ & u(0,t) = f_1(t) \qquad t > 0 \\ & u(L,t) = f_2(t) \qquad t > 0 \\ & u(x,0) = f(x) \qquad 0 \le x \le L \end{aligned}$$

How do we solve this?

Let us first make the substitution

$$z(x,t) = u(x,t) - (1 - \frac{x}{L})f_1(t) - \frac{x}{L}f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
- $z(0,t) = 0$
- $z(L,t) = 0$
- $z(x,0) = g(x)$

It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

Differentiating the above term by term we get that is satisfies the equation

$$z_t - k^2 z_{xx} = \sum_{n\geq 1} \big(Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t)\big)\sin(\frac{n\pi x}{L})$$

Let us write

$$G(x,t) = \sum_{n\geq 1} G_n(t)\sin(\frac{n\pi x}{L})$$

Thus, if we need $z_t - k^2 z_{xx} = G(x,t)$ then we should have that

$$G_n(t) = Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t) \qquad (*)$$

We also need that $z(x,0) = g(x)$.
If

$$g(x) = \sum_{n\geq 1} b_n \sin \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

The solution to the above equation is given by

$$Z_n(t) = Ce^{-\frac{k^2n^2\pi^2}{L^2}t} + e^{-\frac{k^2n^2\pi^2}{L^2}t}\int_0^t G_n(s)e^{\frac{k^2n^2\pi^2}{L^2}s}ds$$

We can find the constant using the initial condition.

Thus, we let $Z_n(t)$ be this unique solution, then the series

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.

Example

Let us now consider the following PDE

$$\begin{aligned} &u_t - u_{xx} = e^t \qquad && 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(1,t) = 0 && t > 0 \\ &u(x,0) = x(x-1) && 0 \leq x \leq 1 \end{aligned}$$

From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

The Fourier sine series of $F(x,t)$ is given by (for $n \geq 1$)

$$\begin{aligned} F_n(t) &= 2\int_0^1 F(x,t)\sin n\pi x\,dx \\ &= 2\int_0^1 e^t \sin n\pi x\,dx \\ &= \frac{2(1-(-1)^n)e^t}{n\pi} \end{aligned}$$

Example (continued ...)

Thus, the Fourier series for e^t is given by

$$e^t = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

The Fourier sine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = \sum_{n\geq 1} \frac{4((-1)^n - 1)}{(n\pi)^3} \sin n\pi x$$

Substitute $u(x,t) = \sum_{n\geq 1} u_n(t) \sin n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 1} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \sin n\pi x = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

Example (continued ...)

Thus, for $n \geq 1$ and even we get

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n e^{-n^2\pi^2 t}$$

If $n \geq 1$ and even, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

For $n \geq 1$ odd we get

$$u_n'(t) + n^2\pi^2 u_n(t) = \frac{4}{n\pi} e^t$$

that is,

$$u_n(t) = e^{-n^2\pi^2 t} \int_0^t \frac{4}{n\pi} e^s e^{n^2\pi^2 s} ds + C_n e^{-n^2\pi^2 t}$$

Example (continued ...)

If $n \geq 1$ and odd, we have the Fourier coefficient of $x(x-1)$ is $\frac{-8}{(n\pi)^3}$. Thus, we get

$$u_n(0) = C_n = \frac{-8}{(n\pi)^3}$$

Thus, the solution we are looking for is

$$u(x,t) = \sum_{n \geq 0} \left(e^{-(2n+1)^2\pi^2 t} \int_0^t \frac{4}{(2n+1)\pi} e^s e^{(2n+1)^2\pi^2 s} ds + \frac{-8}{((2n+1)\pi)^3} e^{-n^2\pi^2 t} \right) \sin(2n+1)\pi x$$

Non homogeneous heat equation:Neumann boundary cond.

Let us now consider the following PDE

$$u_t - k^2 u_{xx} = F(x,t) \qquad 0 < x < L, \quad t > 0$$
$$u_x(0,t) = f_1(t) \qquad t > 0$$
$$u_x(L,t) = f_2(t) \qquad t > 0$$
$$u(x,0) = f(x) \qquad 0 \leq x \leq L$$

How do we solve this?Let us first make the substitution

$$z(x,t) = u(x,t) - (x - \frac{x^2}{2L})f_1(t) - \frac{x^2}{2L}f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
- $z_x(0,t) = 0$
- $z_x(L,t) = 0$
- $z(x,0) = g(x)$

Non homogeneous heat equation:Neumann boundary cond.

It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n\geq 0} Z_n(t)\cos(\frac{n\pi x}{L})$$

Differentiating the above term by term we get that is satisfies the equation

$$z_t - k^2 z_{xx} = \sum_{n\geq 0} \left(Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t)\right)\cos(\frac{n\pi x}{L})$$

Let us write

$$G(x,t) = \sum_{n\geq 0} G_n(t)\cos(\frac{n\pi x}{L})$$

Thus, if we need $z_t - k^2 z_{xx} = G(x,t)$ then we should have that

$$G_n(t) = Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t) \qquad (*)$$

We also need that $z(x,0) = g(x)$.
If

$$g(x) = \sum_{n \geq 0} b_n \cos \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

The solution to the above equation is given by

$$Z_n(t) = Ce^{-\frac{k^2n^2\pi^2}{L^2}t} + e^{-\frac{k^2n^2\pi^2}{L^2}t}\int_0^t G_n(s)e^{\frac{k^2n^2\pi^2}{L^2}s}ds$$

We can find the constant using the initial condition.

Thus, we let $Z_n(t)$ be this unique solution, then the series

$$z(x,t) = \sum_{n\geq 0} Z_n(t)\cos(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.

Example

Let us now consider the following PDE

$$\begin{aligned} &u_t - u_{xx} = e^t && 0 < x < 1, \quad t > 0 \\ &u_x(0,t) = 0 && t > 0 \\ &u_x(1,t) = 0 && t > 0 \\ &u(x,0) = x(x-1) && 0 \le x \le 1 \end{aligned}$$

From the boundary conditions $u_x(0,t) = u_x(1,t) = 0$ it is clear that we should look for solution in terms of Fourier cosine series.

The Fourier cosine series of $F(x,t)$ is given by (for $n \ge 0$)

$$F_0(t) = \int_0^1 F(x,t)\,dx = \int_0^1 e^t dx = e^t$$

$$F_n(t) = 2\int_0^1 F(x,t)\cos n\pi x\,dx = 2\int_0^1 e^t \cos n\pi x\,dx = 0 \quad n > 0$$

Example (continued ...)

Thus, the Fourier series for e^t is simply e^t.

The Fourier cosine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = -\frac{1}{6} + \sum_{n\geq 1} \frac{2((-1)^n + 1)}{(n\pi)^2} \cos n\pi x$$

Substitute $u(x,t) = \sum_{n\geq 0} u_n(t) \cos n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 0} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \cos n\pi x = e^t$$

Example (continued ...)

Thus, for $n = 0$ we get

$$u_0'(t) = e^t$$

that is,

$$u_0(t) = e^t + C_0$$

In the case $n = 0$, we have that the Fourier coefficient of $x(x-1)$ is $\frac{-1}{6}$. Thus, when we put $u_0(0) = -\frac{1}{6}$ we get $C = -\frac{7}{6}$.
For $n \geq 1$

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n e^{-n^2\pi^2 t}$$

Let us now use the initial condition to determine the constants.

Example (continued ...)

In the case $n \geq 1$ and odd, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

In the case $n \geq 1$ even, we have the Fourier coefficient of $x(x-1)$ is $\frac{4}{(n\pi)^2}$. Thus, we get

$$C_n = \frac{4}{(n\pi)^2}$$

Thus, the solution we are looking for is

$$u(x,t) = e^t - \frac{7}{6} + \sum_{n \geq 1} \left(\frac{1}{(n\pi)^2} e^{-4n^2\pi^2 t} \right) \cos(2n\pi x)$$