MA207 - Tutorial Sheet 1

September 15, 2021

1. Find the radius of convergence R and the interval of convergence (if $R > 0$) of the following:

   (a) $\displaystyle\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)(x-1)^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)\dots(n+k+1)} x^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}(x+1)^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^n} x^n$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{2^n(n!)^3} x^n$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

   (i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n(3n+1)(x-1)^{2n+1}$

(j) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(27)^n}(x-3)^{3n+2}$

(k) $\displaystyle\sum_{n=0}^{\infty}\frac{9^n(n+1)}{(n+2)}(x-2)^{2n+2}$

2. Let R be the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.

   (a) Show that $\displaystyle\sum_{n=0}^{\infty} a_n x^{2n}$ has radius of convergence $R^{1/2}$.

   (b) Show that $\displaystyle\sum_{n=0}^{\infty} a_n^2 x^n$ has radius of convergence R^2.

   (c) Let k be a positive integer. Show that $\displaystyle\sum_{n=0}^{\infty} a_n x^{kn}$ has radius of convergence $R^{1/k}$.

   (d) Let k be a positive integer. Show that $\displaystyle\sum_{n=0}^{\infty} a_n^k x^n$ has radius of convergence R^k.

3. Determine the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} n!x^{n^2}$ and $\displaystyle\sum_{n=0}^{\infty} x^{n!}$.

4. Let $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Express the following equations as power series in x

   (a) $(2+x)y'' + xy' + 3y$

   (b) $(1+3x^2)y'' + 3x^2y' - 2y$

   (c) $(1+2x^2)y'' + (2-3x)y' + 4y$

5. Show that the series $y(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2^n n!)^2}x^{2n}$ is a solution of $xy''+y'+xy = 0$.

6. Find the power series in x for the general solution of:

   (a) $(1 + x^2)y'' + 6xy' + 6y = 0$
   (b) $(1 - x^2)y'' - 8xy' - 12y = 0$
   (c) $(1 + 2x^2)y'' + 7xy' + 2y = 0$
   (d) $(1 - 2x^3)y'' - 10x^2y' - 8xy = 0$
   (e) $y'' - xy = 0$
   (f) $(1 - x^2)y'' - xy' + p^2y = 0$
   (g) $y'' - x^2y = 0$

7. Find the power series in $x - 1$ for the general solution of $(2 + 4x - 2x^2)y'' - 12(x - 1)y' - 12y = 0$.

8. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x + 1)^n$ on an open interval around $x = -1$. Find the power series in $(x + 1)$ for $xy'' + (4 + 2x)y' + (2 + x)y$.

9. Compute coefficients $a_0, \ldots, a_4$ in the series solution in x for the IVP:

   (a) $(1 + 2x^2)y'' + 10xy' + 8y = 0, \qquad y(0) = 2,\ y'(0) = -3$
   (b) $(1 + 2x^2)y'' + xy' + y = 0, \qquad y(0) = 1,\ y'(0) = -1$

10. Compute coefficients $a_0, \ldots, a_4$ in the series solution in $x + 1$ for the IVP:

    (a) $(3+x)y''+(1+2x)y'-(2-x)y = 0, \qquad y(-1) = 2,\ y'(-1) = -3$

11. Find the points x_0 around which the following functions are analytic. Find the radius of the following rational functions about a general point x_0

    (a) $\dfrac{1}{(5 - x)(x^2 + 16)}$
    (b) $\dfrac{25x^2 + 13x + 57}{(x^2 + 1)(x^2 + 2x + 27)}$

12. Consider the ODE $a(x)y'' + b(x)y' + c(x)y = 0$. Assume that $a(x)$, $b(x)$, $c(x)$ are analytic in a neighborhood of x_0 and that $a(x_0) \neq 0$. Let $y(x) = \sum_{n=0}^{\infty} \alpha_n (x - x_0)^n$ be a series solution in a neighborhood of x_0. What can you say about the radius of convergence of this series in terms of the radius of convergence of the series of $a(x), b(x), c(x)$ around x_0? Using this, what can you say about the radius of convergence around 0 of the power series solution of the following ODE's

    (a) $(9 - x^2)(x^2 + 2)y'' + x^2 y' + x(2 + x)y = 0$

    (b) $(16 - x^2)(x^2 + 5)y'' + 2xy' + x(2 + x)y = 0$

13. Let $f(x)$ be the function on $\mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable. Show that $f^{(n)}(0) = 0$ for all $n > 0$ and conclude that f is not analytic at 0.