MA-207 Differential Equations II

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For example, we will work out the case where

$$\begin{array}{lll} u(x,0) = f(x) & u(x,b) = 0 & 0 \leq x \leq a \\ u(0,y) = 0 & u(a,y) = 0 & 0 \leq y \leq b \end{array}$$

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Let us apply the method of separation of variables. Let $u(x,y) = A(x)B(y)$. Then the differential equation becomes

$$A''(x)B(y) + A(x)B''(y) = 0$$

Dirichlet boundary conditions: Finding some solutions

Thus, we have

$$\frac{-A''(x)}{A(x)} = \frac{B''(y)}{B(y)} = \text{constant}$$

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For every $n \geq 1$, let

$$\lambda_n = \frac{n\pi}{a}$$

Dirichlet boundary conditions: Finding some solutions

For each $n \geq 1$, we have a solution to

$$A''(x) + \lambda_n^2 A(x) = 0$$
$$A(0) = 0 = A(a)$$

given by

$$A_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

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Next consider for each λ_n the problem

$$B''(y) - \lambda_n^2 B(y) = 0$$
$$B(0) = 1$$
$$B(b) = 0$$

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The solutions to the above equation are given by

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$$u(x,y) = \sum_{n\geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where α_n are real numbers.

This gives that

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Dirichlet boundary conditions: Formal solutions

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Thus, if $f(x)$ has the Fourier expansion

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then we will have solved our Laplace equation with the given boundary conditions.

Dirichlet boundary conditions: Formal solutions

Definition

Consider the Laplace equation with the boundary conditions

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
&u(0,y) = 0 = u(a,y) = 0 && 0 \le y \le b \\
&u(x,0) = f(x) && 0 \le x \le a \\
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The formal solution of the above problem is

$$u(x,t) = \sum_{n \ge 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

Dirichlet boundary conditions: Actual solution

Theorem

Let f be continuous and piecewise smooth on $[0, a]$ such that $f(0) = f(a) = 0$. Consider the Laplace equation with the boundary conditions

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Since f is given by its Fourier series in the above example, it is clear that

$$\alpha_5 = 1$$

$$\alpha_9 = -3$$

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Example (continued)

Thus, the solution to the above problem is given by

$$u(x,t) = \sin\left(\frac{5\pi x}{a}\right)\sinh\left(\frac{5\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{5\pi b}{a}\right) - 3\sin\left(\frac{9\pi x}{a}\right)\sinh\left(\frac{9\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{9\pi b}{a}\right)$$

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where

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Laplace equation in polar coordinates

Consider the Dirichlet problem in a disc of radius r

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To solve this problem write the Laplace operator in polar coordinates.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Laplace equation in polar coordinates

Example

Solve for harmonic function $u(r,\theta)$ in unit disc i.e.

$$\Delta u = 0, \quad r < 1, \;\theta \in [0, 2\pi]$$

with boundary condition

$$u(1,\theta) = f(\theta) = \begin{cases} \sin\theta, & \theta \in [0,\pi] \\ 0, & \theta \in [\pi, 2\pi] \end{cases}$$

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Again we try to analyze using separation of variables. Assume $u(r,\theta) = R(r)\Theta(\theta)$. Then $\Delta u = 0$ becomes

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

From this we get

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda\,.$$

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Thus, we need to solve

$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

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Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic.

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$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

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Recall that this is EVP 5. The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_{n,1}(\theta) = \cos(n\theta), \quad \Theta_{n,2}(\theta) = \sin(n\theta)$$

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The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0\,,$$

is the Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0\,.$$

and for $n \geq 1$,

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For $\lambda = \lambda_0 = 0$ we are in the regular singular repeated roots case. Recall that two linearly independent solutions in this case are

$$R_{0,1}(r) = 1, \quad R_{0,2}(r) = \ln r$$

Laplace equation in polar coordinates

For $\lambda = \lambda_n = n^2 > 0$, $m = \pm n$, two linearly independent solutions are

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Thus, the series solution has the form

$$u(r,\theta) = A_0 + \sum_{n\geq 1} (A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta))$$

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At the boundary we get

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Check that the Fourier series of $f(\theta)$ is

$$f(\theta) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n\geq 1} \frac{\cos(2n\theta)}{4n^2 - 1} + \frac{1}{2} \sin\theta$$

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Therefore, the solution is

$$u(r,\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{4n^2-1}r^{2n}\cos(2n\theta) + \frac{1}{2}r\sin\theta$$

Example

Solve for harmonic function $u(r,\theta)$ in an annulus

$$\Delta u(r,\theta) = 0, \qquad 1 < r < 2,\ \theta \in [0, 2\pi]$$
$$u(1,\theta) = \cos\theta, \quad 0 \le \theta \le 2\pi$$
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This BVP can be interpreted as that for the steady state temperature distribution in an annular region where on the outer boundary the heat flux is prescribed and on the inner boundary, the temperature is prescribed.

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Recall that the Laplace equation in polar coordinates is

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Again we try to analyze using separation of variables. Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

Laplace equation in polar coordinates

From this we get

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

Thus, we need to solve

$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

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Recall that this is EVP 5. The eigenvalues and eigen functions for periodic eigenvalue problem in Θ are

Laplace equation in polar coordinates

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Laplace equation in polar coordinates

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Laplace equation in polar coordinates

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Comparing with $u_r(2,\theta) = \sin 2\theta$, we get $\ B_0 = 0,$
$2(2C_2 - 2^{-3}D_2) = 1$

$$A_n 2^{n-1} - B_n 2^{-n-1} = 0\ (n \geq 1), \quad C_n 2^{n-1} - D_n 2^{-n-1} = 0\ (n \neq 2)$$

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For $n = 1$

$$A_1 + B_1 = 1,\ A_1 - B_1 2^{-2} = 0 \implies A_1 = \frac{1}{5}, B_1 = \frac{4}{5}$$

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Laplace equation in polar coordinates

For $n > 2$,

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Thus the solution is

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$$\frac{Y''(\theta)}{Y(\theta)} = \frac{r^2 T''(t)}{k^2 T(t)} - \frac{r^2 X''(r) + rX'(r)}{X(r)}$$

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If $n = 0$, then the solution is $Y(\theta) = \text{constant}$.

If $n \geq 1$, then the solution is of the type

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Clearly, this solution is unbounded as $r \to 0$.

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Thus for $n, i \geq 1$, we have $u_{n,i}(r, \theta, t) = J_n(\mu_{n,i} r)$.

$$(A\cos(n\theta) + B\sin(n\theta))(C\cos(\mu_{n,i}kt) + D\sin(\mu_{n,i}kt))$$

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$$S := \{J_n(\mu_{n,i}r)\cos(n\theta), J_m(\mu_{m,j}r)\sin(m\theta)\}_{n\geq 0, i\geq 1}$$

is an orthogonal set of functions, under the inner product

$$\langle f, g\rangle := \int_0^R \int_0^{2\pi} f(r,\theta)g(r,\theta)r\,d\theta\,dr$$

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This is because $\{J_n(\mu_{n,i} r)\}_{i \geq 1}$ is a maximal orthogonal set for functions f on $[0, R]$ such that $\langle f, f \rangle < \infty$.

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$$h(r,\theta) = \sum_{n\geq 0, i\geq 1} \Big(A_{n,i}\cos(n\theta) + B_{n,i}\sin(n\theta) \Big) J_n(\mu_{n,i}r) \text{then}$$

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Similarly, we get $B_{n,i}$ by replacing $\cos$ by $\sin$ in the above.

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Solution of the wave equation

Theorem (continued ..)

$$C_{n,i} = \frac{1}{\mu_{n,i}} \frac{\langle g, J_n(\mu_{n,i} r)\cos(n\theta)\rangle}{\langle J_n(\mu_{n,i} r)\cos(n\theta), J_n(\mu_{n,i} r)\cos(n\theta)\rangle}$$

$$D_{n,i} = \frac{1}{\mu_{n,i}} \frac{\langle g, J_n(\mu_{n,i} r)\sin(n\theta)\rangle}{\langle J_n(\mu_{n,i} r)\sin(n\theta), J_n(\mu_{n,i} r)\sin(n\theta)\rangle}$$