# MA-207 Differential Equations II

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September 7, 2021

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# Partial Differential Equations: Some Basics

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Examples of some famous PDEs.

1. $u_t - k^2(u_{xx} + u_{yy}) = 0$ two dimensional Heat equation, order $2$. Here $u$ is a function of three variables.
2. $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$ two dimensional wave equation, order $2$. Here $u$ is a function of three variables.
3. $u_{xx} + u_{yy} = 0$ two dimensional Laplace equation, order $2$. Here $u$ is a function of two variables.
4. $u_{tt} + u_{xxxx}$ Beam equation, order $4$. Here $u$ is a function of two variables.

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Heat equation, Wave equation, Laplace equation and Beam equation are linear PDEs. The example just before the definition is clearly not linear.

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where coefficients $A, B, C, D$ and $f$ are functions of $x, y$ and $z$.

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The operator $L$ is said to be

- elliptic at $(x_0, y_0)$, if $\mathbb{D}(x_0, y_0) > 0$,
- parabolic at $(x_0, y_0)$, if $\mathbb{D}(x_0, y_0) = 0$.
- hyperbolic at $(x_0, y_0)$, if $\mathbb{D}(x_0, y_0) < 0$,

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$$T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$$

The discriminant $\mathbb{D} = x$. Hence $T$ is elliptic in the half-plane $x > 0$, parabolic on the $y$-axis and hyperbolic in the half-plane $x < 0$.

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**Definition**

Let $L$ be a linear differential operator. The PDE $Lu = 0$ is called homogeneous and the PDE $Lu = f$, $(f \neq 0)$ is non-homogeneous.

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Principle 1. If $u_1, \ldots, u_N$ are solutions of $Lu = 0$ and $c_1, \ldots, c_N$ are constants, then $\sum_{i=1}^{N} c_i u_i$ is also a solution of $Lu = 0$.

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Then $w$ is again a solution of $Lu = 0$.

Principle 3 for non-homogeneous PDE.
If $u_i$ is a solution of $Lu = f_i$, then

$$w = \sum_{i=1}^{N} c_i u_i \ ,$$

with constants $c_i$, is a solution of $Lu = \sum_{i=1}^{N} c_i f_i$.

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We can ask to solve this differential equation with various boundary conditions.

# Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Initial-boundary value problem is

$$\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
&u(0,t) = 0 && t \geq 0 \\
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Putting in the initial conditions we get

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As we don't want $B$ to be identically zero, we get

$$A(0) = 0 \quad \text{and} \quad A(L) = 0.$$

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The second problem clearly has solution $B(t) = exp(-k^2 \lambda t)$.

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with associated eigenfunctions

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We get infinitely many solutions for IBVP, one for each $n \geq 1$

$$\begin{aligned} v_n(x,t) &= B_n(t)A_n(x) \\ &= exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L} \end{aligned}$$

Note $\qquad v_n(x,0) = \sin\frac{n\pi x}{L}$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

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More generally, if $\alpha_1, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=1}^{m} \alpha_n \sin\frac{n\pi x}{L}.$$

## Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Let us consider the formal series

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Heuristically, $u(0,t) = u(L,t) = 0$ and the above series satisfies the equation $u_t = k^2 u_{xx}$. Moreover, setting $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

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Heuristically, $u(0,t) = u(L,t) = 0$ and the above series satisfies the equation $u_t = k^2 u_{xx}$. Moreover, setting $t = 0$ we get

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Given $f$ on $[0,L]$, it has a Fourier series

$$f(x) = \sum_{n\ge 1} b_n \sin\frac{n\pi x}{L}$$

# Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

## Definition

The formal solution of IBVP

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is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

where

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$ is the Fourier series of $f$ on $[0, L]$,

that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx.$$

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Both these conditions are satisfied if $f(x)$ is continuous and piecewise smooth on $[0, L]$. Hence we get the next result.

# Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

## Theorem

*Let* $f(x)$ *be continuous and piecewise smooth on* $[0, L]$. *Assume* $f(0) = f(L) = 0$. *Let* $f(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$ *with* $\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$ *be the Fourier series of* $f$ *on* $[0, L]$. *Then the IBVP*

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 && t \geq 0 \\ u(L,t) &= 0 && t \geq 0 \\ u(x,0) &= f(x) && 0 \leq x \leq L \end{aligned}$$

*has a solution*

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n exp\left( \frac{-n^2\pi^2 k^2}{L^2} t \right) \sin \frac{n\pi x}{L}$$

*Here* $u_t$ *and* $u_{xx}$ *can be obtained by term-wise differentiation for* $t > 0$.

### Example

Let $f(x) = x(x^2 - 3Lx + 2L^2)$. Solve IBVP

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The Fourier sine expansion of $f(x)$ is

$$f(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

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# Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$

Initial-boundary value problem is

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As we don't want $B$ to be identically zero, we get

$$A'(0) = 0 \quad \text{and} \quad A'(L) = 0.$$

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We also have $v_t = k^2 v_{xx}$. Putting $u(x,t) = A(x)B(t)$ into this we get

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The second problem clearly has solution $B(t) = exp(-k^2 \lambda t)$.

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The eigenvalues of $(*)$ are

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

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More generally, if $\alpha_0, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=0}^{m} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \cos\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

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is the Fourier sine series of $f$ on $[0,L]$ i.e.

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## Theorem

*$f(x)$ is continuous, piecewise smooth on $[0, L]$; $f'(0) = f'(L) = 0$.*

*Let $S(x) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$ be the Fourier series of $f$ on $[0, L]$.*

*Then the IBVP*

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Let $f(x) = x$ on $[0, L]$. Solve IBVP

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The Fourier cosine expansion of $f(x)$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

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Therefore, the solution of IBVP is

$$u(x,t) =$$

$$\frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} exp\left( \frac{-(2n-1)^2\pi^2 k^2}{L^2} t \right) \cos \frac{(2n-1)n\pi x}{L}.$$

# Non homogeneous heat equation: Dirichlet boundary cond.

Let us now consider the following PDE

$$\begin{aligned}
&u_t - k^2 u_{xx} = F(x,t) \qquad && 0 < x < L, \quad t > 0 \\
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Let us first make the substitution

$$z(x,t) = u(x,t) - (1 - \frac{x}{L})f_1(t) - \frac{x}{L} f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
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Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

The solution to the above equation is given by

$$Z_n(t) = Ce^{-\frac{k^2n^2\pi^2}{L^2}t} + e^{-\frac{k^2n^2\pi^2}{L^2}t} \int_0^t G_n(s)e^{\frac{k^2n^2\pi^2}{L^2}s}ds$$

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Thus, we let $Z_n(t)$ be this unique solution, then the series

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# Non homogeneous heat equation: Dirichlet boundary cond.

## Example

Let us now consider the following PDE

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\begin{aligned}
& u_t - u_{xx} = e^t && 0 < x < 1, \quad t > 0 \\
& u(0,t) = 0 && t > 0 \\
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From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

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The Fourier sine series of $F(x,t)$ is given by (for $n \geq 1$)

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\begin{aligned}
F_n(t) &= 2\int_0^1 F(x,t)\sin n\pi x\,dx\\
&= 2\int_0^1 e^t \sin n\pi x\,dx\\
&= \frac{2(1-(-1)^n)e^t}{n\pi}
\end{aligned}
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## Example (continued ...)

Thus, the Fourier series for $e^t$ is given by

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Substitute $u(x,t) = \sum_{n\geq 1} u_n(t) \sin n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 1} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \sin n\pi x = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

## Example (continued ...)

Thus, for $n \geq 1$ and even we get

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n e^{-n^2\pi^2 t}$$

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If $n \geq 1$ and even, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

For $n \geq 1$ odd we get

$$u_n'(t) + n^2\pi^2 u_n(t) = \frac{4}{n\pi} e^t$$

that is,

$$u_n(t) = e^{-n^2\pi^2 t} \int_0^t \frac{4}{n\pi} e^s e^{n^2\pi^2 s} ds + C_n e^{-n^2\pi^2 t}$$

**Example (continued ...)**

If $n \geq 1$ and odd, we have the Fourier coefficient of $x(x-1)$ is $\frac{-8}{(n\pi)^3}$. Thus, we get

$$u_n(0) = C_n = \frac{-8}{(n\pi)^3}$$

Thus, the solution we are looking for is

$$u(x,t) = \sum_{n \geq 0} \left( e^{-(2n+1)^2\pi^2 t} \int_0^t \frac{4}{(2n+1)\pi} e^s e^{(2n+1)^2\pi^2 s} ds + \frac{-8}{((2n+1)\pi)^3} e^{-n^2\pi^2 t} \right) \sin(2n+1)\pi x$$

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Let us now consider the following PDE

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The Fourier cosine series of $F(x,t)$ is given by (for $n \geq 0$)

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The Fourier cosine series for $f(x) = x(x-1)$ is given by

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In the case $n = 0$, we have that the Fourier coefficient of $x(x-1)$ is $\frac{-1}{6}$. Thus, when we put $u_0(0) = -\frac{1}{6}$ we get $C = -\frac{7}{6}$.
For $n \geq 1$

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

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Let us now use the initial condition to determine the constants.

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Thus, the solution we are looking for is

$$u(x,t) = e^t - \frac{7}{6} + \sum_{n \geq 1} \left( \frac{1}{(n\pi)^2} e^{-4n^2\pi^2 t} \right) \cos(2n\pi x)$$