MA207 - Tutorial Sheet 4

September 20, 2021

1. Using the indicated substitutions, reduce the following differential equations to the Bessel equation and find the general solution in term of the Bessel functions.

   (a) $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0, \quad (\lambda x = z).$

   (b) $xy'' - 5y' + xy = 0, \quad (y = x^3u).$

   (c) $y'' + k^2xy = 0, \quad (y = u\sqrt{x},\ \frac{2}{3}kx^{3/2} = z).$

   (d) $x^2y'' + (1-2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0, \quad (y = x^\nu u,\ \ x^\nu = z).$

2. Prove the following Bessel identities.

   (a) $[x^pJ_p(x)]' = x^pJ_{p-1}(x)$

   (b) $[x^{-p}J_p(x)]' = -x^{-p}J_{p+1}(x).$

   (c) $J_p'(x) + \dfrac{p}{x}J_p(x) = J_{p-1}(x).$

   (d) $J_p'(x) - \dfrac{p}{x}J_p(x) = -J_{p+1}(x).$

   (e) $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x).$

   (f) $J_{p-1}(x) + J_{p+1}(x) = \dfrac{2p}{x}J_p(x).$

3. Show that for $p \in \{0, 1, 2, \ldots\}$ we have $J_{-p}(x) = (-1)^pJ_p(x)$. (HINT: Use vanishing of the Gamma function at certain points)

4. Show that (a) $J_{1/2} = \sqrt{\dfrac{2}{\pi x}}\sin x,$ (b) $J_{-1/2} = \sqrt{\dfrac{2}{\pi x}}\cos x.$

   (c) $J_{3/2} = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\sin x}{x} - \cos x\right),$ (d) $J_{-3/2} = -\sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\cos x}{x} + \sin x\right).$

5. For an integer n, show that $J_n(x)$ is an even (resp. odd) function if n is even (resp. odd).

6. Express $J_2(x), J_3(x), J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

7. Show the following.

   (a) $J_3 + 3J_0' + 4J_0''' = 0.$ (b) $\displaystyle\int J_{\nu+1} dx = \int J_{\nu-1} dx - 2J_\nu.$

8. If y_1 and y_2 are any two solutions of the Bessel equation of order ν, then show that $y_1 y_2' - y_1' y_2 = c/x$ for a suitable constant c.

9. Show that $\displaystyle\int x^\mu J_\nu(x)\, dx = x^\mu J_{\nu+1}(x) - (\mu - \nu - 1) \int x^{\mu-1} J_{\nu+1}(x)\, dx.$

10. Expand the indicated function in Fourier-Bessel series over the given interval and in terms of the Bessel function of given order. (The Bessel expansion theorem applies in each case.)

    (a) $f(x) = 1$ over $[0, 1]$, $\nu = 0$. (b) $f(x) = x$ over $[0, 1]$, $\nu = 1$.

    (c) $f(x) = x^3$ over $[0, 1]$, $\nu = 1$. (d) $f(x) = x^2$ over $[0, 1]$, $\nu = 2$.

    (e) $f(x) = \sqrt{x}$ over $[0, 1]$, $\nu = \frac{1}{2}$.

11. If $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 0 & \text{if } 1/2 < x \le 1 \end{cases}$, show that $\displaystyle f(x) = \sum_{n=1}^{\infty} \frac{J_1(\lambda_{0,n}/2)}{\lambda_{0,n} J_1(\lambda_{0,n})^2} J_0(\lambda_{0,n} x),$
    where $\lambda_{0,n}$'s are positive zeros of $J_0(x)$.