MA-207 Differential Equations II

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The (Dirichlet) boundary conditions are

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Dirichlet boundary conditions: Getting some solutions

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Thus, we get the conditions

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The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad A_n(x) = \sin\frac{n\pi x}{L}, \; n \geq 1.$$

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$$B_n(t) = \alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right),$$

where α_n and β_n are real numbers.

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Thus, we get a solution for each $n \geq 1$

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Dirichlet boundary conditions: Formal solution

From the above we conclude that one possible solution of the wave equation with boundary conditions is,

$$u(x,t) = \sum_{n\geq 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L}t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L}t\right) \right) \sin\frac{n\pi x}{L}.$$

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This function satisfies

$$u(x,0) = \sum_{n\geq 1} \alpha_n \sin\frac{n\pi x}{L} \quad \text{and}$$

$$u_t(x,0) = \sum_{n\geq 1} \beta_n \sin\frac{n\pi x}{L}.$$

Thus, if $f(x)$ and $g(x)$ have Fourier expansions given by

$$f(x) = \sum_{n \geq 1} \alpha_n \sin \frac{n\pi x}{L} \quad \text{and}$$

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Consider the wave equation with initial and boundary values given by

$$\begin{aligned} &u_{tt} = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = u(L,t) = 0 && t > 0 \\ &u(x,0) = f(x) && 0 \leq x \leq L \\ &u_t(x,0) = g(x) && 0 \leq x \leq L \end{aligned}$$

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Definition (continued)

The formal solution of the above problem is

$$u(x,t) = \sum_{n\geq 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L}t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L}t\right)\right) \sin\frac{n\pi x}{L},$$

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$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{and}$$

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We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT make sense, or it may not make sense to differentiate it term wise.

Dirichlet boundary conditions: Actual solution

Theorem

Let f and g be continuous and piecewise smooth functions on $[0, L]$ such that $f(0) = f(L) = 0$. Then the problem given by

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u_{tt} &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
u(0,t) &= u(L,t) = 0 && t \geq 0 \\
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has an actual solution, which is given by

$$u(x,t) = \sum_{n \geq 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right)\right) \sin\frac{n\pi x}{L},$$

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Consider the wave equation with initial and boundary value given by

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Since both f and g are given by their Fourier series in the above example, it is clear that

$$
\begin{aligned}
\alpha_1 &= 1 & \beta_1 &= 0 \\
\alpha_5 &= 3 & \beta_5 &= 1 \\
\alpha_9 &= 0 & \beta_9 &= -26
\end{aligned}
$$

Dirichlet boundary conditions: Example

Example (continued)

Thus, the solution to the above problem is given by

$$u(x,t) = \cos(\sqrt{5}\pi t)\sin(\pi x) + (3\cos(\sqrt{5}\pi t) + \frac{1}{5\pi\sqrt{5}}\sin(\sqrt{5}\pi t))\sin(5\pi x) + \frac{-26}{9\pi\sqrt{5}}\sin(\sqrt{9}\pi t)\sin(9\pi x)$$

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Recall from the section on eigenvalue problems, that we need that $\lambda \geq 0$. The solutions to this problem are given by

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad n \geq 0 \qquad A_n(x) = \cos\frac{n\pi x}{L}, \; n \geq 0.$$

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For each $n \geq 1$ we get a solution for $B(t)$ given by

$$B_n(t) = \alpha_n \cos\left(\frac{kn\pi}{L}t\right) + \frac{\beta_n L}{kn\pi}\sin\left(\frac{kn\pi}{L}t\right),$$

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$$u(x,t) = \beta_0 t + \alpha_0 + \sum_{n\geq 1}\left(\alpha_n \cos\left(\frac{kn\pi}{L}t\right) + \frac{\beta_n L}{kn\pi}\sin\left(\frac{kn\pi}{L}t\right)\right)\cos\frac{n\pi x}{L}.$$

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This function satisfies

$$u(x,0) = \alpha_0 + \sum_{n\geq 1} \alpha_n \cos\frac{n\pi x}{L} \quad \text{and}$$

$$u_t(x,0) = \beta_0 + \sum_{n\geq 1} \beta_n \cos\frac{n\pi x}{L}.$$

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Thus, if $f(x)$ and $g(x)$ have Fourier expansions given by

$$f(x) = \alpha_0 + \sum_{n \geq 1} \alpha_n \cos \frac{n\pi x}{L} \quad \text{and}$$

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Definition

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Definition (continued)

The formal solution of the above problem is

$$u(x,t) = \beta_0 t + \alpha_0 + \sum_{n\geq 1}\left(\alpha_n \cos\left(\frac{kn\pi}{L}t\right) + \frac{\beta_n L}{kn\pi}\sin\left(\frac{kn\pi}{L}t\right)\right)\cos\frac{n\pi x}{L}.$$

where

$$\alpha_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{and}$$

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We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT make sense, or it may not make sense to differentiate it term wise.

Theorem

Let f and g be continuous and piecewise smooth functions on $[0, L]$. Then the problem given by

$$\begin{aligned} &u_{tt} = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = u_x(L,t) = 0 && t \geq 0 \\ &u(x,0) = f(x) && 0 \leq x \leq L \\ &u_t(x,0) = g(x) && 0 \leq x \leq L \end{aligned}$$

has an actual solution, which is given by

$$u(x,t) = \beta_0 t + \alpha_0 + \sum_{n\geq 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right)\right) \cos\frac{n\pi x}{L}.$$

where

$$\alpha_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \textit{and}$$

$$\beta_0 = \frac{1}{L}\int_0^L g(x)\,dx \qquad \beta_n = \frac{2}{L}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx.$$

Neumann boundary conditions: Example

Example

Consider the wave equation with initial and boundary value given by

$$\begin{array}{ll} u_{tt} = 5u_{xx} & 0 < x < 1, \quad t > 0 \\ u_x(0,t) = u_x(L,t) = 0 & t > 0 \\ u(x,0) = 34 + \cos \pi x + 3 \cos 5\pi x & 0 \leq x \leq 1 \\ u_t(x,0) = 23 + \cos 5\pi x - 26 \cos 9\pi x & 0 \leq x \leq 1 \end{array}$$

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$$u_t(x,0) = 23 + \cos 5\pi x - 26\cos 9\pi x \qquad 0 \le x \le 1$$

Since both f and g are given by their Fourier series in the above example, it is clear that

$$\alpha_0 = 34 \qquad \beta_0 = 23$$
$$\alpha_1 = 1 \qquad \beta_1 = 0$$
$$\alpha_5 = 3 \qquad \beta_5 = 1$$
$$\alpha_9 = 0 \qquad \beta_9 = -26$$

Example (continued)

Thus, the solution to the above problem is given by

$$u(x,t) = 23t + 34 + \cos(\sqrt{5}\pi t)\cos(\pi x)$$
$$+ \left(3\cos(\sqrt{5}\pi t) + \frac{1}{5\pi\sqrt{5}}\sin(\sqrt{5}\pi t)\right)\cos(5\pi x)$$
$$\frac{-26}{9\pi\sqrt{5}}\sin(\sqrt{9}\pi t)\cos(9\pi x)$$

Non homogeneous wave equation:Dirichlet boundary cond.

Let us now consider the following PDE

$$u_{tt} - k^2 u_{xx} = F(x,t) \qquad 0 < x < L, \quad t > 0$$
$$u(0,t) = f_1(t) \qquad t > 0$$
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How do we solve this?

Let us first make the substitution

$$z(x,t) = u(x,t) - (1 - \frac{x}{L})f_1(t) - \frac{x}{L}f_2(t)$$

Then clearly

- $z_{tt} - k^2 z_{xx} = G(x,t)$
- $z(0,t) = 0$
- $z(L,t) = 0$
- $z(x,0) = v(x)$
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It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

Non homogeneous wave equation:Dirichlet boundary cond.

It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

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Let us write

$$G(x,t) = \sum_{n\geq 1} G_n(t)\sin(\frac{n\pi x}{L})$$

Thus, if we need $z_{tt} - k^2 z_{xx} = G(x,t)$ then we should have that

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If

$$v(x) = \sum_{n\geq 1} b_n \sin \frac{n\pi x}{L} \qquad w(x) = \sum_{n\geq 1} c_n \sin \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad Z_n'(0) = c_n \quad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

Thus, we let $Z_n(t)$ be this unique solution, then the series

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.

Example

Let us now consider the following PDE

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\begin{aligned}
& u_{tt} - u_{xx} = e^t \qquad 0 < x < 1, \quad t > 0 \\
& u(0,t) = 0 \qquad t > 0 \\
& u(1,t) = 0 \qquad t > 0 \\
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From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

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From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

The Fourier sine series of $F(x,t)$ is given by (for $n \geq 1$)

$$\begin{aligned} F_n(t) &= 2\int_0^1 F(x,t)\sin n\pi x\,dx \\ &= 2\int_0^1 e^t \sin n\pi x\,dx = \frac{2(1-(-1)^n)e^t}{n\pi} \end{aligned}$$

Example (continued ...)

Thus, the Fourier series for e^t is given by

$$e^t = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

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The Fourier sine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = \sum_{n\geq 1} \frac{4((-1)^n - 1)}{(n\pi)^3} \sin n\pi x$$

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Substitute $u(x,t) = \sum_{n\geq 1} u_n(t) \sin n\pi x$ into the equation $u_{tt} - u_{xx} = e^t$

$$\sum_{n\geq 1} \left(u_n''(t) + n^2\pi^2 u_n(t)\right) \sin n\pi x = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

Example (continued ...)

Thus, for $n \geq 1$ and even we get

$$u_n''(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n \cos n\pi t + D_n \sin n\pi t$$

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Since n is even, the nth Fourier coefficient of $f(x)$ is 0. Thus, we get that $C_n = 0$. Further, since $g(x) = 0$, the nth Fourier coefficient is 0. Thus, we get that $D_n = 0$.

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We conclude that $u_n(t) = 0$ for $n \geq 1$ and even.

Example

For $n \geq 1$ and odd we get

$$u_n''(t) + n^2\pi^2 u_n(t) = \frac{4}{n\pi}e^t$$

If we put $u_n(t) = ce^t$ then we get

$$ce^t + n^2ce^t = \frac{4}{n\pi}e^t$$

Solving the above we get that $\dfrac{4}{n(n^2+1)\pi}e^t$ is a solution.

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The general solution is given by

$$u_n(t) = \frac{4}{n(n^2+1)\pi}e^t + C_n \cos n\pi t + D_n \sin n\pi t$$

Let us now use the initial condition to determine the constants.

Example (continued ...)

In the case $n \geq 1$ odd, we have the Fourier coefficient of $x(x-1)$ is $\frac{-8}{(n\pi)^3}$. Thus, we get

$$C_n + \frac{4}{n(n^2+1)\pi} = \frac{-8}{(n\pi)^3}$$

The nth Fourier coefficient of g is 0, and so we get

$$u_n'(0) = \frac{4}{n(n^2+1)\pi} + nD_n = 0$$

Thus, the solution we are looking for is given by

$$u(x,t) = \sum_{n \geq 0} u_{2n+1}(t) \sin(2n+1)\pi x$$

where $u_n(t)$, C_n and D_n are given as above.

Let us now consider the following PDE

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$$G(x,t) = \sum_{n\geq 0} G_n(t)\cos(\frac{n\pi x}{L})$$

Non homogeneous wave equation:Neumann boundary con.

Thus, if we need $z_{tt} - k^2 z_{xx} = G(x,t)$ then we should have that

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$$v(x) = \sum_{n \geq 0} b_n \cos \frac{n\pi x}{L} \qquad w(x) = \sum_{n \geq 0} c_n \cos \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad Z_n'(0) = c_n \qquad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

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Let us now consider the following PDE

$$u_{tt} - u_{xx} = e^{t} \qquad 0 < x < 1, \quad t > 0$$
$$u_x(0,t) = 0 \qquad t > 0$$
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The Fourier cosine series of $F(x,t)$ is given by (for $n \geq 0$)

$$F_0(t) = \int_0^1 F(x,t)\,dx = \int_0^1 e^t dx = e^t$$
$$F_n(t) = 2\int_0^1 F(x,t)\cos n\pi x\,dx = 2\int_0^1 e^t \cos n\pi x\,dx = 0 \quad n > 0$$

Example (continued ...)

Thus, the Fourier series for e^t is simply e^t.

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The Fourier cosine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = -\frac{1}{6} + \sum_{n\geq 1} \frac{2((-1)^n + 1)}{(n\pi)^2} \cos n\pi x$$

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Substitute $u(x,t) = \sum_{n\geq 0} u_n(t) \cos n\pi x$ into the equation $u_{tt} - u_{xx} = e^t$

$$\sum_{n\geq 0} \left(u_n''(t) + n^2\pi^2 u_n(t)\right) \cos n\pi x = e^t$$

Example (continued ...)

Thus, for $n = 0$ we get

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Let us now use the initial condition to determine the constants.

In the case $n = 0$, we have that the Fourier coefficient of $x(x-1)$ is $\frac{-1}{6}$. Thus, when we put $u_0(0) = -\frac{1}{6}$ we get $1 + D = -\frac{1}{6}$.
We also have $u_0'(0) = 0$, that is,$1 + C = 0$.
Thus,

$$u_0(t) = e^t - t - \frac{7}{6}$$

Example (continued ...)

For $n \geq 1$

$$u_n''(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n \cos n\pi t + D_n \sin n\pi t$$

In the case $n \geq 1$ odd, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

We also have $u_n'(0) = 0$, that is, $D_n = 0$. Thus, if n is odd then $u_n(t) = 0$.

Example (continued ...)

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In the case $n \geq 1$ odd, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

We also have $u_n'(0) = 0$, that is, $D_n = 0$. Thus, if n is odd then $u_n(t) = 0$.
In the case $n \geq 1$ even, we have the Fourier coefficient of $x(x-1)$ is $\frac{4}{(n\pi)^2}$. Thus, we get

$$C_n = \frac{4}{(n\pi)^2}$$

We also have $u_n'(0) = 0$, that is, $D_n = 0$.

Example (continued ...)

Thus, when n is even we get

$$u_n(t) = \frac{4}{(n\pi)^2} \cos n\pi t$$

The solution we are looking for is

$$u(x,t) = e^t - t - \frac{7}{6} + \sum_{n\geq 1} \frac{4}{4(n\pi)^2} \cos 2n\pi t \cos 2n\pi x$$