NEF CONES OF SOME QUOT SCHEMES ON A SMOOTH PROJECTIVE CURVE

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ABSTRACT. Let C be a smooth projective curve over $\mathbb{C}$. Let $n, d \geq 1$. Let $\mathcal{Q}$ be the Quot scheme parameterizing torsion quotients of the vector bundle $\mathcal{O}_C^n$ of degree d. In this article we study the nef cone of $\mathcal{Q}$. We give a complete description of the nef cone in the case of elliptic curves. We compute it in the case when $d = 2$ and C very general, in terms of the nef cone of the second symmetric product of C. In the case when $n \geq d$ and C very general, we give upper and lower bounds for the Nef cone. In general, we give a necessary and sufficient criterion for a divisor on $\mathcal{Q}$ to be nef.

1. INTRODUCTION

Throughout this article we assume that the base field to be $\mathbb{C}$. Let X be a smooth projective variety and let $N^1(X)$ be the $\mathbb{R}$-vector space of $\mathbb{R}$-divisors modulo numerical equivalence. It is known that $N^1(X)$ is a finite dimensional vector space. The closed cone $\mathrm{Nef}(X) \subset N^1(X)$ is the cone of all $\mathbb{R}$-divisors whose intersection product with any curve in X is non-negative. It has been an interesting problem to compute $\mathrm{Nef}(X)$. For example, when $X = \mathbb{P}(E)$ where E is a semistable vector bundle over a smooth projective curve, Miyaoka computed the $\mathrm{Nef}(X)$ in [Miy87]. In [BP14], $\mathrm{Nef}(X)$ was computed in the case when X is the Grassmann bundle associated to a vector bundle E on a smooth projective curve C, in terms of the Harder Narasimhan filtration of E. Let $C^{(d)}$ denote the dth symmetric product. In [Pac03], the author computed the $\mathrm{Nef}(C^{(d)})$ in the case when C is a very general curve of even genus and $d = \mathrm{gon}(C) - 1$. In [Kou93] $\mathrm{Nef}(C^{(2)})$ is computed in the case when C is very general and g is a perfect square. In [CK99] $\mathrm{Nef}(C^{(2)})$ was computed assuming the Nagata conjecture. We refer the reader to [Laz04, Section 1.5] for more such examples and details.

The reader is referred to [FGI+05] for the definition and details on Quot schemes. Let E be a vector bundle over a smooth projective curve C. Fix a polynomial $P \in \mathbb{Q}[t]$. Let $\mathcal{Q}(E, P)$ denote the Quot scheme parametrizing quotients of E with Hilbert polynomial P. In [Str87], when $C = \mathbb{P}^1$, the quot scheme $\mathcal{Q}(\mathcal{O}_C^n, P)$ is studied as a natural compactification of the set of all maps from C to some Grassmannians of a fixed degree. In this article we will

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consider the case when $P = d$ a constant, that is, when $\mathcal{Q}(E, d)$ parametrizes torsion quotients of E of degree d. For notational convenience, we will denote $\mathcal{Q}(E, d)$ by $\mathcal{Q}$, when there is no possibility of confusion. It is known that $\mathcal{Q}$ is a smooth projective variety. Many properties of $\mathcal{Q}$ have been studied. In [BDH15] the Brauer group of $\mathcal{Q}(\mathcal{O}^n_C, d)$ is computed. In [BGL94], the Betti cohomologies of $\mathcal{Q}(\mathcal{O}^n_C, d)$ are computed, $\mathcal{Q}(O^n_C, d)$ has been intepreted as the space of higher rank divisors of rank n, and an analogue of the Abel-Jacobi map was constructed. In [Gan19], the automorphism group scheme of $\mathcal{Q}(E, d)$ was computed in the case when either rk $E \geq 3$ or E is semistable and genus of C satisfies $g(C) > 1$. In [GS19], the S-fundamental group scheme of $\mathcal{Q}(E, d)$ was computed.

In this article, we address the question of computing $\mathrm{Nef}(\mathcal{Q})$. Recall that we have a map $\Phi : \mathcal{Q} \to C^{(d)}$ (a precise definition can be found, for example, in [GS19]). For notational convenience, for a divisor $D \in N^1(C^{(d)})$ we will denote its pullback $\Phi^*D \in N^1(\mathcal{Q})$ by D, when there is no possibility of confusion. The line bundle $\mathcal{O}_{\mathcal{Q}}(1)$ is defined in Definition 3.2. In Section 2 we recall the results we need on $\mathrm{Nef}(C^{(d)})$. In Section 3 we compute $\mathrm{Pic}(\mathcal{Q})$.

Theorem (Theorem 3.7)**.** $\mathrm{Pic}(\mathcal{Q}) = \Phi^*\mathrm{Pic}(C^{(d)}) \bigoplus \mathbb{Z}[\mathcal{O}_{\mathcal{Q}}(1)]\,.$

As a corollary (Corollary 3.10) we get that $N^1(\mathcal{Q}) \cong N^1(C^{(d)}) \oplus \mathbb{R}[\mathcal{O}_{\mathcal{Q}}(1)]$. The computation of $N^1(\mathcal{Q})$ can also be found in [BDH15]. As a result, when $C \cong \mathbb{P}^1$, since $C^{(d)} \cong \mathbb{P}^d$, we have that the $N^1(\mathcal{Q})$ is 2-dimensional and we prove that its nef cone is given as follows.

Theorem (Theorem 6.2)**.** *Let* $C = \mathbb{P}^1$. *Let* $E = \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ *with* $a_i \leq a_j$ *for* $i < j$. *Let* $d \geq 1$. *Then*

$$\mathrm{Nef}(\mathcal{Q}(E, d)) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}(E,d)}(1)] + (-a_1 + d - 1)[\mathcal{O}_{\mathbb{P}^d}(1)]) + \mathbb{R}_{\geq 0}[\mathcal{O}_{\mathbb{P}^d}(1)]\,.$$

Note that this theorem was already known in the case when $E = V \otimes \mathcal{O}_{\mathbb{P}^1}$, for a vector space V over k ([Str87, Theorem 6.2]).

For the rest of the introduction, we will assume $E = V \otimes \mathcal{O}_C$ with $\dim_k V = n$ and denote by $\mathcal{Q} = \mathcal{Q}(n, d)$ the Quot scheme $\mathcal{Q}(E, d)$. Let us consider the case $g = 1$. In this case, $N^1(\mathcal{Q})$ is three-dimensional (see Proposition 3.11), and we prove that its nef cone is given as follows (see Definition 2.4 for notations).

Theorem (Theorem 7.15)**.** *Let* $g = 1$, $n \geq 1$ *and* $\mathcal{Q} = \mathcal{Q}(n, d)$. *Then the class* $[\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2] \in N^1(\mathcal{Q})$ *is nef. Moreover,*

$$\mathrm{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[\Delta_d/2]\,.$$

From now on assume that $g \geq 2$ and C is very general. See Definition 7.3 for the definition of t and α_t. When $d = 2$ we have the following result.

Theorem (Theorem 7.5)**.** *Let $g \geq 2$ and C be very general. Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 2)$. Then*

$$\mathrm{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0]) + \mathbb{R}_{\geq 0}[L_0] + \mathbb{R}_{\geq 0}[\alpha_t]\,.$$

Precise values of t are known for small genus. When $g \geq 9$ it is conjectured that $t = \sqrt{g}$. This is known when g is a perfect square. The precise statements have been mentioned after Theorem 7.5.

In general (without any assumptions on n and d), we give a criterion for certain line bundle on $\mathcal{Q}$ to be nef in terms of its pullback along certain natural maps from products $\prod_i C^{(d_i)}$, see subsection 7.6 for notation.

Theorem (Theorem 7.11)**.** *Let $\beta \in N^1(C^{(d)})$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + \beta \in N^1(\mathcal{Q})$ is nef iff the class $[\mathcal{O}(-\Delta_{\mathbf{d}}/2)] + \pi_{\mathbf{d}}^*\beta \in N^1(C^{(\mathbf{d})})$ is nef for all $\mathbf{d} \in \mathcal{P}_d^{\leq n}$.*

Using the above we show that certain classes are in $\mathrm{Nef}(\mathcal{Q})$. Define
(1.1)

$$\kappa_1 := [\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0] + \frac{d+g-2}{dg}[\theta_d] \qquad \kappa_2 = [\mathcal{O}_{\mathcal{Q}}(1)] + \frac{g+1}{2g}[L_0] \in N^1(\mathcal{Q})\,.$$

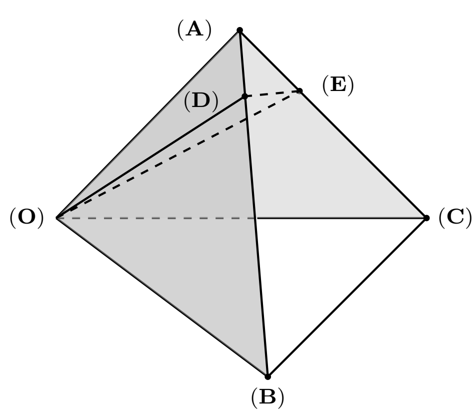
Proposition (Proposition 7.13)**.** *Let $g \geq 1$, $n \geq 1$ and $\mathcal{Q} = \mathcal{Q}(n, d)$. Then*

$$\mathrm{Nef}(\mathcal{Q}) \supset \mathbb{R}_{\geq 0}\kappa_1 + \mathbb{R}_{\geq 0}\kappa_2 + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0]\,.$$

Now consider the case when $n \geq d \geq \mathrm{gon}(C)$. Then $\mathrm{Nef}(C^{(d)})$ is generated by θ_d and L_0 (see Definitions 2.1 and 2.4). In this case we give the following upper bound for the nef cone in Proposition 4.2. Let $\mu_0 := \dfrac{d+g-1}{dg}$. Then

$$\mathrm{Nef}(\mathcal{Q}) \subset \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0]\,.$$

When $d \geq \mathrm{gon}(C)$, in Lemma 5.9 we show that any convex linear combination of the κ_1 and θ_d is nef but not ample. In particular, any such class lies on the boundary of $\mathrm{Nef}(\mathcal{Q})$. Similarly, in Corollary 7.14 we show when $n \geq d$, any convex linear combination of the class κ_2 and $L_0^{(d)}$ is nef but not ample. So any such class lies on the boundary of $\mathrm{Nef}(\mathcal{Q})$.



(1.2)

(1) $(\mathbf{A}) = [\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]$
(2) $(\mathbf{B}) = [\theta_d]$
(3) $(\mathbf{C}) = [L_0]$
(4) $(\mathbf{D}) = \tau\kappa_1 = \tau([\mathcal{O}_{\mathcal{Q}}(1)]/2 + \mu_0[L_0]) + (1-\tau)[\theta_d] \qquad \tau = \dfrac{1}{1+\frac{d+g-2}{dg}}$
(5) $(\mathbf{E}) = \rho\kappa_2 = \rho([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) + (1-\rho)[L_0] \qquad \rho = \dfrac{1}{1+\frac{g+1}{2g}-\frac{d+g-2}{dg}}$

In terms of the above diagram, we have that when $n \geq d \geq \mathrm{gon}(C)$

$$\langle \overline{OD}, \overline{OE}, \overline{OC}, \overline{OB} \rangle \subset \mathrm{Nef}(\mathcal{Q}) \subset \langle \overline{OA}, \overline{OC}, \overline{OB} \rangle .$$

We do not know if the inclusion in the right is an equality when $n \geq d \geq \mathrm{gon}(C)$. This is same as saying that $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]$ is nef when $n \geq d \geq \mathrm{gon}(C)$. In Section 8 we give a sufficient condition for when the pullback of $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]$ along a map $D \to \mathcal{Q}$ is nef. However, when $d = 3$ we have the following result.

Theorem (Theorem 8.6)**.** *Let C be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathcal{Q} = \mathcal{Q}(n,3)$. Let $\mu_0 = \dfrac{g+2}{3g}$ Then*

$$\mathrm{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0] .$$

Some of the results above can be improved in the case when $g = 2k$ using the results in [Pac03]. (See Proposition 5.11.)

2. Nef cone of $C^{(d)}$

We follow [Pac03, §2] for this section. Assume that either C is an elliptic curve or is a very general curve of genus $g \geq 2$. Then it is known that the Neron-Severi space is 2-dimensional. So in this case, to compute the nef cone, it is enough to give two classes in $N^1(C)$ which are nef but not ample.

For any smooth projective curve and $d \geq 2$ (not just a very general curve) there is a natural line bundle L_0 on $C^{(d)}$ which is nef but not ample. This line bundle is constructed in the following manner. Consider the map

$$\phi : C^d \to J(C)^{\binom{d}{2}} ,$$
$$(x_i) \mapsto (x_i - x_j)_{i<j} .$$

Let p_{ij} denote the projections from $J(C)^{\binom{d}{2}}$. Since ϕ is not finite, as it contracts the diagonal, the line bundle $\phi^*(\otimes p_{ij}^*\Theta)$ is nef but not ample. This line bundle is invariant under the action of S_d on C^d. This follows from the fact that Θ in $J(C)$ is invariant under the involution $L \mapsto L^{-1}$.

Definition 2.1. *$\phi^*(\otimes p_{ij}^*\Theta)$ descends to a line bundle L_0 on $C^{(d)}$.*

Since ϕ contracts the small diagonal $\delta : C \hookrightarrow C^{(d)}$, we have $\delta^*[L_0] = 0$. Hence L_0 is nef but not ample [Pac03, Lemma 2.2]. Therefore, in the case when C is very general, computing the nef cone of $C^{(d)}$ boils down to finding another class which is nef but not ample.

In the case when $d \geq \mathrm{gon}(C) =: e$, [Pac03, Lemma 2.3] we can easily construct another line bundle which is nef but not ample: Then we have a map $g_e : C \to \mathbb{P}^1$ of degree e. This induces a closed immersion $\mathbb{P}^1 \to C^{(e)}$ with $v \mapsto [(g_e)^{-1}(v)] \in C^{(e)}$. This in turn gives a closed immersion $\mathbb{P}^1 \to C^{(d)}$ with $v \mapsto [(g_e)^{-1}(v) + (d-e)x]$ for some point $x \in C$.

Definition 2.2. *Denote the class of this $\mathbb{P}^1$ in $N_1(C^{(d)})$ by $[l']$.*

The composition $\mathbb{P}^1 \to C^{(d)} \xrightarrow{u_d} J(C)$ is constant, since there can be no non-constant maps from $\mathbb{P}^1 \to J(C)$. Hence $u_d : C^{(d)} \to J(C)$ is not finite and we get that $u_d^*\Theta$ is nef but not ample.

Definition 2.3. *Define $\theta_d := u_d^*\Theta$.*

Recall that over $C^{(d)}$ we have natural divisors [Pac03, §2]:

Definition 2.4. *Define*

1. θ_d
2. *the big diagonal $\Delta_d \hookrightarrow C^{(d)}$*
3. *If $i_{d-1} : C^{(d-1)} \to C^{(d)}$ is the map given by $D \mapsto D + x$ for a point $x \in C$, then the image $i_{d-1}(C^{(d-1)})$. This divisor will be denoted $[x]$.*

It is known that when $g = 1$ or C is very general of $g \geq 2$, then $N^1(C^{(d)})$ is of dimension 2 and any two of the above three forms a basis.

By abuse of notation, let us denote the class (δ is the small diagonal) $[\delta_*(C)] \in N_1(C^{(d)})$ by δ. We summarise the above discussion in the following theorem.

Proposition 2.5. [Pac03, Proposition 2.4] *When $d \geq \mathrm{gon}(C)$, we have:*

1. $\mathrm{Nef}(C^{(d)}) = \mathbb{R}_{\geq 0}[L_0] \oplus \mathbb{R}_{\geq 0}[\theta_d]$,
2. $\overline{NE}(C^{(d)}) = \mathbb{R}_{\geq 0}[l'] \oplus \mathbb{R}_{\geq 0}[\delta]$.

The above basis are dual to each other.

We will need to write $[L_0]$ in terms of $[x]$ and $[\theta_d]$, for which we need the following computations. Define

$$\delta' : C \xrightarrow{f} C^d \to C^{(d)}$$

where the first map is given by $x \mapsto (x, x_1, \ldots, x_{d-1})$.

Lemma 2.6. *Let $d \geq 1$. We have the following*

1. $\deg(\delta^*[\theta_d]) = d^2 g$
2. $\deg(\delta'^*[\theta_d]) = g$
3. $\deg(\delta^*[x]) = d$
4. $\deg(\delta'^*[x]) = 1$

Proof. Recall that $\theta_d = u_d^*\Theta$, where $u_d : C^{(d)} \to J(C)$ is given by $D \mapsto \mathcal{O}(D - dx_0)$ for a fixed point $x_0 \in C$. Therefore the composition $u_d \circ \delta : C \to J(C)$ is given by $x \mapsto dx \mapsto \mathcal{O}(dx - dx_0)$, which is the map

$$C \xrightarrow{u_1} J(C) \xrightarrow{\times d} J(C)\,.$$

The pullback of Θ under the map $J(C) \xrightarrow{\times d} J(C)$ is Θ^{d^2} and the degree of the pullback of Θ under the map $u_1 : C \to J(C)$ is g. Hence degree of $\delta^*\theta_d = d^2 g$. This proves (1).

The composition $u_d \circ \delta' : C \to J(C)$ is given by $C \to C^{(d)} \to J(C)$

$$x \mapsto x + \sum_{i=1}^{d-1} x_i \mapsto \mathcal{O}(x + \sum_{i=1}^{d-1} x_i - dx_0)$$

which is the composition $C \xrightarrow{u_1} J(C) \xrightarrow{t_a} J(C)$, where t_a is translation by an element in $J(C)$. Hence degree of $\delta'^*\theta_d = g$. This proves (2).

For a line bundle L on C, we will denote by $L^{\boxtimes d}$ to be the unique line bundle on $C^{(d)}$, whose pullback under the quotient map $\pi : C^d \to C^{(d)}$ is $\bigotimes_{i=1}^d p_i^* L$. Recall that by [Pac03, §2], we have that $[x] = [\mathcal{O}(x)^{\boxtimes d}]$ for a point $x \in C$. By definition under the map $\pi : C^d \to C^{(d)}$ the pullback of $\mathcal{O}(x)^{\boxtimes d}$ is $\bigotimes_{i=1}^d p_i^*\mathcal{O}(x)$. Now $\delta : C \hookrightarrow C^{(d)}$ is the composition $C \to C^d \to C^{(d)}$

$$x \mapsto (x, .., x) \mapsto dx\,.$$

Hence we get that the pullback of $\mathcal{O}(x)^{\boxtimes d}$ to δ is $\mathcal{O}(dx)$. Therefore degree of $\delta^*[x] = d$. This proves (3).

We know δ' is the composition $C \to C^d \to C^{(d)}$

$$x \mapsto (x, x_1, .., x_{d-1}) \mapsto x + x_1 + \cdots + x_{d-1}\,.$$

Hence we get that $\delta'^*[x] = \mathcal{O}(x)$. Therefore degree of $\delta'^*[x] = 1$. This proves (4). $\square$

Lemma 2.7. *Let $g, d \geq 1$. Let $\mu_0 := \dfrac{d+g-1}{dg}$. Then*

$$\begin{aligned}[L_0] &= dg[x] - [\theta_d] \\ &= (dg - d - g + 1).[x] + [\Delta_d/2] \\ &= (\frac{1}{\mu_0} - 1)[\theta_d] + \frac{1}{\mu_0}[\Delta_d/2]\,.\end{aligned}$$

Proof. Let $[L_0] = a[\theta_d] + b[x]$. We need two equations to solve for a and b. The first equation is $\delta^*[L_0] = 0$. Recall

$$\delta' : C \xrightarrow{f} C^d \to C^{(d)}$$

where the first map is given by $x \mapsto (x, x_1, \ldots, x_d)$. Hence

$$\delta'^*[L_0] = f^*\phi^*(\otimes p_i^*\Theta)\,.$$

Now the composition

$$C \xrightarrow{f} C^d \xrightarrow{\phi} J(C)^{\binom{d}{2}}$$

is given by $x \mapsto (x - x_1, x - x_2, \ldots, x - x_{d-1}, x_i - x_j)_{i<j}$. Hence

$$\deg(\delta'^*[L_0]) = \sum_{i=1}^{d-1} \deg(\theta_1) = (d-1)g\,.$$

This will be our second equation.

We use these two equations and the preceding computations to compute a and b.

$$\begin{aligned} 0 &= \deg(\delta^*[L_0]) \\ &= a.\deg(\delta^*[\theta_d]) + b.\deg(\delta^*[x]) \\ &= ad^2g + bd\,. \end{aligned}$$

Therefore

$$b = -adg\,.$$

Now using the second equation we get

$$\begin{aligned} (d-1)g &= \deg(\delta'^*[L_0]) \\ &= a.\deg(\delta'^*[\theta_d]) + b.\deg(\delta'^*[x]) \\ &= ag + b \\ &= ag - adg = ag(1-d)\,. \end{aligned}$$

Therefore

$$a = -1, \qquad b = dg\,.$$

Hence we get $[L_0] = dg[x] - [\theta_d]$. For the other two equalities, we use the relation

$$[\theta_d] = (d+g-1)[x] - [\Delta_d/2]$$

between $[x], [\Delta_d/2]$ and $[\theta_d]$ [Pac03, Lemma 2.1]. □

3. Picard group and Neron-Severi group of $\mathcal{Q}$

Let E be a locally free sheaf over C. Throughout this section $\mathcal{Q}$ will denote the Quot scheme $\mathcal{Q}(E, d)$ which parametrizes torsion quotients of E of degree d. In this section we compute the Picard group of $\mathcal{Q}$, and the vector spaces $N^1(\mathcal{Q})$ and $N_1(\mathcal{Q})$.

Lemma 3.1. *Let S be a scheme over k. Let F be a coherent sheaf over $C \times S$ which is S-flat and for all $s \in S$, $F|_{C\times s}$ is a torsion sheaf over C of degree d. Let $p_S : C \times S \to S$ be the projection. Then*

(i) $p_{S*}(F)$ is locally free of rank d and $\forall s \in S$ the natural map $p_{S*}(F)|_s \to H^0(C, F|_{C\times s})$ is an isomorphism.

(ii) *Assume that we are given a morphism $\phi : T \to S$. We have the following diagram:*

$$\begin{array}{ccc} C \times T & \xrightarrow{id\times\phi} & C \times S \\ \downarrow{\scriptstyle p_T} & & \downarrow{\scriptstyle p_S} \\ T & \xrightarrow{\phi} & S \end{array}$$

Then the natural morphism

$$\phi^* p_{S*}(F) \to (p_T)_*(id \times \phi)^* F$$

is an isomorphism.

Proof. Since $F|_{C\times s}$ is a torsion sheaf for all $s \in S$, we have $H^1(C, F|_{C\times s}) = 0$. By [Har77, Chapter III, Theorem 12.11(a)] we get $R^1 p_{S*}(F) = 0$. Using [Har77, Chapter III, Theorem 12.11(b)] (ii) with $i = 1$ we get that the morphism $p_{S*}(F)|_s \to H^0(C, F|_{C\times s})$ is surjective. Again using the same with $i = 0$ we get that $p_{S*}(F)$ is locally free of rank d and the map $p_{S*}(F)|_s \to H^0(C, F|_{C\times s})$ is an isomorphism.

Since F is S-flat it follows that $(id \times \phi)^* F$ is T-flat. Applying the above we see $\phi^* p_{S*}(F)$ and $(p_T)_*(id \times \phi)^* F$ are locally free of rank d. For each $t \in T$ we have the commutative diagram:

$$\begin{array}{ccc} \phi^* p_{S*}(F)|_t = p_{S*}(F)|_{\phi(t)} & \longrightarrow & (p_T)_*(id \times \phi)^* F|_t \\ \downarrow & & \downarrow \\ H^0(C, F|_{C\times\phi(t)}) & = & H^0(C, (id \times \phi^*) F|_{C\times t}) \end{array}$$

By the first part we get that the vertical arrows are isomorphisms. Hence we get that the first row of the diagram is an isomorphism. Therefore

$$\phi^* p_{S*}(F) \to (p_T)_*(id \times \phi)^* F$$

is a surjective morphism of vector bundles of same rank and hence an isomorphism. □

We define a line bundle on $\mathcal{Q}$. Let us denote the projections $C \times \mathcal{Q}$ to C and $\mathcal{Q}$ by p_C and p_Q respectively. Then we have the universal quotient $p_C^* E \to \mathcal{B}_{\mathcal{Q}}$ over $C \times \mathcal{Q}$. By Lemma 3.1, $p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}})$ is a vector bundle of rank d.

Definition 3.2. *Denote the line bundle* $\det(p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}}))$ *by* $\mathcal{O}_{\mathcal{Q}}(1)$.

Denote the d-th symmetric product of C by $C^{(d)}$. Recall the Hilbert-Chow map $\Phi : \mathcal{Q} \to C^{(d)}$ which sends $[E \to B]$ to $\sum l(B_p)p$, where $l(B_p)$ is the length of the $\mathcal{O}_{C,p}$-module B_p. Therefore, we have the pullback $\Phi^* : \mathrm{Pic}(C^{(d)}) \to \mathrm{Pic}(\mathcal{Q})$ which is in fact an inclusion. To see this, recall that the fibres of Φ are projective integral varieties [GS19, Corollary 6.6] and Φ is flat [GS19, Corollary 6.3]. Hence $\Phi_*(\mathcal{O}_{\mathcal{Q}}) = \mathcal{O}_{C^{(d)}}$. Now by projection formula $\Phi_* \Phi^* L \cong L$ for all $L \in \mathrm{Pic}(C^{(d)})$ and the statement follows.

The big diagonal is the image of the map $C \times C^{(d-2)} \to C^{(d)}$ given by $(x, A) \mapsto 2x + A$. Let us denote the big diagonal in $C^{(d)}$ by Δ. Let $U_C := C^{(d)} \setminus \Delta$ and $\mathcal{U} := \Phi^{-1}(U_C)$. Then $\mathcal{U} \subset \mathcal{Q}$.

Lemma 3.3. *For any line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{Q})$, $\exists$ an unique $n \in \mathbb{Z}$ such that $(\mathcal{L} \otimes \mathcal{O}_{\mathcal{Q}}(-n))|_{\Phi^{-1}(p)} \cong \mathcal{O}_{\Phi^{-1}(p)}$ for all $p \in U_C$.*

Proof. Let $\pi : \mathbb{P}(E) \to C$ be the projective bundle associated to E and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the universal line bundle over $\mathbb{P}(E)$. Let $Z = \mathbb{P}(E)^d$. Let $p_i : Z \to \mathbb{P}(E)$ be the i-th projection. Let $\pi_d : Z \to C^d$ be the product map. The symmetric group S_d acts on Z and the map π_d is equivariant for this action. Let $\psi : C^d \to C^{(d)}$ be the quotient map. Define $U_Z := (\psi \circ \pi_d)^{-1}(U)$.

Let $c \in C$ be a closed point and let k_c denote the skyscraper sheaf supported at c. A closed point of $\mathbb{P}(E)$ which maps to $c \in C$ corresponds to a quotient $E \to E_c \to k_c$. Recall that we have a map [Gan19, Theorem 2.2(a)]

$$\tilde{\psi} : U_Z \to \mathcal{U}$$

which sends a closed point

$$(E_{c_i} \to k_{c_i})_{i=1}^d \in U_Z$$

to the quotient

$$E \to \bigoplus_i E_{c_i} \to \bigoplus_i k_{c_i} \in \mathcal{U}\,.$$

So we have a commutative diagram:

$$\begin{tikzcd} U_Z \arrow[r, "\tilde{\psi}"] \arrow[d, "\pi_d"] & \mathcal{U} \arrow[d, "\Phi"] \\ \psi^{-1}(U_C) \arrow[r, "\psi"] & U_C \end{tikzcd}$$

Moreover, if $\underline{c} = (c_1, \ldots, c_d) \in \psi^{-1}(U_C)$, then by [GS19, Lemma 6.5] $\tilde{\psi}$ induces an isomorphism

$$\prod \mathbb{P}(E_{c_i}) = \pi_d^{-1}(\underline{c}) \xrightarrow{\sim} \Phi^{-1}(\psi(\underline{c}))\,.$$

Applying Lemma 3.1 by taking $T = U_Z$, $S = \mathcal{U}$ and $\phi = \tilde{\psi}$ and the definition of the map $\tilde{\psi}$ (see the proof of [Gan19, Theorem 2.2(a)]) we see that

$$\tilde{\psi}^* \mathcal{O}_{\mathcal{Q}}(1) = \bigotimes_{i=1}^d p_i^* \mathcal{O}_{\mathbb{P}(E)}(1)|_{U_Z}\,.$$

Hence it is enough to show that $\exists\, n \in \mathbb{Z}$ such that $\forall\, \underline{c} \in \psi^{-1}(U_C)$

$$\tilde{\psi}^* \mathcal{L}|_{\pi_d^{-1}(\underline{c})} \cong \bigotimes_{i=1}^d p_i^* \mathcal{O}(n)|_{\pi_d^{-1}(\underline{c})}\,.$$

For $\underline{c} \in \psi^{-1}(U_C)$ define $n_i(\underline{c}) \in \mathbb{Z}$ using the equation

$$\tilde{\psi}^*\mathcal{L}|_{\pi_d^{-1}(\underline{c})} = \bigotimes_{i=1}^{d} p_i^*\mathcal{O}_{\mathbb{P}(E_{c_i})}(n_i(\underline{c}))\,.$$

We may view the n_i as functions $n_i : \psi^{-1}(U_C) \to \mathbb{Z}$. Since the line bundle $\tilde{\psi}^*\mathcal{L}$ is invariant under the action of the group S_d, it follows that

$$n_{\sigma(i)}(\underline{c}) = n_i(\sigma(\underline{c})). \tag{3.4}$$

Here $\sigma(\underline{c}) := (c_{\sigma(1)}, \ldots, c_{\sigma(d)})$. Hence it suffices to show that n_1 is a constant function.

Let $c_2, \ldots, c_d$ be distinct points in C. Define $V := C \setminus \{c_2, .., c_d\}$ and a map

$$i : V \hookrightarrow \psi^{-1}(U_C) \qquad\qquad i(c) := (c, c_2, .., c_d)\,.$$

Then $\pi_d^{-1}(V)$ is equal to $\mathbb{P}(E|_V) \times \mathbb{P}(E_{c_2}) \times ... \times \mathbb{P}(E_{c_d})$. The restriction of $\tilde{\psi}^*\mathcal{L}$ to $\mathbb{P}(E|_V) \times \mathbb{P}(E_{c_2}) \times ... \times \mathbb{P}(E_{c_d})$ is isomorphic to

$$\pi^*M \otimes p_1^*\mathcal{O}_{\mathbb{P}(E|_V)}(a_1) \otimes p_2^*\mathcal{O}_{\mathbb{P}(E_{c_2})}(a_2) \ldots \otimes p_d^*\mathcal{O}_{\mathbb{P}(E_{c_d})}(a_d)\,,$$

where M is a line bundle on V. Further restricting to $(c, c_2, \ldots, c_d)$ and $(c', c_2, \ldots, c_d)$, where $c, c' \in V$, we see that

$$n_i(c, c_2, .., c_d) = n_i(c', c_2, \ldots, c_d) \qquad\qquad \forall\ i\,. \tag{3.5}$$

This proves that for distinct points $c, c', c_2, \ldots, c_d \in C$ we have

$$n_i(c, c_2, .., c_d) = n_i(c', c_2, \ldots, c_d) \qquad\qquad \forall\ i\,. \tag{3.6}$$

Choose $2d$ distinct points $c_1, \ldots, c_d, c'_1, \ldots, c'_d$ in C. Then using equations (3.5) and (3.6) we get

$$\begin{aligned} n_1(c_1, c_2, \ldots, c_d) &= n_1(c'_1, c_2, \ldots, c_d) \\ &= n_2(c_2, c'_1, \ldots, c_d) \\ &= n_2(c'_2, c'_1, c_3, \ldots, c_d) \\ &= n_1(c'_1, c'_2, c_3, \ldots, c_d) \\ &= \ldots \\ &= n_1(c'_1, c'_2, \ldots, c'_d)\,. \end{aligned}$$

Finally, for any two points $\underline{c}, \underline{c}' \in \psi^{-1}(U_C)$ choose a third point $\underline{c}''$ such that the coordinates of $\underline{c}''$ are distinct from those of $\underline{c}$ and $\underline{c}'$. Then we see that $n_1(\underline{c}) = n_1(\underline{c}'') = n_1(\underline{c}')$. This proves that n_1 is the constant function. Therefore, $\psi^*\mathcal{L}|_{\pi_d^{-1}(\underline{c})}$ is of the form $\bigotimes p_i^*\mathcal{O}_{\mathbb{P}(E_{c_i})}(n)$, $\forall \underline{c} \in \psi^{-1}(U_C)$. The uniqueness of n is obvious. □

Theorem 3.7. $\mathrm{Pic}(\mathcal{Q}) = \Phi^*\mathrm{Pic}(C^{(d)}) \bigoplus \mathbb{Z}[\mathcal{O}_{\mathcal{Q}}(1)]\,.$

Proof. Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{Q})$. By [GS19, Corollary 6.3] and [GS19, Corollary 6.4] the morphism Φ is flat and fibres of Φ are integral. Then by [MR82, Lemma 2.1.2] and Lemma 3.3 we get that $\mathcal{L} \otimes \mathcal{O}_{\mathcal{Q}}(-n) = \Phi^*\mathcal{M}$ for some

$\mathcal{M} \in \mathrm{Pic}(C^{(d)})$. Hence $\mathcal{L} = \Phi^*\mathcal{M} \otimes \mathcal{O}_{\mathcal{Q}}(n)$. The uniqueness of such an expression follows from the statement on uniqueness in Lemma 3.3. □

For a projective variety X over k recall that $N^1(X)$ (respectively, $N_1(X)$) is the vector space of $\mathbb{R}$-divisors (respectively, 1-cycles) modulo numerical equivalences [Laz04, §1.4]. It is known that $N^1(X)$ and $N_1(X)$ are finite dimensional and the intersection product defines a non-degenerate pairing

$$N^1(X) \times N_1(X) \to \mathbb{R} \qquad ([\beta], [\gamma]) \mapsto [\beta] \cdot [\gamma]\,.$$

We will compute $N^1(\mathcal{Q})$ and $N_1(\mathcal{Q})$. Let $\underline{c} \in U_C \subset C^{(d)}$. As we saw in the proof of Theorem 3.7,

$$\Phi^{-1}(\underline{c}) \cong \prod \mathbb{P}(E_{c_i})\,.$$

Let $\mathbb{P}^1 \hookrightarrow \mathbb{P}(E_{c_1})$ be a line and let $v_i \in \mathbb{P}(E_{c_i})$ for $i \geq 2$. Then we have an embedding:

$$\mathbb{P}^1 \cong \mathbb{P}^1 \times v_2 \times \ldots \times v_d \hookrightarrow \mathbb{P}(E_{c_1}) \times \prod_{i\geq 2} \mathbb{P}(E|_{c_i}) = \Phi^{-1}(\underline{c}) \subset \mathcal{Q}\,. \tag{3.8}$$

Definition 3.9. *Let us denote the class of this curve in $N_1(\mathcal{Q})$ by $[l]$.*

Corollary 3.10. $N^1(\mathcal{Q}) = \Phi^* N^1(C^{(d)}) \bigoplus \mathbb{R}[\mathcal{O}_{\mathcal{Q}}(1)]$.

Proof. Since Φ is surjective, $N^1(C^{(d)}) \to N^1(\mathcal{Q})$ is an inclusion [Laz04, Example 1.4.4]. Note that $\mathcal{O}_{\mathcal{Q}}(1) \neq 0$ in $N^1(\mathcal{Q})$ since $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [l] = 1$. Hence $\mathcal{O}_{\mathcal{Q}}(1) \neq 0$ in $N^1(\mathcal{Q})$. This also shows that $\mathcal{O}_{\mathcal{Q}}(1) \notin \Phi^* N^1(C^{(d)})$.

By theorem 3.7, we know that any $N^1(\mathcal{Q})$ is generated by $\Phi^* N^1(C^{(d)})$ and $[\mathcal{O}_{\mathcal{Q}}(1)]$. The only thing left is to show that

$$\Phi^* N^1(C^{(d)}) \cap \mathbb{R}[\mathcal{O}_{\mathcal{Q}}(1)] = 0\,.$$

For $a \in \mathbb{R}$ if $a[\mathcal{O}_{\mathcal{Q}}(1)] \in N^1(C^{(d)})$, then $a[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [l] = a = 0$. Hence the result follows. □

Hence, it follows from Corollary 3.10 that

Proposition 3.11. *If $g = 1$ or C is very general with $g \geq 2$, then* $\dim_{\mathbb{R}} N^1(\mathcal{Q}) = 3$.

Proof. We already saw that $N^1(C^{(d)})$ is of dimension 2. The Proposition follows. □

To compute $N_1(\mathcal{Q})$ we first construct a section of $\Phi : \mathcal{Q} \to C^{(d)}$. Over $C \times C^{(d)}$ we have the universal divisor Σ which gives us the universal quotient $\mathcal{O}_{C\times C^{(d)}} \to \mathcal{O}_\Sigma$. Choose a surjection $E \to L$ over C, where L is a line bundle on C. This induces a surjection $E \otimes \mathcal{O}_{C\times C^{(d)}} \to L \otimes \mathcal{O}_{C\times C^{(d)}}$. Then the composition

$$E \otimes \mathcal{O}_{C\times C^{(d)}} \to L \otimes \mathcal{O}_{C\times C^{(d)}} \to L \otimes \mathcal{O}_\Sigma$$

gives us a morphism

$$\eta : C^{(d)} \to \mathcal{Q} \tag{3.12}$$

which is easily seen to be a section of Φ.

Corollary 3.13. *$N_1(\mathcal{Q}) = N_1(C^{(d)}) \oplus \mathbb{R}[l]$ where $N_1(C^{(d)}) \hookrightarrow N_1(\mathcal{Q})$ is the morphism given by the pushforward η_*.*

Proof. Since $\Phi \circ \eta = id_{C^{(d)}}$ we have that η_* is an injection. Also since $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [l] = 1$, we have $[l] \neq 0$. We claim that $[l] \notin N_1(C^{(d)})$. If not, assume that $[l] = \eta_*[\gamma]$ for $[\gamma] \in N^1(C^{(d)})$. Then for every $\beta \in N^1(C^{(d)})$ we have

$$[l] \cdot \Phi^*\beta = \Phi_*([l]) \cdot \beta = 0 = \gamma \cdot \beta\,.$$

This proves that $\gamma = 0$.

Let $\gamma \in N_1(\mathcal{Q})$. Then we claim that

$$\gamma = \eta_*\Phi_*\gamma + \Big([\mathcal{O}_{\mathcal{Q}}(1)] \cdot (\gamma - \eta_*\Phi_*\gamma)\Big)[l]\ .$$

This can be seen as follows. It is enough to show that $\forall\ D \in N^1(\mathcal{Q})$,

$$[D] \cdot \gamma = [D] \cdot (\eta_*\Phi_*\gamma) + ([\mathcal{O}_{\mathcal{Q}}(1)] \cdot \gamma)[D] \cdot [l]\,.$$

By Corollary 3.10, it is enough to consider the case when $D = \Phi^*D'$ where $D' \in N^1(C^{(d)})$ or $D = \mathcal{O}_{\mathcal{Q}}(1)$. In the first case the statement follows from projection formula and the second case is by definition. This completes the proof of the Corollary. $\square$

Let $p_C : C \times \mathcal{Q} \to \mathcal{Q}$ and $p_{\mathcal{Q}} : C \times \mathcal{Q} \to C$ be the projections. Let $\mathcal{B}_{\mathcal{Q}}$ denote the universal quotient on $C \times \mathcal{Q}$. For a vector bundle F over C, we define

$$B_{F,\mathcal{Q}} := \det(p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^*F))\,.$$

Lemma 3.14. *Suppose we are given a map $f : T \to \mathcal{Q}$. Let $(id \times f)^*\mathcal{B}_{\mathcal{Q}} = \mathcal{B}_T$. Let $p_T : C \times T \to T$ and $p_{1,T} : C \times T \to C$ be the projections.*

$$\begin{array}{ccc} C \times T & \xrightarrow{id \times f} & C \times \mathcal{Q} \\ \big\downarrow{\scriptstyle p_T} & & \big\downarrow{\scriptstyle p_{\mathcal{Q}}} \\ T & \xrightarrow{\quad f \quad} & \mathcal{Q} \end{array}$$

(i) *$f^*p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^*F) \to p_{T*}(\mathcal{B}_T \otimes p_{1,T}^*F)$ is an isomorphism.*

(ii) *For a vector bundle F on C define $B_{F,T} := \det(p_{T*}(\mathcal{B}_T \otimes p_{1,T}^*F))$. Then $f^*B_{F,\mathcal{Q}} = B_{F,T}$.*

Proof. For (i) take $\mathcal{B}_{\mathcal{Q}} \otimes p_C^*F$ and use Lemma 3.1. The assertion (ii) follows from (i) by applying determinant to the isomorphism

$$f^*p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^*F) \xrightarrow{\sim} p_{T*}(\mathcal{B}_T \otimes p_{1,T}^*F)\,.$$

$\square$

Recall the definition of η from equation (3.12), this is a section of Φ. For a line bundle L on C we have a line bundle $\mathcal{G}_{d,L}$ over $C^{(d)}$ (see [Pac03, page 8] for notation).

Lemma 3.15. *Let η be defined by a quotient $E \to M \to 0$. Then*
$$\eta^* B_{L,\mathcal{Q}} \cong \mathcal{G}_{d,L\otimes M}\,.$$

Proof. We have the diagram:

$$\begin{array}{ccc} C \times C^{(d)} & \xrightarrow{id_C \times \eta} & C \times \mathcal{Q} \\ \big\downarrow & & \big\downarrow \\ C^{(d)} & \xrightarrow{\quad\eta\quad} & \mathcal{Q} \end{array}$$

Recall that by definition of η, the pullback of the universal quotient on $C\times\mathcal{Q}$ to $C \times C^{(d)}$ via the section $(id_C \times \eta)$ is the quotient
$$E \otimes \mathcal{O}_{C\times C^{(d)}} \to L \otimes \mathcal{O}_{C\times C^{(d)}} \to L \otimes \mathcal{O}_\Sigma$$
Hence by Lemma 3.14, we have
$$\eta^* B_{L,\mathcal{Q}} \cong \mathcal{G}_{d,L\otimes M}\,.$$
□

Proposition 3.16. *For any two line bundles L, L' over C*
$$B_{L,\mathcal{Q}} \otimes B^{-1}_{L',\mathcal{Q}} = \Phi^*((L \otimes L'^{-1})^{\boxtimes d})\,.$$

Proof. First we show that $B_{L,\mathcal{Q}}\otimes B^{-1}_{L',\mathcal{Q}} \in \Phi^*\mathrm{Pic}(C^{(d)})$. Since any line bundle over $\mathcal{Q}$ is of the form $\mathcal{O}_\mathcal{Q}(a) \otimes \phi^*\mathcal{L}$, where $\mathcal{L} \in \mathrm{Pic}(C^{(d)})$, it is enough to show that both $B_{L,\mathcal{Q}}$ and $B_{L',\mathcal{Q}}$ have the same $\mathcal{O}_\mathcal{Q}(1)$-th coeffcient.

To compute the coefficient of this component of any line bundle over $\mathcal{Q}$, we can do the following. Fix d distinct points $c_1, \ldots, c_d \in C$. These define a point $\underline{c} \in C^{(d)}$. As we saw in the proof of Theorem 3.7,
$$\Phi^{-1}(\underline{c}) \cong \prod_{i=1}^{d} \mathbb{P}(E_{c_i})\,.$$
Let $v_i \in \mathbb{P}(E_{c_i})$ for $i \geq 2$. Then we have an embedding:
$$f : \mathbb{P}(E_{c_1}) \times v_2 \times \ldots \times v_d \hookrightarrow \mathbb{P}(E_{c_1}) \times \prod_{i\geq 2} \mathbb{P}(E_{c_i}) = \Phi^{-1}(\underline{c})\,.$$
Then the $\mathcal{O}_\mathcal{Q}(1)$-th coefficient of a line bundle $\mathcal{M}$ over $\mathcal{Q}$ is the degree of $f^*\mathcal{M}$ with respect to $\mathcal{O}_{\mathbb{P}(E_{c_1})}(1)$. Let $Y = \mathbb{P}(E_{c_1})$. Using Lemma 3.14, $f^*B_{L,\mathcal{Q}} = \det(p_{Y*}(\mathcal{B}_Y \otimes p^*_{1,Y}L))$.

The $v_j \in \mathbb{P}(E_{c_j})$ correspond to quotients $v_j : E \to E_{c_j} \to k_{c_j}$, for $2 \leq j \leq d$. Over $C \times Y$ we have the inclusions $i_j : Y \cong c_j \times Y \hookrightarrow C \times Y$ for every $1 \leq j \leq d$. We have a map
$$p^*_{1,Y}E \to \bigoplus_{j=1}^{d} i_{j*}(p^*_{1,Y}E|_{c_j\times Y})\,.$$
The bundle $p^*_{1,Y}E|_{c_j\times Y}$ is just the trivial bundle on Y, and using v_j we can get quotients $p^*_{1,Y}E|_{c_j\times Y} \to \mathcal{O}_Y$ for $2 \leq j \leq d$. For $j = 1$ we have the

quotient $p_{1,Y}^*E|_{c_1\times Y}\to i_{1*}(\mathcal{O}_Y(1))$. Since the $c_j\times Y$ are disjoint we can put these together to get a quotient on $C\times Y$

$$p_{1,Y}^*E\to\left(\bigoplus_{j=2}^{d}i_{j*}\mathcal{O}_Y\right)\bigoplus i_{1*}\mathcal{O}_Y(1)\,.$$

By definition, the sheaf $\mathcal{B}_Y$ is the sheaf in the RHS. Then

$$\begin{aligned}\mathcal{B}_Y\otimes p_{1,Y}^*L=&\left(\bigoplus_{j=2}^{d}i_{j*}\mathcal{O}_Y\right)\otimes p_{1,Y}^*L\;\bigoplus\; i_{1*}\mathcal{O}_Y(1)\otimes p_{1,Y}^*L\\ =&\left(\bigoplus_{j=2}^{d}i_{j*}\mathcal{O}_Y\right)\bigoplus i_{1*}\mathcal{O}_Y(1)\\ =&\mathcal{B}_Y\ .\end{aligned}$$

Thus, using the remark in the preceding para, we get that the $\mathcal{O}_{\mathcal{Q}}(1)$-th coefficient of $B_{L,\mathcal{Q}}$ is the same as that of $B_{L',\mathcal{Q}}$. Hence $B_{L,\mathcal{Q}}\otimes B_{L',\mathcal{Q}}^{-1}=\Phi^*\mathcal{L}$.

Recall the section η of Φ from equation (3.12), constructed using some line bundle quotient $E\to M$. Then $\eta^*(B_{L,\mathcal{Q}}\otimes B_{L',\mathcal{Q}}^{-1})=s^*\Phi^*\mathcal{L}=\mathcal{L}$. Now using Lemma 3.15, we get that $\eta^*B_{L,\mathcal{Q}}=\mathcal{G}_{d,L\otimes M}$.

By Göttsche's theorem ([Pac03, page 9]) we get that $\eta^*B_{L,\mathcal{Q}}=\mathcal{G}_{d,L\otimes M}=(L\otimes M)^{\boxtimes d}\otimes\mathcal{O}(-\Delta_d/2)$. Therefore, we get

$$\mathcal{L}=\eta^*(B_{L,\mathcal{Q}}\otimes B_{L',\mathcal{Q}}^{-1})=(L\otimes L'^{-1})^{\boxtimes d}\,.$$

This completes the proof of the Proposition. □

Corollary 3.17. $[B_{L,\mathcal{Q}}]=[\mathcal{O}_{\mathcal{Q}}(1)]+\deg(L)[x]$ *in* $N^1(\mathcal{Q})$.

4. Upper bound on NEF cone

Let V be a vector space of dimension n. From now, unless mentioned otherwise, the notation $\mathcal{Q}$ will be reserved for the space $\mathcal{Q}(V\otimes\mathcal{O}_C,d)$. Sometimes we will also denote this space by $\mathcal{Q}(n,d)$ when we want to emphasize n and d.

Notation. For the rest of this article, except in section 6, the genus of the curve C will be $g(C)\geq 1$. If $g(C)\geq 2$ then we will also assume that C is very general.

Our aim is to compute the NEF cone of $\mathcal{Q}$. Since this cone is dual to the cone of effective curves, it follows that if we take effective curves $C_1,C_2,\dots,C_r$, take the cone generated by these in $N_1(\mathcal{Q})$, and take the dual cone T in $N^1(\mathcal{Q})$, then $\mathrm{Nef}(\mathcal{Q})$ is contained in T. This gives us an upper bound on $\mathrm{Nef}(\mathcal{Q})$. We already know two curves in $\mathcal{Q}$. The first being a line in the fiber of $\Phi:\mathcal{Q}\to C^{(d)}$, see Definition 3.9, which was denoted $[l]$. Recall the section η of Φ from equation (3.12), taking L to be the trivial bundle. The second curve is $\eta_*([l'])$, where $[l']$ is from Definition 2.2. Now we will construct a third curve in $\mathcal{Q}$.

Define a morphism

$$\tilde{\delta} : C \to \mathcal{Q} \tag{4.1}$$

as follows. Let $p_1, p_2 : C \times C \to C$ be the first and second projections respectively. Let $i : C \to C \times C$ be the diagonal. Fix a surjection $k^n \to k^d$ of vector spaces. Then define the quotient over $C \times C$

$$\mathcal{O}^n_{C\times C} \to \mathcal{O}^d_{C\times C} \to i_* i^* \mathcal{O}^d_{C\times C} \,.$$

This induces a morphism $\tilde{\delta} : C \to \mathcal{Q}$ which sends $c \mapsto [\mathcal{O}^n_C \to k^d_c \to 0]$. We will abuse notation and denote the class $[\tilde{\delta}_*(C)] \in N_1(\mathcal{Q})$ by $[\tilde{\delta}]$.

We now give an upper bound for the NEF cone when $n \geq d \geq \mathrm{gon}(C)$.

Proposition 4.2. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Assume $n \geq d \geq \mathrm{gon}(C)$. Let $\mu_0 := \dfrac{d+g-1}{dg}$. Then*

$$\mathrm{Nef}(\mathcal{Q}) \subset \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0] \,.$$

Proof. We claim that the cone dual to $\langle [l], \eta_*([l']), [\tilde{\delta}] \rangle$ is precisely

$$\langle ([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]), [L_0], [\theta_d] \rangle \,.$$

We have the following equalities:

(1) $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [l] = 1$. This is clear.

(2) $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot \eta_*[l'] = 0$. By projection formula and Lemma 3.15, we get that

$$([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [\eta_* l'] = ([-\Delta_d/2] + \mu_0[L_0]) \cdot [l'] \,.$$

By Lemma 2.7 we get that $[-\Delta_d/2] + \mu_0[L_0] = (1 - \mu_0)[\theta_d]$. But as we saw earlier, $[\theta_d] \cdot [l'] = 0$.

(3) $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [\tilde{\delta}] = 0$. By Lemma 3.1, it is easy to see that $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [\tilde{\delta}] = 0$. By projection formula, we get

$$([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [\tilde{\delta}] = [\mu_0 L_0] \cdot [\Phi_* \tilde{\delta}] = [\mu_0 L_0] \cdot [\delta] = 0 \,.$$

(4) $[\theta_d] \cdot [l] = [L_0] \cdot [l] = 0$ follows using the projection formula.

Now the claim follows from Proposition 2.5. As explained before, since $\mathrm{Nef}(\mathcal{Q})$ is contained in the dual to the cone $\langle [l], \eta_*([l']), [\tilde{\delta}] \rangle$, the proposition follows. □

When the genus $g = 1$, we have the following improvement of Proposition 4.2.

Proposition 4.3. *Let C be a smooth projective curve of genus $g = 1$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, d)$. Assume $d \geq \mathrm{gon}(C) = 2$. Then*

$$\mathrm{Nef}(\mathcal{Q}) \subset \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + [L_0]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0] \,.$$

Proof. We claim that the cone dual to $\langle [l], \eta_*([l']), \eta_*[\delta] \rangle$ is precisely

$$\langle ([\mathcal{O}_{\mathcal{Q}}(1)] + [L_0]), [L_0], [\theta_d] \rangle \,.$$

Let us check that $[([\mathcal{O}_{\mathcal{Q}}(1)] + [L_0])] \cdot \eta_*[\delta] = 0$. Since $[L_0] \cdot [\delta] = 0$ it is clear that it suffices to check that $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot \eta_*[\delta] = 0$. Applying the definition of the map $\eta \circ \delta : C \to \mathcal{Q}$ we see that $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot \eta_*[\delta] = \deg(p_{2*}(\mathcal{O}/\mathcal{I}^d))$, where $\mathcal{I}$ is the ideal sheaf of the diagonal in $E \times E$. Since $\mathcal{I}/\mathcal{I}^2$ is trivial and $\mathcal{I}^j/\mathcal{I}^{j+1} = (\mathcal{I}/\mathcal{I}^2)^{\otimes j}$, it follows that $\deg(p_{2*}(\mathcal{O}/\mathcal{I}^d)) = 0$. The rest of the proof is the same as that of Proposition 4.2. □

5. Lower bound on NEF cone

In this section we obtain a lower bound for $\mathrm{Nef}(\mathcal{Q})$ ($\mathcal{Q} = \mathcal{Q}(n,d)$).

Lemma 5.1. *Let $f : D \to \mathcal{Q}$ be a morphism, where D is a smooth projective curve. Fix a point $q \in f(D)$ and an effective divisor A on C containing the scheme theoretic support of $\mathcal{B}_q$. If there is a line bundle L on C such that $H^0(L) \to H^0(L|_A)$ is surjective then $[B_{L,\mathcal{Q}}] \cdot [D] \geq 0$.*

Proof. Consider the map

$$p_{\mathcal{Q}*}(p_C^*(V \otimes \mathcal{O}_C) \otimes p_C^* L) \to p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^* L)$$

on $\mathcal{Q}$. We claim that this map is surjective at the point q. In view of Lemma 3.1 when we restrict this map to q, it becomes equal to the map

$$H^0(V \otimes L) \to H^0(\mathcal{B}_q \otimes L)\,.$$

The map $V \otimes L \to \mathcal{B}_q \otimes L$ on C factors as

$$V \otimes L \to V \otimes L|_A \to \mathcal{B}_q \otimes L\,.$$

Taking global sections we see that the map $H^0(V \otimes L) \to H^0(\mathcal{B}_q \otimes L)$ factors as

$$H^0(V \otimes L) \to H^0(V \otimes L|_A) \to H^0(\mathcal{B}_q \otimes L)\,.$$

The second arrow is surjective since these are coherent sheaves on a zero dimensional scheme. The first arrow is simply

$$V \otimes H^0(L) \to V \otimes H^0(L|_A)\,.$$

Since $H^0(L) \to H^0(L|_A)$ is surjective by our choice of L, it follows that $H^0(V \otimes L) \to H^0(\mathcal{B}_q \otimes L)$ is surjective, and so it follows that $p_{\mathcal{Q}*}(V \otimes p_C^* L) \to p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^* L)$ is surjective at the point q.

The rank of the vector bundle $p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^* L)$ on $\mathcal{Q}$ is d. Taking the dth exterior of $p_{\mathcal{Q}*}(V \otimes p_C^* L) \to p_{\mathcal{Q}*}(\mathcal{B}_{\mathcal{Q}} \otimes p_C^* L)$ we get a map

$$\bigwedge^d (V \otimes H^0(L)) \to B_{L,\mathcal{Q}}\,.$$

This map is nonzero and that can be seen by looking at the restriction to the point q. This shows that there is a global section of $B_{L,\mathcal{Q}}$ whose restriction to q does not vanish. It follows that $[B_{L,\mathcal{Q}}] \cdot [D] \geq 0$. This completes the proof of the lemma. □

Lemma 5.2. *Let A be an effective divisor on C of degree d. Then there is a line bundle L of degree $d+g-1$ such that the natural map*

$$H^0(L) \to H^0(L|_A)$$

is surjective.

Proof. It suffices to find a line bundle of degree $d+g-1$ such that $H^1(L \otimes \mathcal{O}_C(-A)) = 0$. By Serre duality this is same as saying that $H^0(L^\vee \otimes K_C \otimes \mathcal{O}_C(A)) = 0$. The degree of $L^\vee \otimes K_C \otimes \mathcal{O}_C(A)$ is $g-1$. Thus, fixing A we may choose a general L such that $L^\vee \otimes K_C \otimes \mathcal{O}_C(A)$ line bundle has no global sections. □

Definition 5.3. *Define $U \subset \mathcal{Q}$ to be the set of quotients of the form*

$$\mathcal{O}_C^n \to \frac{\mathcal{O}_C}{\prod_{i=1}^r \mathfrak{m}_{C,c_i}^{d_i}} \cong \bigoplus \frac{\mathcal{O}_{C,c_i}}{\mathfrak{m}_{C,c_i}^{d_i}} \qquad c_i \neq c_j\,.$$

We now prove a lemma, which is implicitly contained [GS19, Section 5]. Let $\Sigma \subset C \times C^{(d)}$ denote the closed sub-scheme which is the universal divisor. In the following Lemma we work more generally with $\mathcal{Q}(E,d)$.

Lemma 5.4. *Let E be a locally free sheaf of rank r on C. Let $\mathcal{Q} = \mathcal{Q}(E,d)$ denote the Quot scheme of torsion quotients of length d. The universal quotient $\mathcal{B}_{\mathcal{Q}}$ is supported on $\Phi^*\Sigma \subset C \times \mathcal{Q}$. The set U is open in $\mathcal{Q}$. On $C \times U$ the sheaf $\mathcal{B}_{\mathcal{Q}}$ is a line bundle supported on the scheme $\Phi^*\Sigma \cap (C \times U)$.*

Proof. Let A denote the kernel of the universal quotient on $C \times \mathcal{Q}$

$$0 \to A \xrightarrow{h} p_C^*E \to \mathcal{B}_{\mathcal{Q}} \to 0\,.$$

The map Φ is defined taking the determinant of h, that is, using the quotient

$$0 \to \det(A) \xrightarrow{\det(h)} p_C^*\det(E) \to \mathcal{F} \to 0\,.$$

If $\mathcal{I}_\Sigma$ denotes the ideal sheaf of Σ then this shows that

$$\Phi^*\mathcal{I}_\Sigma = \det(A) \otimes p_C^*\det(E)^{-1}\,.$$

Let $0 \to E' \xrightarrow{h} E$ be locally free sheaves of the same rank on a scheme Y. Let $\mathcal{I}$ denote the ideal sheaf determined by $\det(h)$. Then it is easy to see that $\mathcal{I}E \subset h(E') \subset E$. Applying this we get that $(\Phi^*\mathcal{I}_\Sigma)p_C^*E \subset A$. This proves that $\mathcal{B}$ is supported on $\Phi^*\Sigma$. Let us denote by $Z := \Phi^*\Sigma \subset C \times \mathcal{Q}$. Consider the closed subset $Z_2 \subset Z$ defined as follows

$$Z_2 := \{z = (c,q) \in Z \mid \mathrm{rank}_k(\mathcal{B}_{\mathcal{Q}} \otimes k(z)) \geq 2\}\,.$$

Then the image of Z_2 in $\mathcal{Q}$ is closed and U is precisely the complement of Z_2. This proves that U is open in $\mathcal{Q}$.

Let R be a local ring with maximal ideal $\mathfrak{m}$ and let $R \to S$ be a finite map. Let M be a finite S module, which is flat over R and such that $M/\mathfrak{m}M \cong S/\mathfrak{m}S$. Then it follows easily that $M \cong S$.

Let $q \in U \subset \mathcal{Q}$ be a point. The sheaf $\mathcal{B}_{\mathcal{Q}}$ is a coherent sheaf supported on Z, the map $Z \to \mathcal{Q}$ is finite, the fiber

$$\mathcal{B}_q = \bigoplus \frac{\mathcal{O}_{C,c_i}}{\mathfrak{m}_{C,c_i}^{d_i}} \cong \mathcal{O}_\Sigma|_q \cong \mathcal{O}_Z|_q \,.$$

From the preceding remark it follows that $\mathcal{B}_{\mathcal{Q}}$ is a line bundle over $Z \cap (C \times U)$. □

Lemma 5.5. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Let D be a smooth projective curve and let $D \to \mathcal{Q}$ be a morphism such that its image intersects U. Then $([\mathcal{O}_{\mathcal{Q}}(1)] + [\Delta_d/2]) \cdot [D] \geq 0$.*

Proof. Denote by $\mathcal{B}_D$ the pullback of the universal quotient over $C \times \mathcal{Q}$ to $C \times D$. Denote by $i_D : \Gamma \hookrightarrow C \times D$ the pullback of the universal subscheme $\Sigma \hookrightarrow C \times C^{(d)}$ to $C \times D$. Then $\mathcal{B}_D$ is supported on Γ.

Let Γ_i be the irreducible components of Γ. Since $\Gamma \to D$ is flat each Γ_i dominates D. Let $f : \Gamma \to D$ denote the projection. There is an open subset $U_1 \subset D$ such that

$$f^{-1}(U_1) = \bigsqcup_i \Gamma_i \cap f^{-1}(U_1)$$

and $\mathcal{B}_D$ restricted to $f^{-1}(U_1)$ is a line bundle. Note that by $\Gamma_i \cap f^{-1}(U_1)$ we mean this open sub-scheme of Γ. Fix a closed point $x_i \in \Gamma_i \cap f^{-1}(U_1)$. Consider the quotient

$$V \otimes \mathcal{O}_{C \times D} \to \mathcal{B}_D$$

and restrict it to the point x_i. We get a quotient

$$V \to \mathcal{B}_D \otimes k(x_i) \to 0 \,.$$

If we pick a general line in V, then it surjects onto $\mathcal{B}_D \otimes k(x_i)$. Thus, for the general element $s \in V$, $s \otimes \mathcal{O}_{C \times D}$ surjects onto $\mathcal{B}_D \otimes k(x_i)$. This map factors through $\mathcal{O}_\Gamma$, and we get an exact sequence

$$0 \to \mathcal{O}_\Gamma \to \mathcal{B}_D \to F \to 0$$

where F is supported on a 0 dimensional scheme. Then we have

$$0 \to f_*\mathcal{O}_\Gamma \to f_*\mathcal{B}_D \to f_*F \to 0 \,.$$

Since f_*F is again supported on finitely many points, hence we have

$$\deg(f_*\mathcal{B}_D) - \deg(f_*\mathcal{O}_\Gamma) \geq 0$$

By Lemma 3.1, $\deg(f_*\mathcal{B}_D) = [\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D]$ and by [Pac03, §3] we have

$$\deg(f_*\mathcal{O}_\Gamma) = [\mathcal{O}(-\Delta_d/2)] \cdot [D] \,.$$

Hence the result follows. □

Corollary 5.6. *If the image of $f : D \to \mathcal{Q}$ interects U, then $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0]) \cdot [D] \geq 0$.*

Proof. If its image interects U, then by Lemma 5.5,

$$([\mathcal{O}_{\mathcal{Q}}(1)]+[\Delta_d/2])\cdot[D]\geq 0\,.$$

By Lemma 2.7,

$$[\Delta_d/2]=\mu_0[L_0]-(1-\mu_0)[\theta_d]\,.$$

Since θ_d is nef, we have that

$$([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0])\cdot[D]\geq 0\,.$$

□

Lemma 5.7. *Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,d)$. Let D be a smooth projective curve and let $f:D\to(\mathcal{Q}\setminus U)\subset\mathcal{Q}$ be a morphism. Then $([\mathcal{O}_{\mathcal{Q}}(1)]+(d+g-2)[x])\cdot[D]\geq 0$.*

Proof. Fix a point $q\in f(D)$. Let A be the scheme theoretic support of the quotient $\mathcal{B}_q$ on C. Let $\deg(A)=d'$. Since $q\notin\mathcal{U}$, we have $d'<d$. By Lemma 5.2 we have a line bundle L of degree $d'+g-1$ such that $H^0(L)\to H^0(L|_A)$ is surjective. By Lemma 5.1 and Corollary 3.17 we get that $[B_{L,\mathcal{Q}}]\cdot[D]=([\mathcal{O}_{\mathcal{Q}}(1)]+(d'+g-1)[x])\cdot[D]\geq 0$. Since $[x]$ is nef on $\mathcal{Q}$ and $d'\leq d-1$ we get that $([\mathcal{O}_{\mathcal{Q}}(1)]+(d+g-2)[x])\cdot[D]\geq 0$. □

Proposition 5.8. *Consider the Quot scheme $\mathcal{Q}=\mathcal{Q}(n,d)$. Let $\mu_0=\dfrac{d+g-1}{dg}$. Then the class $\kappa_1:=[\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0]+\dfrac{d+g-2}{dg}[\theta_d]$ is nef.*

Proof. Let $D\to\mathcal{Q}$ is a morphism, where D is a smooth projective curve. If the image of this morphism intersects U then by Lemma 5.5 we have $([\mathcal{O}_{\mathcal{Q}}(1)]+[\Delta_d/2])\cdot[D]\geq 0$. By Lemma 2.7 we have $[\Delta_d/2]=\mu_0[L_0]-(1-\mu_0)[\theta_d]$. Hence we get

$$([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0])\cdot[D]\geq(1-\mu_0)[\theta_d]\cdot[D]\geq 0\,.$$

Since $[\theta_d]$ is nef, we get

$$([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0])\cdot[D]+\frac{d+g-2}{dg}[\theta_d]\cdot[D]\geq 0\,.$$

Now assume $D\to\mathcal{Q}$ does not intersect U. Then by Lemma 5.7 we get

$$([\mathcal{O}_{\mathcal{Q}}(1)]+(d+g-2)[x])\cdot[D]\geq 0\,.$$

By Lemma 2.7 we have $[x]=\dfrac{1}{dg}[L_0]+\dfrac{1}{dg}[\theta_d]$. Therefore

$$\begin{aligned}(d+g-2)[x]=&\frac{d+g-2}{dg}[L_0]+\frac{d+g-2}{dg}[\theta_d]\\=&\mu_0[L_0]-\frac{1}{dg}[L_0]+\frac{d+g-2}{dg}[\theta_d]\,.\end{aligned}$$

Since L_0 is nef we get that

$$([\mathcal{O}_{\mathcal{Q}}(1)]+\mu_0[L_0]+\frac{d+g-2}{dg}[\theta_d])\cdot[D]\geq 0\,.$$

□

Lemma 5.9. *Let L be a line bundle on C of degree $d+g-1$. If $d \geq \mathrm{gon}(C)$ then the line bundle $B_{L,\mathcal{Q}}$ is not ample. Moreover, for any $t \in [0,1]$ the class $t[B_{L,\mathcal{Q}}] + (1-t)[\theta_d]$ is nef but not ample.*

Proof. We saw in the last para of the proof of Proposition 3.16 that $\eta^* B_{L,\mathcal{Q}} = L^{\boxtimes d} \otimes \mathcal{O}(-\Delta_d/2)$. Its class in the nef cone is $(d+g-1)[x] - [\Delta_d/2]$. It follows from Lemma 2.7 that this is equal to $[\theta_d]$. Since $d \geq \mathrm{gon}(C)$ we have θ_d is not ample on $C^{(d)}$. That $t[B_{L,\mathcal{Q}}] + (1-t)[\theta_d]$ is nef is clear since both $[B_{L,\mathcal{Q}}]$ and $[\theta_d]$ are nef. This is not ample since η^* of this class is $[\theta_d]$ on $C^{(d)}$, which is not ample. $\square$

Proposition 5.10. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + (d+g-1)[x] \in N^1(\mathcal{Q})$ is nef.*

Proof. It is easily checked that the class $[\mathcal{O}_{\mathcal{Q}}(1)] + (d+g-1)[x]$ can be written as a positive linear combination of $[\theta_d]$ and the class in Proposition 5.8. $\square$

We may slightly improve Proposition 5.10 in a special case using the results in [Pac03]. For this we first recall the main results in [Pac03, §4]. Let C be a very general curve of genus $g(C) = 2k$. Since the gonality is given by $\lfloor \frac{g+3}{2} \rfloor$, in this case it is $k+1$. Let L'_i denote the finitely many g^1_{k+1}'s on C and define $L_i = K_C - L'_i$. Then $\deg(L_i) = 3(k-1)$. It is proved in [Pac03, Proposition 3.6, Theorem 4.1] that $\mathcal{G}_{k,L_i}$ is nef but not ample.

Proposition 5.11. *Let C be a very general curve of genus $g(C) = 2k$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,k)$. The line bundle $B_{L,\mathcal{Q}}$ is nef when $\deg(L) \geq 3(k-1)$. When $\deg(L) = 3(k-1)$ the class $t[B_{L,\mathcal{Q}}] + (1-t)[\mathcal{G}_{k,L}]$ is nef but not ample for any $t \in [0,1]$.*

We remark that this is an improvement since Proposition 5.10 only shows that $B_{L,\mathcal{Q}}$ is nef when $\deg(L) \geq 3k-1$.

Proof. It follows from Proposition 3.16 that the class of $B_{L,\mathcal{Q}}$ in $N^1(\mathcal{Q})$ is $[\mathcal{O}_{\mathcal{Q}}(1)] + \deg(L)[x]$, since $B_{\mathcal{O}_C,\mathcal{Q}} = \mathcal{O}_{\mathcal{Q}}(1)$. Notice that this class only depends on the degree of L. Since the sum of nef line bundles is nef, it suffices to show that $[B_{L,\mathcal{Q}}] = [\mathcal{O}_{\mathcal{Q}}(1)] + \deg(L)[x]$ is nef when $\deg(L) = 3(k-1)$.

The set $V(\sigma_{L_i})$ is defined in equation [Pac03, equation (18)]. Then (A) in [Pac03, Theorem 4.1] says that for every $A \in C^{(k)}$ there is an L_i such that $H^0(C, L_i) \to H^0(C, L_i|_A)$ is surjective.

Let $f: D \to \mathcal{Q}$ be morphism, where D is a smooth projective curve. Fix a point $q \in f(D)$. Let A be the divisor corresponding to $\Phi(q)$, then A is an effective divisor of degree k. For this A, choose a line bundle L_i such that

$$H^0(C, L_i) \to H^0(C, L_i|_A)$$

is surjective. The scheme theoretic support of $\mathcal{B}_q$ is contained in A. It follows from Lemma 5.1 that

$$f^* B_{L_i,\mathcal{Q}} = f^*([\mathcal{O}_{\mathcal{Q}}(1)] + 3(k-1)[x]) \geq 0\,.$$

It follows that $B_{L,\mathcal{Q}}$ is nef.

Note that

$$\begin{aligned}\eta^* B_{L,\mathcal{Q}} &= \eta^*[\mathcal{O}_{\mathcal{Q}}(1)] + \deg(L)\eta^*[x] \\ &= [\mathcal{O}(-\Delta_k/2)] + 3(k-1)[x] \\ &= [\mathcal{G}_{k,L}]\,.\end{aligned}$$

Thus, when $t \in [0,1]$ the pullback along η of $t[B_{L,\mathcal{Q}}] + (1-t)[\mathcal{G}_{k,L}]$ is $[\mathcal{G}_{k,L}]$, which is not ample. □

6. The genus 0 case

Throughout this section we will work with $C = \mathbb{P}^1$. Let us first compute the nef cone of $\mathcal{Q}(n,d)$.

Note that we have $C^{(d)} \cong \mathbb{P}^d$. Hence $N^1(C^{(d)}) = \mathbb{R}[\mathcal{O}_{\mathbb{P}^d}(1)]$. By Corollary 3.10 it follows that $N^1(\mathcal{Q})$ is two dimensional. Hence, it suffices to find a line bundle on $\mathcal{Q}$ which is different from the pullback of $\mathcal{O}_{\mathbb{P}^d}(1)$ and which is nef but not ample. The following result is proved in [Str87, Theorem 6.2], but we include it for the benefit of the reader.

Proposition 6.1.

$$\begin{aligned}\mathrm{Nef}(\mathcal{Q}(n,d)) =& \mathbb{R}_{\geq 0}[B_{\mathcal{O}(d-1),\mathcal{Q}}] + \mathbb{R}_{\geq 0}[\mathcal{O}_{\mathbb{P}^d}(1)] \\ =& \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + (d-1)[\mathcal{O}_{\mathbb{P}^d}(1)]) + \mathbb{R}_{\geq 0}[\mathcal{O}_{\mathbb{P}^d}(1)]\,.\end{aligned}$$

Proof. Let $W := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. There is a natural isomorphism $\mathbb{P}W^* \xrightarrow{\sim} C^{(d)}$. The universal sub-scheme $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}W^*$ is given by the tautological section

$$p_2^*\mathcal{O}_{\mathbb{P}W^*}(-1) \to p_2^*W = p_1^*W \to p_1^*\mathcal{O}_{\mathbb{P}^1}(d)\,.$$

By Lemma 5.1 and Lemma 5.2 we get that $B_{\mathcal{O}(d-1),\mathcal{Q}}$ is nef. To show $B_{\mathcal{O}(d-1),\mathcal{Q}}$ is not ample, consider a section $\eta : C^{(d)} \to \mathcal{Q}$ constructed as in (3.12) with L the trivial bundle. Let p_i denote the two projections from $\mathbb{P}^1 \times \mathbb{P}W^*$. By definition and Lemma 3.14 it follows that $\eta^* B_{\mathcal{O}(d-1),\mathcal{Q}} = \det(p_{2*}(\mathcal{O}_\Sigma \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(d-1)))$. Tensoring the exact sequence

$$0 \to p_1^*\mathcal{O}_{\mathbb{P}^1}(-d) \otimes p_2^*\mathcal{O}_{\mathbb{P}W^*}(-1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}W^*} \to \mathcal{O}_\Sigma \to 0$$

with $p_1^*\mathcal{O}_{\mathbb{P}^1}(d-1)$ and applying p_{2*} it easily follows that $p_{2*}(\mathcal{O}_\Sigma \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(d-1))$ is the trivial bundle and so $\eta^* B_{\mathcal{O}(d-1),\mathcal{Q}}$ is trivial. This proves that $B_{\mathcal{O}(d-1),\mathcal{Q}}$ is nef but not ample.

By restricting to a fiber of Φ and using Corollary 3.17 we see that $[B_{\mathcal{O}(d-1),\mathcal{Q}}]$ is linearly independent from $[\mathcal{O}_{\mathbb{P}^d}(1)]$. This completes the proof of the first equality. The second equality will follow from the first equality once we show that

$$[B_{\mathcal{O}(d-1),\mathcal{Q}}] = [\mathcal{O}_{\mathcal{Q}}(1)] + (d-1)[\mathcal{O}_{\mathcal{P}^d}(1)]\,.$$

By Corollary 3.17, we have that $[B_{\mathcal{O}(d-1),\mathcal{Q}}] = [\mathcal{O}_{\mathcal{Q}}(1)] + (d-1)[x]$. Now recall that given $x \in \mathbb{P}^1$, $[x]$ is the class of the divisor in $C^{(d)}$ whose underlying set consists of effective divisors of degree d containing x (see (2.4)). Hence, $[x]$ is the class of the hyperplane section

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d) \otimes \mathcal{O}(-x)))^*) \subset \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d))^*) = C^{(d)} .$$

Therefore $[x] = [\mathcal{O}_{\mathbb{P}^1}(1)]$ and this completes the proof of the second equality. $\square$

Theorem 6.2. *Let $C = \mathbb{P}^1$. Let $E = \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ with $a_i \leq a_j$ for $i < j$. Let $d \geq 1$. Let $L = \mathcal{O}(-a_1 + d - 1)$. Then*

$$\begin{aligned} \mathrm{Nef}(\mathcal{Q}(E,d)) =& \mathbb{R}_{\geq 0}[B_{L,\mathcal{Q}(E,d)}] + \mathbb{R}_{\geq 0}[\mathcal{O}_{\mathbb{P}^d}(1)] \\ =& \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}(E,d)}(1)] + (-a_1 + d - 1)[\mathcal{O}_{\mathbb{P}^d}(1)]) + \mathbb{R}_{\geq 0}[\mathcal{O}_{\mathbb{P}^d}(1)] . \end{aligned}$$

Proof. By Corollary 3.10 we get that $N^1(\mathcal{Q}(E,d))$ is 2-dimensional. Hence it is enough to give two line bundles which are nef but not ample. Clearly $\Phi^*_{\mathcal{Q}(E,d)}\mathcal{O}_{\mathbb{P}^d}(1)$ is nef but not ample. So it is enough to show that $B_{L,\mathcal{Q}(E,d)}$ is nef but not ample.

Since $a_j - a_1 \geq 0 \ \forall \ j \geq 1$, we get that $E(-a_1)$ is globally generated. Let $V := H^0(C, E(-a_1))$ and let $\dim V = n$. Then we have a surjection $V \otimes \mathcal{O}_C \to E(-a_1)$. Then gives us a surjection

$$V \otimes \mathcal{O}_C \to p_C^* E(-a_1) \to \mathcal{B}_{\mathcal{Q}(E,d)} \otimes p_C^* \mathcal{O}_C(-a_1) \to 0 .$$

This defines a map $f : \mathcal{Q}(E,d) \to \mathcal{Q}(n,d)$. By Lemma 3.14 we get that

$$f^* B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)} = B_{L,\mathcal{Q}(E,d)} = \det(p_{\mathcal{Q}(E,d)*}(\mathcal{B}_{\mathcal{Q}(E,d)} \otimes p_C^* L)) .$$

Since $B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}$ is nef we get that $B_{L,\mathcal{Q}(E,d)}$ is nef. We next show that the $B_{L,\mathcal{Q}(E,d)}$ is not ample. Consider the section $\eta_{\mathcal{Q}(E,d)}$ of $\Phi_{\mathcal{Q}(E,d)} : \mathcal{Q}(E,d) \to C^{(d)}$ defined by the quotient $p_C^* E \to p_C^* \mathcal{O}(a_1) \otimes \mathcal{O}_\Sigma$ on $C \times C^{(d)}$ (see (3.12)). Then $f \circ \eta_{\mathcal{Q}(E,d)}$ is a section of $\Phi : \mathcal{Q}(n,d) \to C^{(d)}$ defined by a quotient $\mathcal{O}_C^n \to \mathcal{O}_\Sigma \to 0$ on $C \times C^{(d)}$. Therefore $\eta^*_{\mathcal{Q}(E,d)} B_{L,\mathcal{Q}(E,d)} = \eta^* B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}$. As $\eta^* B_{\mathcal{O}(d-1),\mathcal{Q}(n,d)}$ is not ample, we get that $B_{L,\mathcal{Q}(E,d)}$ is not ample. The second equality follows again from the fact that $[x] = [\mathcal{O}_{\mathbb{P}^d}(1)]$. $\square$

7. Some cases of equality

Now we are back to the assumption that the genus of the curve satisfies $g(C) \geq 1$ and if $g(C) \geq 2$ then we also assume that C is very general.

Definition 7.1. *Let $U' \subset \mathcal{Q}$ be the open set consisting of quotients $\mathcal{O}_C^n \to B \to 0$ such that the induced map $H^0(C, \mathcal{O}_C^n) \to H^0(C, B)$ is surjective.*

Lemma 7.2. *Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n,d)$. Let D be a smooth projective curve and let $D \to \mathcal{Q}$ be a morphism such that its image intersects U'. Then $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] \geq 0$.*

Proof. We continue with the notations of Lemma 5.5. Let $p_D : C \times D \to D$ be the projection. Then applying $(p_D)_*$ to the quotient $\mathcal{O}^n_{C\times\mathcal{Q}} \to \mathcal{B}_D$ we get that the morphism

$$(p_D)_*\mathcal{O}^n_{C\times D} = \mathcal{O}^n_D \to (p_D)_*\mathcal{B}_D$$

is generically surjective by our assumption and Lemma 3.1. Hence we get that

$$[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D] = \deg((p_D)_*\mathcal{B}_D) \geq 0\,.$$

□

One extremal ray in $\mathrm{Nef}(C^{(2)})$ is given by L_0. Let other extremal ray of $\mathrm{Nef}(C^{(2)})$ be given by

$$\alpha_t = (t+1)x - \Delta_2/2\,, \tag{7.3}$$

(see [Laz04, page 75]). Then using Lemma 2.7, we get that

$$\Delta_2/2 = \frac{t+1}{g+t}L_0 - \frac{g-1}{g+t}\alpha_t\,. \tag{7.4}$$

Theorem 7.5. *Let $d = 2$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 2)$. Then*

$$\mathrm{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0]) + \mathbb{R}_{\geq 0}[L_0] + \mathbb{R}_{\geq 0}[\alpha_t]\,.$$

Proof. We first prove that $[\mathcal{O}_{\mathcal{Q}}(1)]+\frac{t+1}{g+t}[L_0]$ is nef. Since $d = 2$, then there are only three types of quotients:

(1) $\mathcal{O}^n_C \to \frac{\mathcal{O}_{C,c_1}}{\mathfrak{m}_{C,c_1}} \oplus \frac{\mathcal{O}_{C,c_2}}{\mathfrak{m}_{C,c_2}}$ with $c_1 \neq c_2$,
(2) $\mathcal{O}^n_C \to \frac{\mathcal{O}_{C,c_1}}{\mathfrak{m}^2_{C,c_1}}$,
(3) $\mathcal{O}^n_C \to \frac{\mathcal{O}_{C,c}}{\mathfrak{m}_{C,c}} \oplus \frac{\mathcal{O}_{C,c}}{\mathfrak{m}_{C,c}}$.

The first two quotients are in U while the third one is in U', that is, we get $U \cup U' = \mathcal{Q}$. Now let D be a smooth projective curve and $D \to \mathcal{Q}$ be a morphism. If its image interects U, then by Corollary 5.6, $([\mathcal{O}_{\mathcal{Q}}(1)]+\Delta_2/2)\cdot[D] \geq 0$. Using (7.4) and the fact that α_t is nef, we get that $([\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0])\cdot[D] \geq 0$. If D does not intersect U then $D \subset U'$. Hence by Lemma 7.2, we have

$$[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D] \geq 0\,.$$

Since $[L_0]$ is nef we have that

$$([\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0])\cdot[D] \geq 0\,.$$

Also $([\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0])\cdot[\tilde{\delta}] = 0$. Hence any convex linear combination of $[\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0]$ and $[L_0]$ is nef but not ample. By (7.4) $\eta^*([\mathcal{O}_{\mathcal{Q}}(1)] +$

$\frac{t+1}{g+t}[L_0]) = \frac{g-1}{g+t}\alpha_t$. Hence any convex linear combination of $[\mathcal{O}_{\mathcal{Q}}(1)] + \frac{t+1}{g+t}[L_0]$ and $[\alpha_t]$ is not ample. Hence the result follows. □

Precise values for t depending on g are known when

(1) When $g = 1$, $t = 1$.
(2) When $g = 2$, $t = 2$.
(3) When $g = 3$, $t = 9/5$.
(4) When g is a perfect square $t = \sqrt{g}$, see [Kou93, Theorem 2].
(5) In [CK99, Propn. 3.2], when $g \geq 9$, assuming the Nagata conjecture, they prove that $t = \sqrt{g}$.

Thus, in all these cases using Theorem 7.5 we get the Nef cone of $\mathcal{Q}(n, 2)$.

7.6. Criterion for nefness. In the remainder of this section, we will need to work with $C^{(d)}$ for different values of d. The line bundles L_0 on $C^{(d)}$ will therefore be denoted by $L_0^{(d)}$ when we want to emphasize the d. Similarly, we will denote $\mu_0^{(d)} = \frac{d+g-1}{dg}$. Let $\mathcal{P}_{(d)}^{\leq n}$ be the set of all partitions $(d_1, d_2, \ldots, d_k)$ of d of length at most n. Given an element $\mathbf{d} \in \mathcal{P}_{(d)}^{\leq n}$ define

$$C^{(\mathbf{d})} := C^{(d_1)} \times C^{(d_2)} \times \ldots \times C^{(d_k)}$$

and if $p_i : C^{(\mathbf{d})} \to C^{(d_i)}$ is the i-th projection we define a class

$$[\mathcal{O}(-\Delta_{\mathbf{d}}/2)] := [\sum p_i^* \mathcal{O}(-\Delta_{d_i}/2)] \in N^1(C^{(\mathbf{d})})\,.$$

Note that we have a natural addition

$$\pi_{\mathbf{d}} : C^{(\mathbf{d})} \to C^{(d)}\,.$$

For a partition $\mathbf{d} \in \mathcal{P}_d^{\leq n}$ define a morphism

$$\eta_{\mathbf{d}} : C^{(\mathbf{d})} \to \mathcal{Q}$$

as follows. For any $l \geq 1$, we define the universal subscheme of $C^{(l)}$ over $C \times C^{(l)}$ by Σ_l. Then over $C \times C^{(\mathbf{d})}$ we have the subschemes $(id \times p_i)^* \Sigma_{d_i}$. We have a quotient

$$q_{\mathbf{d}} : \mathcal{O}^n_{C \times C^{(d)}} \to \bigoplus_i \mathcal{O}_{(id \times p_{i,\mathbf{d}})^* \Sigma_{d_i}}$$

defined by taking direct sum of morphisms $\mathcal{O}_{C \times C^{(d)}} \to \mathcal{O}_{(id \times p_{i,\mathbf{d}})^* \Sigma_{d_i}}$. Then $q_{\mathbf{d}}$ defines a map $C^{(\mathbf{d})} \to \mathcal{Q}$. By Lemma 3.14, we have

$$[\eta_{\mathbf{d}}^* \mathcal{O}_{\mathcal{Q}}(1)] = [\mathcal{O}(-\Delta_{\mathbf{d}}/2)]\,. \tag{7.7}$$

Lemma 7.8. *Let D be a smooth projective curve. Let $D \to \mathcal{Q}$ be a morphism. Then there exists a partition $\mathbf{d} \in \mathcal{P}_{(d)}^{\leq n}$ such that the composition $D \to \mathcal{Q} \to C^{(d)}$ factors as $D \to C^{(\mathbf{d})} \to C^{(d)}$ and $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] \geq [\mathcal{O}(-\Delta_{\mathbf{d}}/2)] \cdot [D]$.*

Proof. We will proceed by induction on d. When $d = 1$ the statement is obvious.

Let us denote the pullback of the universal quotient on $C \times \mathcal{Q}$ to $C \times D$ by $\mathcal{B}_D$ and let $f : C \times D \to D$ be the natural projection. Consider a section such that the composite $\mathcal{O}_{C\times D} \to \mathcal{O}^n_{C\times D} \to \mathcal{B}_D$ is non-zero and let $\mathcal{F}$ denote the cokernel of the composite map. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C\times D} \arrow[r] \arrow[d, two heads] & \mathcal{O}^n_{C\times D} \arrow[r] \arrow[d, two heads] & \mathcal{O}^{n-1}_{C\times D} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\Gamma'} \arrow[r] & \mathcal{B}_D \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd} \tag{7.9}$$

Let $T_0(\mathcal{F}) \subset \mathcal{F}$ denote the maximal subsheaf of dimension 0, see [HL10, Definition 1.1.4]. Define $\mathcal{F}' := \mathcal{F}/T_0(\mathcal{F})$. Now, either $\mathcal{F}' = 0$ or $\mathcal{F}'$ is torsion free over D, and hence, flat over D. In the first case, it follows that D meets the open set U in Lemma 5.5. Then we take $\mathbf{d} = (d)$ and the statement follows from Lemma 5.5. So we assume $\mathcal{F}'$ is flat over D and let d' be the degree of $\mathcal{F}'|_{C\times x}$, for $x \in D$. So $0 < d' < d$. By (7.9) we have

$$\deg\, f_*\mathcal{B}_D = \deg\, f_*\mathcal{O}_{\Gamma'} + \deg\, f_*\mathcal{F}\,.$$

Since $T_0(\mathcal{F})$ is supported on finitely many points, we have $\deg\,\mathcal{F} \geq \deg\,\mathcal{F}'$. In other words, we have

$$\deg\, f_*\mathcal{B}_D \geq \deg\, f_*\mathcal{O}_{\Gamma'} + f_*\mathcal{F}'\,. \tag{7.10}$$

Now Γ' defines a morphism $D \to C^{(d-d')}$ and note that

$$\deg\, f_*\mathcal{O}_{\Gamma'} = [\mathcal{O}(-\Delta_{d-d'}/2)]\cdot[D]\,.$$

The quotient $\mathcal{O}^{n-1}_{C\times D} \to \mathcal{F}' \to 0$ defines a map $D \to \mathcal{Q}(n-1, d')$. By induction hypothesis, we get that there exists a partition $\mathbf{d}' \in \mathcal{P}^{\leq n-1}_{d'}$ such that the composition $D \to \mathcal{Q}(n-1, d') \to C^{(d')}$ factors as $D \to C^{(\mathbf{d}')} \to C^{(d')}$ and

$$[\mathcal{O}_{\mathcal{Q}(n-1,d')}(1)]\cdot[D] \geq [\mathcal{O}(-\Delta_{\mathbf{d}'}/2)]\cdot[D]\,.$$

Since $\deg\, f_*\mathcal{F}' = [\mathcal{O}_{\mathcal{Q}(n-1,d')}(1)]\cdot[D]$ we have that $\deg\, f_*\mathcal{F}' \geq [\mathcal{O}(-\Delta_{\mathbf{d}'}/2)]\cdot[D]$. From (7.10) we get that

$$[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D] \geq [\mathcal{O}(-\Delta_{d-d'}/2)]\cdot D + [\mathcal{O}(-\Delta_{\mathbf{d}'}/2)]\cdot[D]\,.$$

Now we define $\mathbf{d} := (d-d', \mathbf{d}')$ and the statement follows from the above inequality. □

Theorem 7.11. *Let $\beta \in N^1(C^{(d)})$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + \beta \in N^1(\mathcal{Q})$ is nef iff the class $[\mathcal{O}(-\Delta_{\mathbf{d}}/2)] + \pi^*_{\mathbf{d}}\beta \in N^1(C^{(\mathbf{d})})$ is nef for all $\mathbf{d} \in \mathcal{P}^{\leq n}_d$.*

Proof. From (7.7) it is clear that if $[\mathcal{O}_{\mathcal{Q}}(1)]+\beta$ is nef, then $\eta^*_{\mathbf{d}}([\mathcal{O}_{\mathcal{Q}}(1)]+\beta) = [\mathcal{O}(-\Delta_{\mathbf{d}}/2)] + \pi^*_{\mathbf{d}}\beta$ is nef.

For the converse, we assume $[\mathcal{O}(-\Delta_{\mathbf{d}}/2)]+\pi^*_{\mathbf{d}}\beta$ is nef for all $\mathbf{d} \in \mathcal{P}^{\leq n}_d$. Let D be a smooth projective curve and $D \to \mathcal{Q}$ be a morphism. By Lemma

7.8 we have that there exists $\mathbf{d} \in \mathcal{P}_d^{\leq n}$ such that $D \to C^{(d)}$ factors as $D \to C^{(\mathbf{d})} \to C^{(d)}$ and

$$[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] \geq [\mathcal{O}(-\Delta_{\mathbf{d}}/2)] \cdot [D]\,.$$

Now by assumption we have that

$$[\mathcal{O}(-\Delta_{\mathbf{d}}/2)] \cdot [D] \geq -\beta \cdot [D]\,.$$

Therefore we get

$$[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] \geq -\beta \cdot [D]\,.$$

Hence we get that the class $[\mathcal{O}_{\mathcal{Q}}(1)] + \beta$ is nef. □

Lemma 7.12. *Suppose we are given a map $D \to C^{(\mathbf{d})} \xrightarrow{\pi_{\mathbf{d}}} C^{(d)}$. Then we have*

$$[L_0^{(d)}] \cdot [D] \geq \sum_i [L_0^{(d_i)}] \cdot [D]\,.$$

Proof. By $[L_0^{d_i}] \cdot [D]$ we mean the degree of the pullback of $[L_0^{(d_i)}]$ along $D \to C^{(\mathbf{d})} \xrightarrow{p_i} C^{(d_i)}$. The lemma follows easily from the definition of $L_0^{(d)}$ and is left to the reader. □

Proposition 7.13. *Let $n \geq 1$, $g \geq 1$ and $\mathcal{Q} = \mathcal{Q}(n,d)$. Then the class $\kappa_2 := [\mathcal{O}_{\mathcal{Q}}(1)] + \dfrac{g+1}{2g}[L_0^{(d)}] \in N^1(\mathcal{Q})$ is nef. As a consequence we get that*

$$\mathrm{Nef}(\mathcal{Q}) \supset \mathbb{R}_{\geq 0}\kappa_1 + \mathbb{R}_{\geq 0}\kappa_2 + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0^{(d)}]\,.$$

Proof. Recall $\mu_0^{(2)} = \dfrac{g+1}{2g}$. By Theorem 7.11 it suffices to show that for all $\mathbf{d} \in \mathcal{P}_{(d)}^{\leq n}$ we have $[\mathcal{O}(-\Delta_{\mathbf{d}}/2)] + \mu_0^{(2)}\pi_{\mathbf{d}}^*[L_0^{(d)}]$ is nef. Using Lemma 2.7, $[L_0^{(1)}] = 0$ and Lemma 7.12 we get

$$\begin{aligned}
([\mathcal{O}(-\Delta_{\mathbf{d}}/2)] + \mu_0^{(2)}\pi_{\mathbf{d}}^*[L_0^{(d)}]) \cdot [D] &= \Big(\sum_i (1-\mu_0^{(d_i)})[\theta_{d_i}] - \mu_0^{(d_i)}[L_0^{d_i}]\Big) \cdot [D] \\
&\quad + \mu_0^{(2)}[L_0^{(d)}] \cdot [D] \\
&\geq \sum_i (\mu_0^{(2)} - \mu_0^{(d_i)})[L_0^{d_i}] \cdot [D]\,.
\end{aligned}$$

This proves that κ_2 is nef. That κ_1 is nef is proved in Proposition 5.8. This completes the proof of the theorem. □

Corollary 7.14. *Let $n \geq d$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(2)}[L_0^{(d)}] \in N^1(\mathcal{Q})$ is nef but not ample.*

Proof. By Proposition 7.13 we have that $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(2)}[L_0^{(d)}]$ is nef. Now recall that when $n \geq d$ we have the curve $\tilde{\delta} \hookrightarrow \mathcal{Q}$ (4.1). From the definition of $\tilde{\delta}$ and Lemma 3.14 we have $[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [\tilde{\delta}] = 0$. Also $\Phi_*\tilde{\delta} = \delta$. Hence $[L_0^{(d)}] \cdot [\tilde{\delta}] = [L_0^{(d)}] \cdot [\delta] = 0$. From this we get $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(2)}[L_0^{(d)}] \cdot [\tilde{\delta}] = 0$ and hence $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(2)}L_0^{(d)}$ is not ample. □

As a corollary we get the following result. When $g=1$ note that $\mu_0^{(2)}=1$.

Theorem 7.15. *Let $g=1$, $n\geq 1$ and $\mathcal{Q}=\mathcal{Q}(n,d)$. Then the class $[\mathcal{O}_{\mathcal{Q}}(1)]+[\Delta_d/2]\in N^1(\mathcal{Q})$ is nef. Moreover,*

$$\mathrm{Nef}(\mathcal{Q})=\mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)]+[\Delta_d/2])+\mathbb{R}_{\geq 0}[\theta_d]+\mathbb{R}_{\geq 0}[\Delta_d/2]\,.$$

8. Curves over the small diagonal

Throughout this section the genus of the curve C will be $g(C)\geq 2$ and C is a very general curve. Recall that $\Phi:\mathcal{Q}\to C^{(d)}$ is the Hilbert-Chow map.

Proposition 8.1. *Let $f:D\to\mathcal{Q}(n,d)$ be such that $\Phi\circ f$ factors through the small diagonal. Then $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D]\geq 0$.*

Proof. Since $\Phi\circ f$ factors through the small diagonal, there is a map $g:D\to C$ such that if $\Gamma:=\Gamma_g$ denotes the graph of g in $C\times D$, and $\mathcal{O}^n_{C\times D}\to\mathcal{B}_D$ is the quotient on $C\times D$, then $\mathcal{B}_D$ is supported on $\mathcal{O}_{C\times D}/\mathscr{I}(\Gamma)^d$. Denote $\mathcal{I}:=\mathscr{I}(\Gamma)$. Then $\mathcal{B}_D/\mathcal{I}\mathcal{B}_D$ is a globally generated sheaf on D and so its determinant has degree ≥ 0. Now consider the sheaf

$$\mathcal{I}^i\mathcal{B}_D/\mathcal{I}^{i+1}\mathcal{B}_D\cong(\mathcal{I}/\mathcal{I}^2)^{\otimes i}\otimes\mathcal{B}_D/\mathcal{I}\mathcal{B}_D\,.$$

Using adjunction it is easily seen that $\mathcal{I}/\mathcal{I}^2\cong g^*\omega_C$. Since $\det(\mathcal{B}_D/\mathcal{I}\mathcal{B}_D)$ has degree ≥ 0, it follows that $\det(\mathcal{I}^i\mathcal{B}_D/\mathcal{I}^{i+1}\mathcal{B}_D)$ has degree ≥ 0. From the filtration

$$\mathcal{B}_D\supset\mathcal{I}\mathcal{B}_D\supset\mathcal{I}^2\mathcal{B}_D\supset\ldots\supset\mathcal{I}^d\mathcal{B}_D=0$$

we easily conclude that $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D]\geq 0$. □

Lemma 8.2. *Let $D\to C^{(d)}$ be a morphism. Then we can find a cover $\tilde{D}\to D$ such that the composite $\tilde{D}\to D\to C^{(d)}$ factors through C^d.*

Proof. Let D_1 be a component of $D\times_{C^{(d)}}C^d$ which dominates D. Take $\tilde{D}$ to be a resolution of D_1. □

Corollary 8.3. *Let $D\to\mathcal{Q}$ be a morphism. Replacing D by a cover $\tilde{D}$ we may assume that the map $\tilde{D}\to D\to\mathcal{Q}\to C^{(d)}$ factors through C^d.*

In view of the above, given a map $D\to Q$ we may assume that the composite $D\to\mathcal{Q}\to C^{(d)}$ factors through C^d. Let each component be given by a map $f_i:D\to C$. Denote by $i_D:\Gamma\hookrightarrow C\times D$ the pullback of the universal subscheme $\Sigma\hookrightarrow C\times C^{(d)}$ to $C\times D$. The ideal sheaf of Γ is the product $\mathscr{I}(\Gamma_{f_i})$, the ideal sheaves of the graphs $\Gamma_{f_i}\subset C\times D$. Moreover, $\mathcal{B}_D$ is supported on Γ. Let $g_1,g_2,\ldots,g_r$ be the distinct maps in the set $\{f_1,f_2,\ldots,f_d\}$ and assume that g_i occurs d_i many times. Then we have $\mathscr{I}(\Gamma)=\prod_{i=1}^r\mathscr{I}(\Gamma_{g_i})^{d_i}$. There is a natural map

$$\psi:\mathcal{B}_D\to\bigoplus\mathcal{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D\,.$$

Lemma 8.4. *Let $f:D\to\mathcal{Q}$ be such that $\Phi\circ f$ factors through $C^d\to C^{(d)}$. If ψ is an isomorphism then $[\mathcal{O}_{\mathcal{Q}}(1)]\cdot[D]\geq 0$.*

Proof. Since $\mathcal{B}_D$ is a quotient of $\mathcal{O}^n_{C\times D}$ it follows that each $\mathcal{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D$ is a quotient of $\mathcal{O}^n_{C\times D}$. Thus, each $\mathcal{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D$ defines a map $D \to \mathcal{Q}(n, d'_i)$ such that the image under the map $\Phi : \mathcal{Q}(n, d'_i) \to C^{(d'_i)}$ is the small diagonal. By Proposition 8.1 it follows that degree of $\det(p_{D*}(\mathcal{B}_D/\mathscr{I}(\Gamma_{g_i})^{d_i}\mathcal{B}_D))$ is ≥ 0. Since ψ is an isomorphism it follows that degree of $\det(p_{D*}(\mathcal{B}_D))$ is ≥ 0. □

We can use the above method to prove a result similar to Theorem 7.5 when $d = 3$.

Corollary 8.5. *Let $d = 3$. Consider the Quot scheme $\mathcal{Q} = \mathcal{Q}(n, 3)$. Let $\mu_0^{(3)} = \dfrac{g+2}{3g}$. Then $[\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(3)}[L_0^{(3)}]$ is nef.*

Proof. If $d = 3$ there are only these types of quotients:

(1) $\mathcal{O}^n_C \to \mathcal{O}_C/\mathfrak{m}_{C,c_1}\mathfrak{m}_{C,c_2}\mathfrak{m}_{C,c_3}$,

(2) $\mathcal{O}^n_C \to \mathcal{O}_{C,c_1}/\mathfrak{m}_{C,c_1} \oplus \mathcal{O}_C/\mathfrak{m}_{C,c_1}\mathfrak{m}_{C,c_2}$,

(3) $\mathcal{O}^n_C \to \dfrac{\mathcal{O}_{C,c}}{\mathfrak{m}_{C,c}} \oplus \dfrac{\mathcal{O}_{C,c}}{\mathfrak{m}_{C,c}} \oplus \dfrac{\mathcal{O}_{C,c}}{\mathfrak{m}_{C,c}}$.

Let $f : D \to \mathcal{Q}$ be a map. If D contains a quotient of type (1) or (3) then D meets U or U'(see Definition 5.3 and Definition 7.1). Thus, in these cases $([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(3)}[L_0^{(3)}]) \cdot [D] \geq 0$ by Corollary 5.6 and Lemma 7.2.

Now consider the case when all points in the image of D are of type (2). After replacing D by a cover, using Corollary 8.3, we may assume that the map $D \to \mathcal{Q}$ factors through C^3. Since the images of points of D represent quotients of type (2), we may assume that the map from $D \to C^3$ looks like $d \mapsto (g_1(d), g_1(d), g_2(d))$. Now consider a general section $\mathcal{O}_{C\times D} \to \mathcal{B}_D$. Arguing as in the proof of Lemma 5.5 we get a diagram as in equation (7.9), such that $\mathcal{O}_{\Gamma'}$ defines a map $D \to C^{(2)}$ and $\mathcal{F}' = \mathcal{F}/T_0(\mathcal{F})$ is a line bundle on D which is globally generated. Hence

$$\begin{aligned}[\mathcal{O}_{\mathcal{Q}}(1)] \cdot [D] &\geq [\mathcal{O}(-\Delta_2/2)] \cdot [D] + [c_1(p_{D*}(\mathcal{F}))] \cdot [D] \\ &\geq -\mu_0^{(2)}[L_0^{(2)}] \cdot [D] \,.\end{aligned}$$

One easily checks using the definition of L_0 that in this case $[L_0^{(3)}] \cdot [D] = 2[L_0^{(2)}] \cdot [D]$. Thus,

$$([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0^{(3)}[L_0^{(3)}]) \cdot [D] \geq (2\mu_0^{(3)} - \mu_0^{(2)})[L_0^{(2)}] \cdot [D] \geq 0 \,.$$

This completes the proof of the Corollary. □

Combining this with Proposition 4.2 we get the following result.

Theorem 8.6. *Let C be a very general curve of genus $2 \leq g(C) \leq 4$. Let $n \geq 3$ and let $\mathcal{Q} = \mathcal{Q}(n, 3)$. Let $\mu_0 = \dfrac{g+2}{3g}$ Then*

$$\mathrm{Nef}(\mathcal{Q}) = \mathbb{R}_{\geq 0}([\mathcal{O}_{\mathcal{Q}}(1)] + \mu_0[L_0^{(3)}]) + \mathbb{R}_{\geq 0}[\theta_d] + \mathbb{R}_{\geq 0}[L_0^{(3)}]\,.$$

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