MA207 - Tutorial Sheet 6

August 24, 2021

1. Solve the following heat equations.

   (a) $L = 1, \quad u_t = u_{xx}$, with $u(0,t) = 0 = u(1,t),\ u(x,0) = x(1-x)$

   (b) $L = \pi,\ u_t = 3u_{xx}$, with $u(0,t) = 0 = u(\pi,t),\ u(x,0) = x\sin x$

   (c) $L = 2,\ u_t = 4u_{xx}$, with $u_x(0,t) = 0 = u_x(2,t),\ u(x,0) = \cos(\frac{\pi x}{2})$

   (d) $L = 1,\ u_t = u_{xx}$, with $u_x(0,t) = 0 = u_x(1,t),\ u(x,0) = x^2(3x^2 - 8x + 6)$

   (e) $L = 1,\ u_t = u_{xx}$, with $u_x(0,t) = 0 = u_x(1,t),\ u(x,0) = \cos \pi x$

2. Solve the following non-homogeneous IBVP.

   (a) $L = 4,\ u_t = 9u_{xx} - 54x$, with $u(0,t) = 1,\ u(4,t) = 61,\ u(x,0) = 1 - x + x^3$

   (b) $L = 1,\ u_t = u_{xx} - 2$, with $u(0,t) = 1,\ u(1,t) = 3,\ u(x,0) = 2x^2 + 1$

   (c) $L = 1,\ u_t = 3u_{xx} - 18x$, with $u_x(0,t) = -1,\ u_x(1,t) = -1,\ u(x,0) = -x$

   (d) $L = 1, \quad u_t = 3u_{xx} + \pi^2 \sin \pi x$, with $u_x(0,t) = 0,\ u_x(1,t) = -\pi,\ u(x,0) = 2\cos \pi x$.

   (e) $L = \pi, \quad u_t - u_{xx} = 8e^{-t}\sin 3x$, with $u(0,t) = 0 = u(\pi,t),\ u(x,0) = 2\sin 2x$

   (f) $L = \pi, \quad u_t - u_{xx} = e^{-t}\cos 2x$, with $u_x(0,t) = e^{-t},\ u_x(\pi,t) = -e^{-t},\ u(x,0) = \sin x$.

   Remaining problems are not for the exam but only for your intellectual curiosity.

3. For the heat equation: $u_t - ku_{xx} = 0,\ 0 < x < \ell,\ t > 0$ with $u(x,0) = u_0(x)$ and $u_x(0,t) = u_x(\ell,t) = 0$, show that $\displaystyle\int_0^\ell u(x,t)\,dx = C,$ where C is a constant. In other words, the average temperature stays constant.

   Further, show that $\lim_{t\to\infty} u(x,t) = \dfrac{1}{\ell}\displaystyle\int_0^\ell u_0(x)\,dx.$

   Compute the solution, when u_0 is: $(i)\, u_0(x) = x, \qquad (ii)\, u_0(x) = \sin^2(\frac{\pi x}{\ell})$.

4. Compute the solution of $u_t - ku_{xx} + a^2 u = 0, \quad 0 < x < \ell,\ t > 0$ with $u(x,0) = u_0(x)$ and $u(0,t) = u(\ell,t) = 0$. Find $\lim_{t\mapsto\infty} u(x,t)$.