MA-207 Differential Equations II

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Two dimensional Laplace equation

Consider the following differential equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b,$$

called the Laplace equation in two variables.

We can can ask for solutions to the above equation, which satisfy certain boundary conditions.

For example, we will work out the case where

$$\begin{array}{lll} u(x,0) = f(x) & u(x,b) = 0 & 0 \le x \le a \\ u(0,y) = 0 & u(a,y) = 0 & 0 \le y \le b \end{array}$$

Let us apply the method of separation of variables. Let $u(x,y) = A(x)B(y)$. Then the differential equation becomes

$$A''(x)B(y) + A(x)B''(y) = 0$$

Thus, we have

$$\frac{-A''(x)}{A(x)} = \frac{B''(y)}{B(y)} = \text{constant}$$

Since $u(0, y) = A(0)B(y) = 0$, $u(a, y) = A(a)B(y) = 0$ and we do not want $B(y)$ to be identically zero, we get that $A(0) = 0$ and $A(a) = 0$.

This boundary condition on A forces that the constant above should be positive. Let us denote this positive constant by λ^2.

For every $n \geq 1$, let

$$\lambda_n = \frac{n\pi}{a}$$

Dirichlet boundary conditions: Finding some solutions

For each $n \geq 1$, we have a solution to

$$A''(x) + \lambda_n^2 A(x) = 0$$
$$A(0) = 0 = A(a)$$

given by

$$A_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

Since we do not want $A(x)$ to be identically 0 and $u(x,b) = A(x)B(b) = 0$, this forces that $Y(b) = 0$. Let us also impose the condition that $Y(0) = 1$.

Next consider for each λ_n the problem

$$B''(y) - \lambda_n^2 B(y) = 0$$
$$B(0) = 1$$
$$B(b) = 0$$

The solutions to the above equation are given by

$$B_n(y) = \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right).$$

Thus, for each $n \geq 1$ we get a solution

$$u_n(x,y) = \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right)$$

Now consider the series

$$u(x,y) = \sum_{n\geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where α_n are real numbers.

This gives that

$$u(x,0) = f(x) = \sum_{n\geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right),$$

Thus, if $f(x)$ has the Fourier expansion

$$f(x) = \sum_{n\geq 1} \alpha_n \sin\frac{n\pi x}{a}$$

then we will have solved our Laplace equation with the given boundary conditions.

Dirichlet boundary conditions: Formal solutions

Definition

Consider the Laplace equation with the boundary conditions

$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & 0 < x < a, \quad 0 < y < b \\ u(0, y) = 0 = u(a, y) = 0 & 0 \leq y \leq b \\ u(x, 0) = f(x) & 0 \leq x \leq a \\ u(x, b) = 0 & \end{array}$$

The formal solution of the above problem is

$$u(x, t) = \sum_{n \geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

Dirichlet boundary conditions: Actual solution

Theorem

Let f be continuous and piecewise smooth on $[0, a]$ such that $f(0) = f(a) = 0$. Consider the Laplace equation with the boundary conditions

$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & 0 < x < a, \quad 0 < y < b \\ u(0, y) = 0 = u(a, y) = 0 & 0 \leq y \leq b \\ u(x, 0) = f(x) & 0 \leq x \leq a \\ u(x, b) = 0 & \end{array}$$

The solution to the above problem is given by

$$u(x, t) = \sum_{n \geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Dirichlet boundary condition: Example

Example

Consider the Laplace equation with boundary conditions given by

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
&u(0, y) = 0 = u(a, y) = 0 && 0 \leq y \leq b \\
&u(x, 0) = \sin\left(\frac{5\pi x}{a}\right) - 3\sin\left(\frac{9\pi x}{a}\right) && 0 \leq x \leq a \\
&u(x, b) = 0
\end{aligned}$$

Since f is given by its Fourier series in the above example, it is clear that

$$\alpha_5 = 1$$

$$\alpha_9 = -3$$

Dirichlet boundary condition: Example

Example (continued)

Thus, the solution to the above problem is given by

$$
\begin{aligned}
u(x,t) = &\sin\left(\frac{5\pi x}{a}\right)\sinh\left(\frac{5\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{5\pi b}{a}\right)\\
&-3\sin\left(\frac{9\pi x}{a}\right)\sinh\left(\frac{9\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{9\pi b}{a}\right)
\end{aligned}
$$

Consider the following differential equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \ 0 < y < b,$$

called the Laplace equation in two variables.

Consider the boundary conditions

$$\begin{array}{lll} u(x,0) = f(x) & \qquad u(x,b) = 0 & \qquad 0 \leq x \leq a \\ u_x(0,y) = 0 & \qquad u_x(a,y) = 0 & \qquad 0 \leq y \leq b \end{array}$$

Let $u(x,y) = A(x)B(y)$. Then the differential equation becomes

$$A''(x)B(y) + A(x)B''(y) = 0$$

Thus, we have

$$\frac{-A''(x)}{A(x)} = \frac{B''(y)}{B(y)} = \text{constant}$$

Since $u_x(0, y) = A'(0)B(y) = 0$, $u_x(a, y) = A'(a)B(y) = 0$ and we do not want Y to be identically zero, we get that $A'(0) = 0$ and $A'(a) = 0$.

This boundary condition on A forces that the constant above should be positive. Let us denote this positive constant by λ^2.

For every $n \geq 0$, let

$$\lambda_n = \frac{n\pi}{a}$$

Neumann boundary conditions: Finding some solutions

For each $n \geq 0$, we have a solution to

$$\begin{aligned} A''(x) + \lambda_n^2 A(x) &= 0 \\ A'(0) = 0 &= A'(a) \end{aligned}$$

given by

$$A_n(x) = \cos\left(\frac{n\pi x}{a}\right)$$

Since we do not want $A(x)$ to be identically 0 and $u(x,b) = A(x)B(b) = 0$, this forces that $B(b) = 0$. Let us also impose the condition that $B(0) = 1$.

Next consider for each λ_n the problem

$$\begin{aligned} B''(y) - \lambda_n^2 B(y) &= 0 \\ B(0) &= 1 \\ B(b) &= 0 \end{aligned}$$

The solutions to the above equation are given by

For $n \geq 0$

$$B_0(y) = \frac{-1}{b} y + 1$$

and for $n \geq 1$

$$B_n(y) = \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right).$$

Thus, for each $n \geq 0$ we get a solution

$$u_n(x, y) = \cos\left(\frac{n\pi x}{a}\right) B_n(y)$$

Now consider the series

$$u(x, y) = \sum_{n \geq 0} \alpha_n \cos\left(\frac{n\pi x}{a}\right) B_n(y),$$

where α_n are real numbers.

This gives that

$$u(x,0) = f(x) = \alpha_0 + \sum_{n\geq 1} \alpha_n \cos\left(\frac{n\pi x}{a}\right),$$

Thus, if $f(x)$ has the Fourier expansion

$$f(x) = \alpha_0 + \sum_{n\geq 1} \alpha_n \cos\left(\frac{n\pi x}{a}\right)$$

then we will have solved our Laplace equation with the given boundary conditions.

Neumann boundary conditions: Formal solution

Definition

Consider the Laplace equation with the boundary conditions

$$
\begin{aligned}
&u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
&u_x(0,y) = 0 = u_x(a,y) = 0 && 0 \le y \le b \\
&u(x,0) = f(x) && 0 \le x \le a \\
&u(x,b) = 0 && 0 \le x \le a
\end{aligned}
$$

The formal solution of the above problem is

$$
u(x,y) = \alpha_0\Big(\frac{-1}{b}y + 1\Big) + \sum_{n\geq 1} \alpha_n \cos\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),
$$

where

$$
\alpha_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx
$$

Neumann boundary conditions: Actual solution

Theorem

Let f be continuous and piecewise smooth on $[0,a]$.
Consider the Laplace equation with the boundary conditions

$$\begin{array}{ll} u_{xx}+u_{yy}=0 & 0<x<a, \quad 0<y<b \\ u_x(0,y)=0=u_x(a,y)=0 & 0\le y\le b \\ u(x,0)=f(x) & 0\le x\le a \\ u(x,b)=0 & 0\le x\le a \end{array}$$

The solution to the above problem is given by

$$u(x,y)=\alpha_0\Big(\frac{-1}{b}y+1\Big)+ \sum_{n\geq 1}\alpha_n\cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_0=\frac{1}{L}\int_0^L f(x)\,dx \qquad \alpha_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Example

Consider the Laplace equation with boundary conditions given by

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
&u_x(0,y) = 0 = u_x(a,y) = 0 && 0 \leq y \leq b \\
&u(x,0) = \cos\left(\frac{5\pi x}{a}\right) - 3\cos\left(\frac{9\pi x}{a}\right) && 0 \leq x \leq a \\
&u(x,b) = 0
\end{aligned}$$

Since f is given by its Fourier series in the above example, it is clear that

$$\alpha_5 = 1$$

$$\alpha_9 = -3$$

Neumann boundary conditions: Actual solution

Example (continued)

Thus, the solution to the above problem is given by

$$u(x,t) = \cos\left(\frac{5\pi x}{a}\right)\sinh\left(\frac{5\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{5\pi b}{a}\right) \\ - 3\cos\left(\frac{9\pi x}{a}\right)\sinh\left(\frac{9\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{9\pi b}{a}\right)$$

Laplace equation in polar coordinates

Consider the Dirichlet problem in a disc of radius r

$$u_{xx} + u_{yy} = 0$$

with

$$u = f$$

on the boundary of the disc, which is a circle of radius r.
To solve this problem write the Laplace operator in polar coordinates.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Laplace equation in polar coordinates

Example

Solve for harmonic function $u(r,\theta)$ in unit disc i.e.

$$\Delta u = 0, \quad r < 1, \;\; \theta \in [0, 2\pi]$$

with boundary condition

$$u(1,\theta) = f(\theta) = \begin{cases} \sin\theta, & \theta \in [0,\pi] \\ 0, & \theta \in [\pi, 2\pi] \end{cases}$$

Laplace equation in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Again we try to analyze using separation of variables. Assume $u(r,\theta) = R(r)\Theta(\theta)$. Then $\Delta u = 0$ becomes

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

Laplace equation in polar coordinates

From this we get

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda\,.$$

Thus, we need to solve

$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic. Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

Recall that this is EVP 5. The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_{n,1}(\theta) = \cos(n\theta), \quad \Theta_{n,2}(\theta) = \sin(n\theta)$$

The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0\,,$$

is the Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0\,.$$

$$\implies m = \pm\sqrt{\lambda}$$

For $\lambda = \lambda_0 = 0$ we are in the regular singular repeated roots case. Recall that two linearly independent solutions in this case are

$$R_{0,1}(r) = 1, \quad R_{0,2}(r) = \ln r$$

Laplace equation in polar coordinates

For $\lambda = \lambda_n = n^2 > 0$, $m = \pm n$, two linearly independent solutions are

$$R_{n,1}(r) = r^n, \quad R_{n,2}(r) = r^{-n}$$

Let us look for a solution of the Laplace equation in the disc which is a linear combinations of

$$\{1,\ \ln r\} \cup \{r^n \cos(n\theta),\ r^n \sin(n\theta),\ r^{-n} \cos(n\theta),\ r^{-n} \sin(n\theta)\}_{n \geq 1}$$

Since we are looking for solutions that are bounded in the disc, we will discard $\ln r$, $r^{-n} \cos(n\theta)$ and $r^{-n} \sin(n\theta)$.
Thus, the series solution has the form

$$u(r, \theta) = A_0 + \sum_{n \geq 1} (A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta))$$

At the boundary we get

$$u(1,\theta) = A_0 + \sum_{n\geq 1}(A_n\cos(n\theta) + B_n\sin(n\theta))$$

But we need that $u(1,\theta) = f(\theta)$. Hence, A_i and B_i are Fourier coefficients of $f(\theta)$.
Check that the Fourier series of $f(\theta)$ is

$$f(\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2n\theta)}{4n^2-1} + \frac{1}{2}\sin\theta$$

Therefore, the solution is

$$u(r,\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{4n^2-1}r^{2n}\cos(2n\theta) + \frac{1}{2}r\sin\theta$$

Example

Solve for harmonic function $u(r,\theta)$ in an annulus

$$\begin{aligned} \Delta u(r,\theta) &= 0, \qquad 1 < r < 2,\ \theta \in [0,2\pi] \\ u(1,\theta) &= \cos\theta, \quad 0 \le \theta \le 2\pi \\ u_r(2,\theta) &= \sin 2\theta, \quad 0 \le \theta \le 2\pi \end{aligned}$$

This BVP can be interpreted as that for the steady state temperature distribution in an annular region where on the outer boundary the heat flux is prescribed and on the inner boundary, the temperature is prescribed.

Laplace equation in polar coordinates

Recall that the Laplace equation in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

As the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point in the plane, any function u defined in the plane is 2π-periodic in θ. Therefore,

$$u(r, 0) = u(r, 2\pi), \quad u_r(r, 0) = u_r(r, 2\pi)\,.$$

Again we try to analyze using separation of variables. Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) = 0$$

From this we get

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

Thus, we need to solve

$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic. Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

Recall that this is EVP 5. The eigenvalues and eigen functions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_{n,1}(\theta) = \cos(n\theta), \quad \Theta_{n,1}(\theta) = \sin(n\theta)$$

The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

is the Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0\,.$$

$$\implies m = \pm\sqrt{\lambda}$$

For $\lambda = \lambda_0 = 0$, we are in the regular singular repeated roots case, two linearly independent solutions are

$$R_{0,1}(r) = 1, \quad R_{0,2}(r) = \ln r$$

For $\lambda = \lambda_n = n^2 > 0$, $m = \pm n$, the general solutions are

$$R_{n,1}(r) = r^n, \quad R_{n,2}(r) = r^{-n}$$

Let us look for a solution of the Laplace equation in the disc which is a linear combinations of

$$\{1,\ \ln r\} \cup \{r^n \cos(n\theta),\ r^n \sin(n\theta),\ r^{-n} \cos(n\theta),\ r^{-n} \sin(n\theta)\}_{n \geq 1}$$

Hence the general solution is

$$u(r,\theta) = (A_0 + B_0 \ln r) + \sum_{n \geq 1} (A_n r^n \cos(n\theta) + B_n n r^{-n} \cos(n\theta)) + \sum_{n \geq 1} (C_n r^n \sin(n\theta) + D_n r^{-n} \sin(n\theta))$$

We are given

$$u(1,\theta) = \cos\theta, \quad u_r(2,\theta) = \sin 2\theta$$

$$u(1,\theta) = A_0 + \sum_{n\geq 1}(A_n + B_n)\cos(n\theta) + (C_n + D_n)\sin(n\theta)$$

Compare with $u(1,\theta) = \cos\theta$, we get $\quad A_0 = 0,$

$$A_1 + B_1 = 1,\ A_n + B_n = 0\ (n \geq 2),\ C_n + D_n = 0\ (n \geq 1)$$

$$u_r(r,\theta) = \frac{B_0}{r} + \sum_{n\geq 1} n(A_n r^{n-1} - B_n r^{-n-1})\cos n\theta$$

$$+n(C_n r^{n-1} - D_n r^{-n-1})\sin n\theta$$

Comparing with $u_r(2,\theta) = \sin 2\theta$, we get $\; B_0 = 0,$
$2(2C_2 - 2^{-3}D_2) = 1$

$$A_n 2^{n-1} - B_n 2^{-n-1} = 0\ (n \geq 1), \quad C_n 2^{n-1} - D_n 2^{-n-1} = 0\ (n \neq 2)$$

Using these equations we can solve for the values of the constants.

$$A_0 = 0 = B_0$$

For $n = 1$

$$A_1 + B_1 = 1,\ A_1 - B_1 2^{-2} = 0 \implies A_1 = \frac{1}{5}, B_1 = \frac{4}{5}$$

$$C_1 + D_1 = 0, C_1 - D_1 2^{-2} = 0 \implies C_1 = 0, D_1 = 0$$

For $n = 2$,

$$A_2 + B_2 = 0,\ A_2 2 - B_2 2^{-3} = 0 \implies A_2 = 0 = B_2$$

$$C_2 + D_2 = 0,\ 2C_2 - \frac{1}{2^3} D_2 = \frac{1}{2} \implies C_2 = \frac{4}{17}, D_2 = \frac{-4}{17}$$

Laplace equation in polar coordinates

For $n > 2$,

$$A_n + B_n = 0,\ A_n 2^{n-1} - B_n 2^{-n-1} = 0 \implies A_n^1 = 0 = B_n^1$$

$$C_n + D_n = 0,\ C_n 2^{n-1} - D_n 2^{-n-1} = 0 \implies C_n = 0 = D_n$$

Thus the solution is

$$u(r,\theta) = (\frac{1}{5}r + \frac{4}{5}r^{-1})\cos\theta + (\frac{4}{17}r^2 + \frac{-4}{17}r^{-2})\sin 2\theta$$

Consider the two dimensional wave equation given by

$$u_{tt} = k^2(u_{xx} + u_{yy}) \qquad k > 0$$

In polar coordinates (r, θ) in the region $\mathbb{R}^2$ this equation becomes

$$u_{tt} = k^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

We impose the following initial conditions

$$u(r, \theta, 0) = f(r, \theta) \qquad\qquad u_t(r, \theta, 0) = g(r, \theta)$$

and the following boundary condition

$$u(R, \theta, t) = 0.$$

Let use the method of separation of variables to get some solutions.

If we divide the differential equation by u, then we get

$$\frac{u_{tt}}{u} = k^2\Big(\frac{u_{rr} + r^{-1}u_r}{u} + r^{-2}\frac{u_{\theta\theta}}{u}\Big).$$

Let $u(r, \theta, t) = X(r)Y(\theta)T(t)$.

We get (after multiplying everything with $\frac{r^2}{k^2}$)

$$\frac{Y''(\theta)}{Y(\theta)} = \frac{r^2T''(t)}{k^2T(t)} - \frac{r^2X''(r) + rX'(r)}{X(r)}$$

Thus, both the above have to be a constant.

Also, we are looking for periodic solutions in θ,
This forces that the above constant has to be $-n^2$, for some integer $n \geq 0$.

Thus, $Y(\theta)$ satisfies the differential equation

$$Y''(\theta) + n^2 Y(\theta) = 0$$

If $n = 0$, then the solution is $Y(\theta) = \text{constant}$.

If $n \geq 1$, then the solution is of the type

$$Y(\theta) = A\cos(n\theta) + B\sin(n\theta).$$

The second equation becomes

$$\frac{r^2 T''(t)}{k^2 T(t)} - \frac{r^2 X''(r) + rX'(r)}{X(r)} = -n^2$$

We rewrite this as

$$\frac{T''(t)}{k^2 T(t)} = \frac{r^2 X''(r) + rX'(r)}{r^2 X(r)} - \frac{n^2}{r^2}$$

Again, this forces both sides to be a constant c.

The function X satisfies

$$r^2 X''(r) + rX'(r) - (r^2 c + n^2) X(r) = 0$$

We shall later rule out the case $c \geq 0$.
Let us assume that $c < 0$ and put $c = -a^2$

$$r^2 X''(r) + rX'(r) + (r^2 a^2 - n^2) X(r) = 0$$

Let us assume that $n \geq 1$.

Let $y(r)$ be a solution of the equation

$$r^2 y''(r) + ry'(r) + (r^2 - \lambda^2) y(r) = 0$$

Define $f(r) := y(ar)$
Then $f(r)$ satisfies the differential equation

$$r^2 f''(r) + rf'(r) + ((ra)^2 - \lambda^2) y(r) = 0$$

Thus, the first Bessel solution of

$$r^2X''(r) + rX'(r) + (r^2a^2 - n^2)X(r) = 0$$

$$J_n(ar) = \left(\frac{ar}{2}\right)^n \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{r}{2}\right)^{2m} \quad r > 0$$

Since $2n$ is an even integer, the second solution is

$$y_2(r) = \sum_{n=0}^{p-1} \frac{1}{2^{2n}n!(p-n)!} r^{2n-p} +$$

$$\sum_{n\geq p} \frac{(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!} \Big(H_n - H_{p-1} + H_{n-p}\Big) r^{2n-p} +$$

$$-\sum_{n\geq p} \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!} r^{2n-p} \log r$$

Clearly, this solution is unbounded as $r \to 0$.

Thus, the second solution is to be discarded.

We have the boundary condition $u(R, \theta, t) = 0 = J_n(aR)Y(\theta)T(t)$

Since we do not want Y or T to be identically 0,

we get the condition $J_n(aR) = 0$.

This forces that $a = \mu_{n,i}$ for some $i \geq 1$

where $\mu_{n,i} > 0$ are the zeros of the Bessel function $J_n(ar)$.

With $a = \mu_{n,i}$ let us consider the equation

$$T''(t) + (\mu_{n,i}k)^2 T(t) = 0$$

The solutions to this equation are given by

$$T(t) = C\cos(\mu_{n,i}kt) + D\sin(\mu_{n,i}kt)$$

Thus for $n, i \geq 1$, we have $u_{n,i}(r, \theta, t) = J_n(\mu_{n,i} r)$.

$$(A\cos(n\theta) + B\sin(n\theta))(C\cos(\mu_{n,i}kt) + D\sin(\mu_{n,i}kt))$$

For $n = 0$, we only allow constant solutions for Y (since we want the solutions to be periodic),

So $u_{0,i}(r, \theta, t) = J_0(\mu_{0,i} r)\Big(C\cos(\mu_{n,i}kt) + D\sin(\mu_{n,i}kt)\Big)$

Thus, consider

$$u(r, \theta, t) = \sum_{n \geq 0, i \geq 1} \Big(A_{n,i}\cos(n\theta)\cos(\mu_{n,i}t) +$$
$$+B_{n,i}\sin(n\theta)\cos(\mu_{n,i}t) + C_{n,i}\cos(n\theta)\sin(\mu_{n,i}t)$$
$$+D_{n,i}\sin(n\theta)\sin(\mu_{n,i}t)\Big) J_n(\mu_{n,i} r)$$

If we put $t = 0$ in the above, then we get

$$\sum_{n \geq 0, i \geq 1} (A_{n,i} \cos(n\theta) + B_{n,i} \sin(n\theta)) J_n(\mu_{n,i} r)$$

The initial condition that we have is $u(r, \theta, 0) = f(r, \theta)$
Question: Can we write f as an expansion as above?

Let us first observe that the functions

$$S := \{J_n(\mu_{n,i} r) \cos(n\theta), J_m(\mu_{m,j} r) \sin(m\theta)\}_{n \geq 0, i \geq 1}$$

is an orthogonal set of functions, under the inner product

$$\langle f, g \rangle := \int_0^R \int_0^{2\pi} f(r, \theta) g(r, \theta) r d\theta dr$$

Next let us observe that for any function $f(r)$ and $n \geq 0$ fixed, the function $f(r)\cos(n\theta)$ has a Fourier-Bessel series in terms of $J_n(\mu_{n,i}r)\cos(n\theta)$

This is because $\{J_n(\mu_{n,i}r)\}_{i\geq 1}$ is a maximal orthogonal set for functions f on $[0, R]$ such that $\langle f, f\rangle < \infty$.

Recall that for f as above, $f(r) = \sum_{i \geq 1} c_i J_n(\mu_{n,i} r)$

where

$$c_i = \frac{1}{\langle J_n(\mu_{n,i} r), J_n(\mu_{n,i} r) \rangle}$$

Thus,

$$f(r)\cos(n\theta) = \sum_{i \geq 1} c_i J_n(\mu_{n,i} r)\cos(n\theta)$$

Similarly, the function $f(r)\sin(n\theta)$ has a Fourier-Bessel series in terms of $J_n(\mu_{n,i} r)\sin(n\theta)$

If $g(\theta)$ is any function in $L^2([0, \theta])$, then $g(\theta)$ has a Fourier expansion in terms of $\{\cos(n\theta), \sin(n\theta)\}_{n \in \mathbb{Z}}$, we get that any function of the type $f(r)g(\theta)$ can be approximated by the functions in S.

Since functions of the type $f(r)g(\theta)$ approximate functions of the type $h(r,\theta)$, we get that $h(r,\theta)$ can be approximated by functions in S.

If we write

$$h(r,\theta) = \sum_{n\geq 0, i\geq 1} \Big(A_{n,i}\cos(n\theta) + B_{n,i}\sin(n\theta) \Big) J_n(\mu_{n,i} r) \text{then}$$

$$A_{n,i} = \frac{\langle h, J_n(\mu_{n,i} r)\cos(n\theta)\rangle}{\langle J_n(\mu_{n,i} r)\cos(n\theta), J_n(\mu_{n,i} r)\cos(n\theta)\rangle}$$

Similarly, we get $B_{n,i}$ by replacing $\cos$ by $\sin$ in the above.

Solution of the wave equation

We now return to our initial condition $u(r, \theta, 0) = f(r, \theta)$.

Because of the above discussion, we can solve for $A_{n,i}, B_{n,i}$.

The condition $u_t(r, \theta, 0) = g(r, \theta)$ determines $C_{n,i}, D_{n,i}$.

Theorem

Consider the differential equation

$$u_{tt} = k^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}) \qquad k > 0$$

in the disc $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < R^2\}$*,*
with initial conditions

$$u(r,\theta,0) = f(r,\theta) \qquad u_t(r,\theta,0) = g(r,\theta)$$

where f *and* g *are smooth functions in the disc,*
and boundary condition $u(R,\theta,t) = 0$*.*

Theorem (continued ..)

The above differential equation with initial and boundary conditions has a solution given by

$$u(r,\theta,t) = \sum_{n\geq 0, i\geq 1} \Big(A_{n,i}\cos(n\theta)\cos(\mu_{n,i}t) + \\ + B_{n,i}\sin(n\theta)\cos(\mu_{n,i}t) + C_{n,i}\cos(n\theta)\sin(\mu_{n,i}t) \\ + D_{n,i}\sin(n\theta)\sin(\mu_{n,i}t)\Big) J_n(\mu_{n,i}r)$$

where

$$A_{n,i} = \frac{\langle f, J_n(\mu_{n,i}r)\cos(n\theta)\rangle}{\langle J_n(\mu_{n,i}r)\cos(n\theta), J_n(\mu_{n,i}r)\cos(n\theta)\rangle}$$

$$B_{n,i} = \frac{\langle f, J_n(\mu_{n,i}r)\sin(n\theta)\rangle}{\langle J_n(\mu_{n,i}r)\sin(n\theta), J_n(\mu_{n,i}r)\sin(n\theta)\rangle}$$

Solution of the wave equation

Theorem (continued ..)

$$C_{n,i} = \frac{1}{\mu_{n,i}} \frac{\langle g, J_n(\mu_{n,i} r) \cos(n\theta) \rangle}{\langle J_n(\mu_{n,i} r) \cos(n\theta), J_n(\mu_{n,i} r) \cos(n\theta) \rangle}$$

$$D_{n,i} = \frac{1}{\mu_{n,i}} \frac{\langle g, J_n(\mu_{n,i} r) \sin(n\theta) \rangle}{\langle J_n(\mu_{n,i} r) \sin(n\theta), J_n(\mu_{n,i} r) \sin(n\theta) \rangle}$$