MA207 - Tutorial Sheet 2

September 7, 2021

1. Express x^2, x^3 , and x^4 as linear combinations of Legendre polynomials.

2. (a) Let $f(x)$ be a polynomial of the type $(x-a)^2(x-b)^2g(x)$ where $a<b$. Use Rolle's theorem to show that $f''(x)$ has at least two distinct roots in the interval (a,b).

   (b) Generalize the above and show that the Legendre polynomial $P_n(x)$ has exactly n distinct roots in the interval $(-1,1)$.

3. Recall that Rodrigues formula says that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\Big((x^2-1)^n\Big).$$

   Evaluate the RHS to show that

$$P_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}. \qquad (0.0.1)$$

4. Show using Rodrigues formula and/or equation (0.0.1) that

   (a) $P_n(-x) = (-1)^n P_n(x)$

   (b) $P'_n(-x) = (-1)^{n+1} P'_n(x)$

   (c) $P_n(1) = 1$

   (d) $P_n(-1) = (-1)^n$

   (e) $P_{2n+1}(0) = 0$

   (f) $P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$

(g) $P_n'(1) = \dfrac{n(n+1)}{2}$

(h) $P_n'(-1) = (-1)^{n+1}\dfrac{n(n+1)}{2}$

(i) $P_{2n}'(0) = 0$

(j) $P_{2n+1}'(0) = (-1)^n\dfrac{(2n+1)!}{2^{2n}(n!)^2}$

5. Show that

$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x) = \begin{cases} \frac{2n(n+1)}{2n+1} & m = n \\ 0 & m \neq n \end{cases}$$

6. Prove the following identities

   (a) Let $n - m = 2k$ where $k \geq 0$. Show that

$$\int_{-1}^{1} P_m'(x)P_n'(x) = m(m+1)\,.$$

   (b) Let $n - m = 2k$ where $k \geq 0$. Show that

$$\int_{-1}^{1} x^m P_n'(x) = 0\,.$$

   (c) Let $n - m = 2k + 1$ where $k \geq 0$. Show that

$$\int_{-1}^{1} P_m'(x)P_n'(x) = 0\,.$$

   (d) Let $n - m = 2k + 1$ where $k \geq 0$. Evaluate

$$\int_{-1}^{1} x^m P_n'(x)\,.$$

7. If $x^n = \sum_{r=0}^{n} a_r P_r(x)$ then show that

$$a_n = \frac{2^n(n!)^2}{(2n)!}\,.$$

8. Let $n \geq 1$ be an integer. Let $p(x)$ be a polynomial of degree n such that $\int_{-1}^{1} x^k p(x)dx = 0$ for $k \in \{0, 1, \ldots, n-1\}$. Show that $p(x) = cP_n(x)$ for some constant c.

9. Find the Legendre series of the following polynomials and the values to which the series converges for points in $[-1, 1]$

   (a) $f_1(x) = \begin{cases} -1 & -1 \leq x \leq 0 \\ 1 & 0 < x \leq 1 \end{cases}$

   (b) $f_2(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ 1 & 0 < x \leq 1 \end{cases}$

   (c) $f_3(x) = \begin{cases} -x & -1 \leq x \leq 0 \\ x & 0 < x \leq 1 \end{cases}$

   (d) $f_4(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ x & 0 < x \leq 1 \end{cases}$