IRREDUCIBILITY OF SOME NESTED HILBERT SCHEMES

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Abstract. Let S be a smooth projective surface over $\mathbb{C}$. Let $S^{[n_1,\dots,n_k]}$ denote the nested Hilbert scheme which parametrizes zero-dimensional subschemes $\xi_{n_1} \subset \ldots \subset \xi_{n_k}$ where ξ_i is a closed subscheme of S of length i. We show that $S^{[n,m]}$, $S^{[n,m,m+1]}$, $S^{[n,n+1,m]}$, $S^{[n,n+1,m,m+1]}$, $S^{[n,n+2,m]}$ and $S^{[n,n+2,m,m+1]}$ are irreducible.

1. Introduction

Let S be a smooth projective surface over $\mathbb{C}$. The Hilbert scheme $S^{[n]}$ which parametrizes closed zero-dimensional subschemes of S of length n is a well studied space. It was shown by Fogarty in [Fog68, Theorem 2.4] that the Hilbert scheme $S^{[n]}$ is a smooth projective variety of dimension $2n$. A natural generalization of $S^{[n]}$ is the nested Hilbert Scheme, about which far less is known. For an increasing tuple of positive integers $n_1 < \ldots < n_k$, the nested Hilbert scheme $S^{[n_1,\dots,n_k]}$ parametrizes nested zero-dimensional subschemes $\xi_{n_1} \subset \ldots \subset \xi_{n_k}$ where ξ_i is a subscheme of S of length i. In recent years the nested Hilbert schemes $S^{[n,m]}$ have received growing attention. They have been studied by several authors using techniques from commutative algebra, representation theory and Lie algebras. In a recent article, [RS21], Ramkumar and Sammartano introduce methods to study $S^{[n,m]}$. They use these methods to show that the scheme $S^{[2,n]}$ is smooth in codimension 3 and has rational singularities. In particular, $S^{[2,n]}$ is normal and Cohen-Macaulay. They also mention several interesting questions related to the schemes $S^{[n,m]}$, one of them being the irreducibility of these schemes. The purpose of this article is to show that $S^{[n,m]}$ is irreducible.

Before we state our results, we mention a few already existing results related to irreducibility of nested Hilbert schemes. The nested Hilbert scheme $S^{[1,n]}$ is irreducible of dimension $2n$ by [Fog73, Corollary 7.3]. The scheme $S^{[n,n+1]}$ is smooth and irreducible, as shown in [Che98, Theorem 3.0.1]. In [GH04, Proposition 6], the authors show that $S^{[n,n+2]}$ is irreducible of dimension $2n + 4$. In [BE16], Bulois and Evain studied irreducible components of nested Hilbert schemes supported at a single point using the connection between nested Hilbert schemes and commuting varieties of parabolic subalgebras. In [Add16, §3.A] the irreducibility of $S^{[n,n+1,n+2]}$ is proved. In [RT22], Ryan and Taylor study the irreducibility, singularities and Picard groups of $S^{[n,n+1,n+2]}$. In [RS21, Theorem 3.1], Ramkumar and Sammartano have shown that $S^{[2,n]}$ is irreducible of dimension $2n$.

The following two results limit the collection of tuples $(n_1, \dots, n_k)$ for which the nested Hilbert scheme $S^{[n_1,\dots,n_k]}$ is irreducible. By [RT22, Corollary 3.17] the nested Hilbert scheme $S^{[n_1,\dots,n_k]}$ is reducible for $k > 22$. In [RS21, Proposition 3.7] the authors prove the existence

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of tuples $n_1 < \cdots < n_k$, for each $k \geqslant 5$, such that the nested Hilbert scheme $(\mathbb{A}^2)^{[n_1,\dots,n_k]}$ is reducible. We refer the reader to [RT22], [RS21] and the references therein for more results related to the geometry of nested Hilbert schemes.

In [RS21], the authors pose the problem of irreducibility of the two step nested Hilbert schemes, see [RS21, Question 9.4]. Our goal in this paper is to prove the following results on irreducibility of nested Hilbert schemes.

Theorem (Theorem 3.8)**.** *Let n and m be two positive integers such that $n < m$. Then $S^{[n,m,m+1]}$ and $S^{[n,m]}$ are irreducible.*

Theorem (Theorem 4.7)**.** *Let n and m be two positive integers such that $n+1 < m$. Then $S^{[n,n+1,m,m+1]}$ and $S^{[n,n+1,m]}$ are irreducible.*

Theorem (Theorem 5.2)**.** *Let n and m be two positive integers such that $n+2 < m$. Then $S^{[n,n+2,m,m+1]}$ and $S^{[n,n+2,m]}$ are irreducible.*

Let E be a locally free sheaf on S and let $\mathrm{Quot}(E,d)$ denote the Grothendieck Quot scheme of quotients of E of length d. In [EL99, Theorem 1] it is proved that this Quot scheme is irreducible. The proofs of the above results proceed by combining some of the ideas in [EL99], [BE16] and [RT22], and using an induction argument. We assume that $S^{[n,m]}$ is irreducible and show that $S^{[n,m,m+1]}$ is irreducible. Using the surjectivity of the natural map $S^{[n,m,m+1]} \to S^{[n,m+1]}$ we see that $S^{[n,m+1]}$ is irreducible.

A crucial input in all the proofs is that the dimension of some of the spaces of the type $S_p^{[l',l]}$ (this notation is explained before Lemma 3.3) satisfy a certain upper bound. These dimensions have been computed in [BE16] when $0 \leqslant l - l' \leqslant 2$. It is natural to ask if the methods in this article can be used to proved the irreduciblity of $S^{[n_1,n_2,n_3]}$ for all triples. One of the obstacles is the non-existence of similar bounds on the dimension of $S_p^{[l',l]}$ for all pairs (l', l) with $l - l' \geqslant 0$.

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2. Preliminaries

Let S be a smooth projective surface over $\mathbb{C}$. For a pair of positive integers n, m with $n < m$, the nested Hilbert scheme $S^{[n,m]}$ parametrizes nested subschemes $\xi_n \subset \xi_m$ of S, where ξ_i is a finite scheme of length i. Recall that the scheme $S^{[n,m]}$ represents the functor of nested flat families $\mathfrak{h}ilb_S^{[n,m]}$

$$\mathfrak{h}ilb_S^{[n,m]} : \mathrm{Sch}/\mathbb{C} \longrightarrow \mathrm{Sets}\,,$$

where $\mathfrak{h}ilb_S^{[n,m]}(T)$ is the set of isomorphism classes of T-flat subschemes $X_n \subset X_m \subset S \times T$ such that for each point $t \in T$, the length of the subscheme $X_n \otimes k(t)$ is n and the length of the subscheme $X_m \otimes k(t)$ is m. In particular, we have universal nested families of closed subschemes $Z_n \subset Z_m \subset S \times S^{[n,m]}$. The closed points of Z_n and Z_m have the following descriptions:

$$
\begin{aligned}
Z_n &= \{(p, \xi_n, \xi_m) \in S \times S^{[n,m]} \quad | \quad p \in \xi_n \subset \xi_m\}\,, \\
Z_m &= \{(p, \xi_n, \xi_m) \in S \times S^{[n,m]} \quad | \quad p \in \xi_m\}\,.
\end{aligned}
$$

We have the projection map

$$\pi_m : S^{[n,m]} \longrightarrow S^{[m]}\,.$$

Let $\mathscr{I}_m$ denote the ideal sheaf of the universal subscheme inside $S \times S^{[m]}$. Consider the map

$$\mathrm{Id}_S \times \pi_m : S \times S^{[n,m]} \longrightarrow S \times S^{[m]}\,.$$

Denote the pullback

$$\tilde{\mathscr{I}}_m := (\mathrm{Id}_S \times \pi_m)^* \mathscr{I}_m\,.$$

Consider the projective bundle

$$\varphi : \mathbb{P}(\tilde{\mathscr{I}}_m) \longrightarrow S \times S^{[n,m]}\,. \tag{2.1}$$

On $\mathbb{P}(\tilde{\mathscr{I}}_m)$, we have the tautological quotient

$$\varphi^* \tilde{\mathscr{I}}_m \longrightarrow \mathcal{O}_{\mathbb{P}(\tilde{\mathscr{I}}_m)}(1)\,.$$

Let φ_1 denote the composite $\mathbb{P}(\tilde{\mathscr{I}}_m) \xrightarrow{\varphi} S \times S^{[n,m]} \longrightarrow S$, where the second map is the projection to S. Similarly, let φ_2 denote the composite $\mathbb{P}(\tilde{\mathscr{I}}_m) \xrightarrow{\varphi} S \times S^{[n,m]} \longrightarrow S^{[n,m]}$, where the second map is the projection to $S^{[n,m]}$. Consider the graph of φ_1,

$$\mathbb{P}(\tilde{\mathscr{I}}_m) \overset{\iota}{\hookrightarrow} S \times \mathbb{P}(\tilde{\mathscr{I}}_m)\,.$$

Since ι is the graph of φ_1, it follows that the composite map $\mathbb{P}(\tilde{\mathscr{I}}_m) \overset{\iota}{\hookrightarrow} S \times \mathbb{P}(\tilde{\mathscr{I}}_m) \to \mathbb{P}(\tilde{\mathscr{I}}_m)$ is the identity. This shows that the sheaf $\iota_* \mathcal{O}_{\mathbb{P}(\tilde{\mathscr{I}}_m)}(1)$ on $S \times \mathbb{P}(\tilde{\mathscr{I}}_m)$ is flat over $\mathbb{P}(\tilde{\mathscr{I}}_m)$.

Now consider the map

$$(\mathrm{Id}_S \times \varphi_2) : S \times \mathbb{P}(\tilde{\mathscr{I}}_m) \longrightarrow S \times S^{[n,m]}\,.$$

On $S \times \mathbb{P}(\tilde{\mathscr{I}}_m)$, there is a canonical surjection

$$\delta : (\mathrm{Id}_S \times \varphi_2)^* \tilde{\mathscr{I}}_m \longrightarrow \iota_* \iota^* (\mathrm{Id}_S \times \varphi_2)^* \tilde{\mathscr{I}}_m = \iota_* \varphi^* \tilde{\mathscr{I}}_m \longrightarrow \iota_* \mathcal{O}_{\mathbb{P}(\tilde{\mathscr{I}}_m)}(1)\,.$$

Using δ we define a sheaf $\mathcal{T}$ on $S \times \mathbb{P}(\tilde{\mathscr{I}}_m)$ by the push-out diagram below

$$
\begin{tikzcd}
0 \arrow[r] & (\mathrm{Id}_S \times \varphi_2)^* \mathscr{I}_m \arrow[r] \arrow[d, "\delta"] & \mathcal{O}_{S \times \mathbb{P}(\tilde{\mathscr{I}}_m)} \arrow[r] \arrow[d] & (\mathrm{Id}_S \times \varphi_2)^* \mathcal{O}_{Z_m} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \iota_* \mathcal{O}_{\mathbb{P}(\tilde{\mathscr{I}}_m)}(1) \arrow[r] & \mathcal{T} \arrow[r] & (\mathrm{Id}_S \times \varphi_2)^* \mathcal{O}_{Z_m} \arrow[r] & 0\,.
\end{tikzcd}
\tag{2.2}
$$

Remark 2.3. Recall the following general fact. Let $X \to Y$ be a map of schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over Y. Let $f : Y' \to Y$ be a morphism of schemes and consider the Cartesian square

$$\begin{tikzcd} X' \arrow[r, "\tilde{f}"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

Then one easily checks that the sheaf $\tilde{f}^*\mathcal{F}$ is flat over Y'. □

Applying Remark 2.3 to the diagram

$$\begin{tikzcd} S \times \mathbb{P}(\tilde{\mathscr{I}}_m) \arrow[r, "\mathrm{Id}_S \times \varphi_2"] \arrow[d] & S \times S^{[n,m]} \arrow[d] \\ \mathbb{P}(\tilde{\mathscr{I}}_m) \arrow[r, "\varphi_2"] & S^{[n,m]} \end{tikzcd}$$

and the sheaf $\mathcal{O}_{Z_m}$ on $S \times S^{[n,m]}$ we see that $(\mathrm{Id}_S \times \varphi_2)^*\mathcal{O}_{Z_m}$ is flat over $\mathbb{P}(\tilde{\mathscr{I}}_m)$. We already saw that $\iota_*\mathcal{O}_{\mathbb{P}(\tilde{\mathscr{I}}_m)}(1)$ is flat over $\mathbb{P}(\tilde{\mathscr{I}}_m)$. Thus, it follows that the sheaf $\mathcal{T}$ on $S \times \mathbb{P}(\tilde{\mathscr{I}}_m)$ is flat over $\mathbb{P}(\tilde{\mathscr{I}}_m)$. It is clear that $\mathcal{T}$ is a family of quotients of length $m+1$. This gives a nested family of quotients

$$\mathcal{O}_{S \times \mathbb{P}(\tilde{\mathscr{I}}_m)} \longrightarrow \mathcal{T} \longrightarrow (\mathrm{Id}_S \times \varphi_2)^*\mathcal{O}_{Z_m} \longrightarrow (\mathrm{Id}_S \times \varphi_2)^*\mathcal{O}_{Z_n}$$

on $S \times \mathbb{P}(\tilde{\mathscr{I}}_m)$. Using the universal property for $S^{[n,m+1]}$ and the quotients

$$\mathcal{O}_{S \times \mathbb{P}(\tilde{\mathscr{I}}_m)} \longrightarrow \mathcal{T} \longrightarrow (\mathrm{Id}_S \times \varphi_1)^*\mathcal{O}_{Z_n}$$

we get a map

$$\psi : \mathbb{P}(\tilde{\mathscr{I}}_m) \longrightarrow S^{[n,m+1]}\,. \tag{2.4}$$

A pointwise description of this map is given as follows. Let $(p, \xi_n, \xi_m) \in S \times S^{[n,m]}$ be a closed point. So we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_{\xi_m} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{\xi_m} \longrightarrow 0\,.$$

A point in $\mathbb{P}(\tilde{\mathscr{I}}_m)$ over (p, ξ_n, ξ_m) is given by a quotient $\lambda : \mathcal{I}_{\xi_m} \longrightarrow k(p)$. We shall represent such a point by the tuple $(p, \xi_n, \xi_m, \lambda)$. We get the quotient $\mathcal{O}_S \longrightarrow \mathcal{O}_{\xi_{m+1}}$ by the push-out diagram below in which the columns are short exact sequences.

$$\begin{tikzcd} & \mathcal{I}_{\xi_{m+1}} \arrow[r, equal] \arrow[d] & \mathcal{I}_{\xi_{m+1}} \arrow[d] & & \\ 0 \arrow[r] & \mathcal{I}_{\xi_m} \arrow[r] \arrow[d, "\lambda"] & \mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_{\xi_m} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k(p) \arrow[r] & \mathcal{O}_{\xi_{m+1}} \arrow[r] & \mathcal{O}_{\xi_m} \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

The map ψ takes the point $(p, \xi_n, \xi_m, \lambda)$ of $\mathbb{P}(\mathscr{I}_m)$ to the point $(p, \xi_n, \xi_{m+1}) \in S \times S^{[n,m+1]}$.

We note the following maps

$$\begin{tikzcd} \mathbb{P}(\tilde{\mathscr{I}}_m) \arrow[r, "\psi"] \arrow[d, "\varphi"'] & S^{[n,m+1]} \\ S \times S^{[n,m]} & \end{tikzcd} \tag{2.6}$$

For an $\mathcal{O}_S$ module $\mathcal{F}$, we shall denote by $\mathcal{F}_p$ the localization $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S,p}$. Here $\mathcal{O}_{S,p}$ is the local ring of S at the closed point p.

Lemma 2.7. *The map ψ is surjective on closed points.*

Proof. A closed point in $S^{[n,m+1]}$ corresponds to subschemes $\xi_n \subset \xi_{m+1}$ with $\text{length}(\xi_n) = n$ and $\text{length}(\xi_{m+1}) = m+1$. Let K denote the kernel of the map $\mathcal{O}_{\xi_{m+1}} \longrightarrow \mathcal{O}_{\xi_n}$. Then we may write

$$K = \bigoplus_{p \in \text{Supp}(K)} K_p\,.$$

Choose any map $k(p) \longrightarrow K_p$ of $\mathcal{O}_{S,p}$ modules and form the diagram

$$\begin{tikzcd} & k(p) \arrow[r, equal] \arrow[d] & k(p) \arrow[d, "\lambda"] & & \\ 0 \arrow[r] & K \arrow[r] & \mathcal{O}_{\xi_{m+1}} \arrow[r] \arrow[d, "\theta"] & \mathcal{O}_{\xi_n} \arrow[r] \arrow[d, equal] & 0 \\ & & \mathcal{O}_{\xi_m} \arrow[r] & \mathcal{O}_{\xi_n} \arrow[r] & 0 \end{tikzcd}$$

Note that the middle column is a short exact sequence. Using this observation and applying Snake Lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_{\xi_{m+1}} \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & \mathcal{O}_{\xi_m+1} \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & \mathcal{I}_{\xi_m} \arrow[r] & \mathcal{O}_S \arrow[r] & \mathcal{O}_{\xi_m} \arrow[r] & 0 \end{tikzcd} \tag{2.8}$$

one easily concludes that we have a short exact sequence of ideal sheaves

$$0 \longrightarrow \mathcal{I}_{\xi_{m+1}} \longrightarrow \mathcal{I}_{\xi_m} \xrightarrow{\lambda} k(p) \longrightarrow 0\,.$$

The reader will easily check that when we take the push-out of the lower row in (2.8) along the map λ, we get diagram (2.5). One easily concludes that the closed point $(p, \xi_n, \xi_m, \lambda) \in \mathbb{P}(\tilde{\mathscr{I}}_m)$ is mapped to the closed point $(\xi_n, \xi_{m+1}) \in S^{[n,m+1]}$ under ψ. This completes the proof of the Lemma. $\square$

3. Irreducibility of $S^{[n,m]}$

Let $W_{i,[n,m]}$ denote the following locus in $S \times S^{[n,m]}$

$$W_{i,[n,m]} := \{(p, \xi_n, \xi_m) \in S \times S^{[n,m]} \quad | \quad \dim(\mathcal{I}_{\xi_m} \otimes k(p)) = i \quad \}\,.$$

In other words, it is the locus of points (p,ξ_n,ξ_m) such that the ideal $\mathcal{I}_{\xi_m}$ is generated by exactly i elements at the point p. Since ξ_m is a zero dimensional scheme on a smooth surface, it follows that if p is in the support of ξ_m, then $\dim(\mathcal{I}_{\xi_m}\otimes k(p))\geqslant 2$. In other words, $W_{1,[n,m]}$ is the complement of the universal family Z_m in $S\times S^{[n,m]}$. For $i\geqslant 2$ we define subsets $W_{i,[n,m],l',l}\subset W_{i,[n,m]}$ as follows.

Definition 3.1. *Let $i\geqslant 2$. Let $W_{i,[n,m],l',l}\subset W_{i,[n,m]}$ be the subset consisting of points (p,ξ_n,ξ_m) such that* $\mathrm{length}(\mathcal{O}_{\xi_n,p})=l'$ *and* $\mathrm{length}(\mathcal{O}_{\xi_m,p})=l$.

Notice that for the set $W_{i,[n,m],l',l}$ to be nonempty we need that $0\leqslant l'\leqslant n$, $0\leqslant l'\leqslant l$ and $1\leqslant l\leqslant m$. As $i\geqslant 2$, we have that $p\in\mathrm{Supp}(\xi_m)$, which implies that $1\leqslant l$. Note that $l'=0$ is allowed as it may happen that p is not in the support of ξ_n.

Clearly,

$$W_{i,[n,m]}=\bigcup_{l',l}W_{i,[n,m],l',l}\,. \tag{3.2}$$

In the next lemma, using the sets $W_{i,[n,m],l',l}$, we shall obtain a bound on the dimension of $W_{i,[n,m]}$. We need the following notations. Let $p\in S$ denote a closed point.

- By $S^{[0,m]}$ we mean $S^{[m]}$.
- Let $S^{[l]}_{p,i}$ denote the subset of $S^{[l]}$ corresponding to subschemes η satisfying the following two conditions: $\mathrm{Supp}(\eta)=\{p\}$ and $\dim(\mathcal{I}_\eta\otimes k(p))=i$.
- Let $S^{[l',l]}_{p,i}$ denote the subset of $S^{[l',l]}$ consisting of pairs $(\xi_{l'},\xi_l)$ satisfying the following two conditions: $\mathrm{Supp}(\xi_l)=\{p\}$ and $\dim(\mathcal{I}_{\xi_l}\otimes k(p))=i$.
- By $S^{[0,l]}_{p,i}$ we mean $S^{[l]}_{p,i}$.

Lemma 3.3. *Fix integers $n<m$. Consider pairs of integers (l',l) for which the following three conditions hold:*

- $0\leqslant n-l'\leqslant m-l$,
- $0\leqslant l'\leqslant l$,
- $1\leqslant l$.

Assume that for each such pair, the locus $S^{[n-l',m-l]}$ is irreducible of dimension $2(m-l)$. Let $i\geqslant 2$. Then $\dim(W_{i,[n,m]})\leqslant 2m+2-i$.

Proof. In view of (3.2) it suffices to show that if $W_{i,[n,m],l',l}$ is nonempty then we have $\dim(W_{i,[n,m],l',l})\leqslant 2m+2-i$. The argument is similar to that of [RT22, Lemma 3.3], along with a key input from [BE16]. Consider the projection map $p_1:W_{i,[n,m],l',l}\longrightarrow S$ which sends $(p,\xi_n,\xi_m)\mapsto p$. We shall find an upper bound for the dimension of the fiber over a closed point $p\in S$. Let U denote the open subset $S\setminus\{p\}$. Given a point $(p,\xi_n,\xi_m)\in p_1^{-1}(p)$, we may write

$$\mathcal{O}_{\xi_m}=\mathcal{O}_{\xi_m,p}\bigoplus\left(\bigoplus_{q\in U}\mathcal{O}_{\xi_m,q}\right)\,,\qquad \mathcal{O}_{\xi_n}=\mathcal{O}_{\xi_n,p}\bigoplus\left(\bigoplus_{q\in U}\mathcal{O}_{\xi_n,q}\right)$$

The quotient $\mathcal{O}_{\xi_m} \longrightarrow \mathcal{O}_{\xi_n}$ gives rise to quotients

$$\mathcal{O}_{\xi_m,p} \longrightarrow \mathcal{O}_{\xi_n,p}\,, \qquad \left(\bigoplus_{q\in U}\mathcal{O}_{\xi_m,q}\right) \longrightarrow \left(\bigoplus_{q\in U}\mathcal{O}_{\xi_n,q}\right)$$

This gives rise to the following map which is an inclusion on closed points

$$p_1^{-1}(p) \longrightarrow S^{[l',l]}_{p,i} \times U^{[n-l',m-l]}\,. \tag{3.4}$$

When $l' = 0$ the above map is

$$p_1^{-1}(p) \longrightarrow S^{[l]}_{p,i} \times U^{[n,m-l]}\,. \tag{3.5}$$

As $U^{[n-l',m-l]}$ is an open subset of $S^{[n-l',m-l]}$, and the latter is irreducible of dimension $2(m-l)$ by our hypothesis, it follows that $\dim(U^{[n-l',m-l]}) = 2(m-l)$. Next we a bound on the dimension of $S^{[l',l]}_{p,i}$. To do this we shall first give a bound on the dimension of $S^{[l]}_{p,i}$.

First we consider the case $l' \neq 0$. Fix a point $\xi_l \in S^{[l]}_{p,i}$. Let M be a module over the local ring $\mathcal{O}_{S,p}$ whose support is zero dimensional. By $\mathrm{Soc}(M)$ we mean the space $\mathrm{Hom}_{\mathcal{O}_{S,p}}(k(p), M)$. Since the only closed point in the support of ξ_l is p, it follows that if we have a subscheme $\xi_{l-1} \subset \xi_l$, then the kernel of the map $\mathcal{O}_{\xi_l} \to \mathcal{O}_{\xi_{l-1}}$ is isomorphic to $k(p)$. Conversely, taking the quotient of an inclusion of $\mathcal{O}_S$ modules $k(p) \to \mathcal{O}_{\xi_l}$ gives a length $l-1$ subscheme of ξ_l. This shows that the set of subschemes of length $l-1$ of ξ_l is in bijective correspondence with $\mathbb{P}(\mathrm{Soc}(\mathcal{O}_{\xi_l})^\vee)$. By [EL99, Lemma 2], we have $\dim(\mathbb{P}(\mathrm{Soc}(\mathcal{O}_{\xi_l})^\vee)) = i-2$. Thus, all fibers of the map $S^{[l-1,l]}_{p,i} \to S^{[l]}_{p,i}$ have dimension $i-2$. From this, it follows that

$$\dim(S^{[l-1,l]}_{p,i}) = \dim(S^{[l]}_{p,i}) + i - 2\,.$$

As $S^{[l-1,l]}_{p,i} \subset S^{[l-1,l]}_{p}$ it follows that $\dim(S^{[l-1,l]}_{p,i}) \leqslant \dim(S^{[l-1,l]}_{p})$. In [BE16, Corollary 5.9], it is proved that $\dim(S^{[l-1,l]}_{p}) = l-1$. Thus, we get

$$\dim(S^{[l]}_{p,i}) + i - 2 = \dim(S^{[l-1,l]}_{p,i}) \leqslant \dim(S^{[l-1,l]}_{p}) = l-1\,.$$

The above gives the following bound on the dimension of $S^{[l]}_{p,i}$,

$$\dim(S^{[l]}_{p,i}) \leqslant l - i + 1\,. \tag{3.6}$$

The natural map $S^{[l',l]}_{p,i} \longrightarrow S^{[l]}_{p,i} \times S^{[l']}_{p}$ is an inclusion on closed points. As $l' \geqslant 1$, we have $\dim(S^{[l']}_{p}) = l'-1$, see [Bri77]. Thus, it follows that

$$\dim(S^{[l',l]}_{p,i}) \leqslant l - i + 1 + l' - 1 = l + l' - i \leqslant 2l - i\,. \tag{3.7}$$

Thus, using (3.4) it follows that

$$\dim(p_1^{-1}(p)) \leqslant 2l - i + 2(m-l) = 2m - i\,,$$

from which it follows that

$$\dim(W_{i,[n,m],l',l}) \leqslant 2m + 2 - i\,.$$

Next we consider the case $l' = 0$. Using (3.5) and (3.6) we get

$$\dim(p_1^{-1}(p)) \leqslant 2(m-l) + l - i + 1 = 2m - l - i + 1\,.$$

It follows that

$$\dim(W_{i,[n,m],0,l}) \leqslant 2m - l - i + 3\,.$$

In the proof of [RT22, Lemma 3.2] it is proved that $l \geqslant \binom{i}{2}$. Since $i \geqslant 2$, we have that $\binom{i}{2} - 1 \geqslant 0$. Thus, we get

$$\dim(W_{i,[n,m],0,l}) \leqslant 2m - l - i + 3 \leqslant 2m + 2 - i - \binom{i}{2} + 1 \leqslant 2m + 2 - i\,.$$

This completes the proof of the Lemma. □

Theorem 3.8. *Let n and m be two positive integers such that $n < m$. Then $S^{[n,m,m+1]}$ and $S^{[n,m]}$ are irreducible.*

Proof. Let $\mathcal{A}$ be the set of pairs of integers (a,b) with $1 \leqslant a < b$ and $S^{[a,b]}$ reducible. Assume $\mathcal{A}$ is nonempty. By [Fog73, Corollary 7.3] for every $b \geqslant 2$ the pair $(1,b) \notin \mathcal{A}$. Similarly, by [Che98, Theorem 3.0.1] for every $a \geqslant 1$ the pair $(a,a+1) \notin \mathcal{A}$. Consider the projection map to the first coordinate $\mathcal{A} \longrightarrow \mathbb{Z}_{\geqslant 1}$, where $\mathbb{Z}_{\geqslant 1}$ denotes the set of positive integers. Let n be the smallest integer in the image of this map. Clearly, $n > 1$. Among the set of integers b such that $(n,b) \in \mathcal{A}$, let b_0 be the smallest. Clearly, $b_0 > n+1$. Let $m = b_0 - 1$. Then $m \geqslant n+1$. We conclude that for all pairs of integers (a,b) with $1 \leqslant a < b$, if $a < n$ then $S^{[a,b]}$ is irreducible, and for all integers b such that $n < b \leqslant m$, $S^{[n,b]}$ is irreducible. Further $S^{[n,m+1]}$ is reducible. We will arrive at a contradiction, which will prove that $\mathcal{A}$ is empty, and hence prove the theorem.

Note that if $S^{[a,b]}$ is irreducible then its dimension is $2b$. This can be seen as follows. Consider the open subset consisting of pairs (ξ_a, ξ_b) such that the support of ξ_b has b distinct points. The natural map from this open set to $S^{[b]}$ is dominant and quasi-finite and so this open set has dimension $2b$. Since $S^{[a,b]}$ is irreducible, it follows that it has dimension $2b$.

The method of proof is identical to the method in [EL99, Proposition 5]. Consider the map φ in (2.6). We claim that we can find locally free sheaves $\mathcal{F}$ of rank r and $\mathcal{E}$ of rank $r+1$ on $S \times S^{[n,m]}$ which fit into a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \tilde{\mathscr{I}}_m \longrightarrow 0$$

on $S \times S^{[n,m]}$. Let $\mathcal{E}$ be a locally free sheaf which surjects onto $\tilde{\mathscr{I}}_m$ and let $\mathcal{F}$ be the kernel of this surjection. As $\tilde{\mathscr{I}}_m$ is flat over $S^{[n,m]}$ and $\mathcal{E}$ is flat, it follows that $\mathcal{F}$ is flat over $S^{[n,m]}$. If $x \in S^{[n,m]}$ is a closed point, then the restriction to $S \times x$ gives a short exact sequence

$$0 \longrightarrow \mathcal{F}|_{S\times x} \longrightarrow \mathcal{E}|_{S\times x} \longrightarrow \tilde{\mathscr{I}}_m|_{S\times x} \longrightarrow 0\,.$$

As $\tilde{\mathscr{I}}_m|_{S\times x}$ is the ideal sheaf of a zero dimensional scheme, it follows this has projective dimension 1. Thus, it follows that $\mathcal{F}|_{S\times x}$ is locally free on S. Using the following result from commutative algebra, we see that $\mathcal{F}$ is locally free. Let $A \to B$ be a local homomorphism of local rings, M a finite B module which is flat over A and $M/(\mathfrak{m}_A M)$ is a free $B/(\mathfrak{m}_A B)$ module. Then M is a free B module. It is clear that if $\mathcal{F}$ has rank r then $\mathcal{E}$ has rank $r+1$. This completes the proof of the claim.

Let X be a scheme and suppose that $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of coherent sheaves on X. Let $\mathrm{Sym}^*(\mathcal{B})$ denote the sheaf of algebras on X associated to $\mathcal{B}$. Let $\mathcal{J} \subset \mathrm{Sym}^*(\mathcal{B})$ denote the sheaf of ideals generated by $\mathcal{A}$. Then we have

$$\mathrm{Sym}^*(\mathcal{C}) = \mathrm{Sym}^*(\mathcal{B})/\mathcal{J}\,.$$

On $\mathrm{Proj}(\mathrm{Sym}^*(\mathcal{B})) \xrightarrow{\pi} X$ we have the map of sheaves $\pi^*\mathcal{B} \to \mathcal{O}(1)$. The sheaf of ideals of $\mathrm{Proj}(\mathrm{Sym}^*(\mathcal{C})) \subset \mathrm{Proj}(\mathrm{Sym}^*(\mathcal{B}))$ is the image of the composite $\pi^*\mathcal{A} \to \pi^*\mathcal{B} \to \mathcal{O}(1)$.

Let $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow S \times S^{[n,m]}$ denote the projective bundle. It follows that $\mathbb{P}(\tilde{\mathscr{I}}_m) \subset \mathbb{P}(\mathcal{E})$ is the vanishing locus of the composite homomorphism $\pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. As $S\times S^{[n,m]}$ is irreducible, it follows that $\mathbb{P}(\mathcal{E})$ is irreducible of dimension $2m+2+r$. As $\mathbb{P}(\tilde{\mathscr{I}}_m)$ is locally cut out by r equations, it follows that each irreducible component of $\mathbb{P}(\tilde{\mathscr{I}}_m)$ has dimension at least $2m+2$.

Let $i \geqslant 2$. The hypothesis of Lemma 3.3 holds and so we get that $\dim(W_{i,[n,m]}) \leqslant 2m+2-i$. The dimension of the fiber of $\varphi : \mathbb{P}(\tilde{\mathscr{I}}_m) \longrightarrow S \times S^{[n,m]}$ over a point $(p, \xi_n, \xi_m) \in W_{i,[n,m]}$ is $i-1$. Thus,

$$\dim(\varphi^{-1}(W_{i,[n,m]})) \leqslant 2m+2-i+i-1 = 2m+1\,.$$

Let T be an irreducible component of $\mathbb{P}(\tilde{\mathscr{I}}_m)$. As $\dim(T) \geqslant 2m+2$, it follows that T cannot be contained in $\varphi^{-1}(W_{i,[n,m]})$ for any $i \geqslant 2$. Thus, T meets the set $\varphi^{-1}(W_{1,[n,m]})$. Note that $W_{1,[n,m]}$ is the complement of Z_m and so is an open subset of $S \times S^{[n,m]}$. Moreover, it is clear that

$$\varphi : \varphi^{-1}(W_{1,[n,m]}) \longrightarrow W_{1,[n,m]}$$

is an isomorphism. Let $\widetilde{W}_1$ denote the open and irreducible subset $\varphi^{-1}(W_{1,[n,m]})$. It follows that $T \cap \widetilde{W}_1$ is open in T and so is also dense in T. It follows that T is contained in the closure of $\widetilde{W}_1$. Thus, every irreducible component is contained in the closure of $\widetilde{W}_1$. As $\widetilde{W}_1$ is irreducible, so is its closure. It follows that every irreducible component of $\mathbb{P}(\tilde{\mathscr{I}}_m)$ is contained in the closure of $\widetilde{W}_1$. Thus, there is only one irreducible component, that is, $\mathbb{P}(\tilde{\mathscr{I}}_m)$ is irreducible.

We saw in Lemma 2.7 that ψ is surjective. It follows that $S^{[n,m+1]}$ is irreducible. This is a contradiction and so $\mathcal{A}$ is empty. As $\mathbb{P}(\tilde{\mathscr{I}}_m) \cong S^{[n,m,m+1]}$, the above discussion also shows that $S^{[n,m,m+1]}$ is irreducible. The completes the proof. □

4. Irreducibility of $S^{[n,n+1,m]}$

For a tuple of positive integers a, b, c with $a < b < c$, the nested Hilbert scheme $S^{[a,b,c]}$ parametrizes nested closed subschemes $\xi_a \subset \xi_b \subset \xi_c$ of S, where ξ_i is a finite scheme of length i. We have the universal nested family of closed subschemes $Z_c \subset S \times S^{[a,b,c]}$. The closed points of Z_c have the following descriptions.

$$Z_c = \{(p, \xi_a, \xi_b, \xi_c) \in S \times S^{[a,b,c]} \mid p \in \xi_c\}\,.$$

We have the projection map

$$\pi_c : S^{[a,b,c]} \longrightarrow S^{[c]}\,.$$

Let $\mathscr{I}_c$ denote the ideal sheaf of the universal subscheme inside $S \times S^{[c]}$. Consider the map

$$\mathrm{Id}_S \times \pi_c : S \times S^{[a,b,c]} \longrightarrow S \times S^{[c]}\,.$$

Denote the pullback

$$\tilde{\mathscr{I}}_c := (\mathrm{Id}_S \times \pi_c)^*\mathscr{I}_c\,.$$

Consider the projective bundle

$$\varphi : \mathbb{P}(\tilde{\mathscr{I}}_c) \longrightarrow S \times S^{[a,b,c]}\,. \tag{4.1}$$

We define the map $\psi : \mathbb{P}(\tilde{\mathscr{I}_c}) \longrightarrow S^{[a,b,c+1]}$ in the same way as defined in (2.4) in §2. We have the following maps

$$\begin{tikzcd} \mathbb{P}(\tilde{\mathscr{I}_c}) \arrow[r, "\psi"] \arrow[d, "\varphi"'] & S^{[a,b,c+1]} \\ S \times S^{[a,b,c]} & \end{tikzcd} \tag{4.2}$$

The pointwise description of the map ψ is similar to the one given in §2 and is left to the reader. By similar argument as in the proof of Lemma 2.7, we conclude that the map ψ is surjective on closed points.

As in the case of $S^{[n,m]}$, here also we define the subsets $W_{i,[n,n+1,m]}$ in a similar manner. Let $W_{i,[a,b,c]}$ denote the locus in $S \times S^{[a,b,c]}$ where the ideal sheaf $\mathscr{I}_c$ of Z_c is generated by i elements, that is,

$$W_{i,[a,b,c]} := \{(p, \xi_a, \xi_b, \xi_c) \in S \times S^{[a,b,c]} \mid \dim(\mathcal{I}_{\xi_c} \otimes k(p)) = i \}\,.$$

The set $W_{1,[a,b,c]}$ is the complement of the universal family Z_c in $S \times S^{[a,b,c]}$. Define subsets $W_{i,[a,b,c],l'',l',l} \subset W_{i,[a,b,c]}$ as follows.

Definition 4.3. *Let $i \geqslant 2$. Let $W_{i,[a,b,c],l'',l',l} \subset W_{i,[a,b,c]}$ be the subset consisting of points (p, ξ_a, ξ_b, ξ_c) such that* $\mathrm{length}(\mathcal{O}_{\xi_a,p}) = l''$, $\mathrm{length}(\mathcal{O}_{\xi_b,p}) = l'$ *and* $\mathrm{length}(\mathcal{O}_{\xi_c,p}) = l$.

Notice that for the set $W_{i,[a,b,c],l'',l',l}$ to be nonempty we need that $0 \leqslant a-l'' \leqslant b-l' \leqslant c-l$, $0 \leqslant l'' \leqslant l' \leqslant l$ and $1 \leqslant l$. As $i \geqslant 2$, we have that $p \in \mathrm{Supp}(\xi_m)$, which implies that $1 \leqslant l$. Clearly,

$$W_{i,[a,b,c]} = \bigcup_{l,l',l''} W_{i,[a,b,c],l'',l',l}\,. \tag{4.4}$$

Let $p \in S$ denote a closed point. Let $S^{[l'',l',l]}_{p,i}$ denote the subset of $S^{[l'',l',l]}$ consisting of the tuples $(\xi_{l''}, \xi_{l'}, \xi_l)$ satisfying the following two conditions: $\mathrm{Supp}(\xi_l) = \{p\}$ and $\dim(\mathcal{I}_{\xi_l} \otimes k(p)) = i$.

Lemma 4.5. *Let n and m be two positive integers such that $n+1 < m$. Consider triples of integers (l'', l', l) which satisfy the following three conditions*

- $0 \leqslant n - l'' \leqslant n+1-l' \leqslant m-l$,
- $0 \leqslant l'' \leqslant l' \leqslant l$, *and*
- $1 \leqslant l$.

Assume that $S^{[n-l'',n+1-l',m-l]}$ is irreducible of dimension $2(m-l)$ for all such triples. Let $i \geqslant 2$. Then $\dim(W_{i,[n,n+1,m]}) \leqslant 2m+2-i$.

Proof. It suffices to prove that for $i \geqslant 2$, if $W_{i,[n,n+1,m],l'',l',l}$ is nonempty then

$$\dim(W_{i,[n,n+1,m],l'',l',l}) \leqslant 2m+2-i\,.$$

The proof is very similar to the proof of Lemma 3.3 and so we omit some details. Consider $p_1 : W_{i,[n,n+1,m],l'',l',l} \longrightarrow S$ which sends $(p, \xi_n, \xi_{n+1}, \xi_m)$ to p. We find an upper bound for the dimension of the fiber over a closed point $p \in S$. Let U be the open subset $S \setminus \{p\}$. Given a point $(p, \xi_n, \xi_{n+1}, \xi_m) \in p_1^{-1}(p)$, the quotient $\mathcal{O}_{\xi_m} \longrightarrow \mathcal{O}_{\xi_{n+1}}$ gives rise to quotients

$$\mathcal{O}_{\xi_m,p} \longrightarrow \mathcal{O}_{\xi_{n+1},p}\,, \qquad \left(\bigoplus_{q\in U} \mathcal{O}_{\xi_m,q}\right) \longrightarrow \left(\bigoplus_{q\in U} \mathcal{O}_{\xi_{n+1},q}\right)$$

and the quotient $\mathcal{O}_{\xi_{n+1}} \longrightarrow \mathcal{O}_{\xi_n}$ gives rise to quotients

$$\mathcal{O}_{\xi_{n+1},p} \longrightarrow \mathcal{O}_{\xi_n,p}\,, \qquad \left(\bigoplus_{q\in U} \mathcal{O}_{\xi_{n+1},q}\right) \longrightarrow \left(\bigoplus_{q\in U} \mathcal{O}_{\xi_n,q}\right).$$

This gives rise to the following map which is an inclusion on closed points

$$p_1^{-1}(p) \longrightarrow S_{p,i}^{[l'',l',l]} \times U^{[n-l'',n+1-l',m-l]}\,. \tag{4.6}$$

We note that $n+1-l' \geqslant n-l''$, that is, $l' \leqslant l''+1$. As $l'' \leqslant l'$, there are only the following two possibilities: either $l' = l''$ or $l' = l''+1$.

If $l' = l''$ then by our hypothesis $S^{[n-l'',n+1-l',m-l]}$ is irreducible of dimension $2(m-l)$. If $l' = l''+1$ then $S^{[n-l'',n+1-l',m-l]}$ is same as $S^{[n+1-l',m-l]}$, which is irreducible of dimension $2(m-l)$ by Theorem 3.8. So it follows that $\dim U^{[n-l'',n+1-l',m-l]} = 2(m-l)$.

Now we need to find an upper bound of $\dim(S_{p,i}^{[l'',l',l]})$. We have two cases: $l'' = l'-1$ and $l'' = l'$. We first consider the case $l'' = l'-1$. There is a natural map

$$S_{p,i}^{[l'',l',l]} \longrightarrow S_{p,i}^{[l]} \times S_p^{[l'',l']}$$

which is an inclusion on closed points. As $l'' = l'-1$, by [BE16, Corollary 5.9] we have $\dim(S_p^{[l'',l']}) = l'-1$. Also from (3.6), we get $\dim(S_{p,i}^{l}) \leqslant l+1-i$. So it follows that

$$\dim S_{p,i}^{[l'',l',l]} \leqslant (l+1-i) + (l'-1) \leqslant 2l - i\,.$$

This gives

$$\dim(p_1^{-1}(p)) \leqslant 2(m-l) + 2l - i = 2m - i\,.$$

Thus, we get

$$\dim(W_{i,[n,n+1,m],l'',l',l}) \leqslant 2m + 2 - i\,.$$

Next we consider the case $l'' = l'$. In this case $S_{p,i}^{[l'',l',l]}$ is same as $S_{p,i}^{[l',l]}$ which has dimension at most $2l - i$ by (3.7). Thus again we get

$$\dim(W_{i,[n,n+1,m],l'',l',l}) \leqslant 2m + 2 - i\,.$$

This proves the lemma. □

Theorem 4.7. *Let n and m be two positive integers such that $n+1 < m$. Then $S^{[n,n+1,m,m+1]}$ and $S^{[n,n+1,m]}$ is irreducible.*

Proof. We follow the same method as we used in the proof of Theorem 3.8. Let $\mathcal{A}$ be the set of pairs of integers (a, b) with $1 \leqslant a$, $a+1 < b$ and $S^{[a,a+1,b]}$ reducible. Assume that $\mathcal{A}$ is nonempty. By [RT22, Theorem 3.10] for every $a \geqslant 1$ the pair $(a, a+2) \notin \mathcal{A}$. Consider the projection map to the first coordinate $\mathcal{A} \longrightarrow \mathbb{Z}_{\geqslant 1}$. Let n be the smallest integer in the image of this map. Among the set of integers b such that $(n, b) \in \mathcal{A}$, let b_0 be the smallest.

Clearly, $b_0 > n+2$. Let $m = b_0 - 1$. Then $m \geqslant n+2$. We conclude that for all pairs of integers (a,b) with $1 \leqslant a$, $a+1 < b$, if $a < n$ then $S^{[a,a+1,b]}$ is irreducible and $S^{[n,n+1,b]}$ is irreducible if $b \leqslant m$. Further $S^{[n,n+1,m+1]}$ is reducible. Note that if $S^{[a,a+1,b]}$ is irreducible then its dimension is $2b$. A similar argument as in the proof of Theorem 3.8, after replacing Lemma 3.3 with Lemma 4.5, concludes the proof of the Theorem. □

5. Irreducibility of $S^{[n,n+2,m]}$

We begin with the following Lemma.

Lemma 5.1. *Fix integers $1 \leqslant n$ and $n+2 < m$. Consider triples of integers (l'', l', l) which satisfy the following three conditions*

- $0 \leqslant n - l'' \leqslant n+2-l' \leqslant m-l$,
- $0 \leqslant l'' \leqslant l' \leqslant l$, *and*
- $1 \leqslant l$.

Assume that $S^{[n-l'',n+2-l',m-l]}$ is irreducible of dimension $2(m-l)$ for all such triples. Let $i \geqslant 2$. Then $\dim(W_{i,[n,n+2,m]}) \leqslant 2m+2-i$.

Proof. From (4.4), we have,

$$W_{i,[n,n+2,m]} = \bigcup_{l'',l',l} W_{i,[n,n+2,m],l'',l',l}\,.$$

So it suffices to prove that $\dim(W_{i,[n,n+2,m],l'',l',l}) \leqslant 2m+2-i$ for $i \geqslant 2$. Consider the projection $p_1 : W_{i,[n,n+2,m],l'',l',l} \longrightarrow S$ which sends $(p,\xi_n,\xi_{n+2},\xi_m)$ to p. We find an upper bound for the dimension of the fiber over a closed point $p \in S$. Let U be the open subset $S \setminus \{p\}$. Given a point $(p,\xi_n,\xi_{n+2},\xi_m) \in p_1^{-1}(p)$, the quotient $\mathcal{O}_{\xi_m} \longrightarrow \mathcal{O}_{\xi_{n+2}}$ gives rise to quotients

$$\mathcal{O}_{\xi_m,p} \longrightarrow \mathcal{O}_{\xi_{n+2},p}\,, \qquad \left(\bigoplus_{q\in U}\mathcal{O}_{\xi_m,q}\right) \longrightarrow \left(\bigoplus_{q\in U}\mathcal{O}_{\xi_{n+2},q}\right)$$

and the quotient $\mathcal{O}_{\xi_{n+2}} \longrightarrow \mathcal{O}_{\xi_n}$ gives rise to quotients

$$\mathcal{O}_{\xi_{n+2},p} \longrightarrow \mathcal{O}_{\xi_n,p}\,, \qquad \left(\bigoplus_{q\in U}\mathcal{O}_{\xi_{n+2},q}\right) \longrightarrow \left(\bigoplus_{q\in U}\mathcal{O}_{\xi_n,q}\right).$$

This gives rise to the following map which is an inclusion on closed points

$$p_1^{-1}(p) \longrightarrow S_{p,i}^{[l'',l',l]} \times U^{[n-l'',n+2-l',m-l]}\,.$$

We note that $n+2-l' \geqslant n-l''$, that is, $l' \leqslant l''+2$. As $l'' \leqslant l'$, there are only the following three possibilities: $l' = l''$ or $l' = l''+1$ or $l' = l''+2$.

If $l'' = l'$ then by our hypothesis $S^{[n-l'',n+2-l',m-l]}$ is irreducible of dimension $2(m-l)$. If $l'' = l'-1$ then $S^{[n-l'',n+2-l',m-l]}$ is the same as $S^{[n-l'',n+1-l'',m-l]}$, which is irreducible of dimension $2(m-l)$ by Theorem 4.7. If $l'' = l'-2$ then $S^{[n-l'',n+2-l',m-l]}$ is same as $S^{[n+2-l',m-l]}$, which is irreducible of dimension $2(m-l)$ by Theorem 3.8. So it follows that $\dim(U^{[n-l'',n+2-l',m-l]}) = 2(m-l)$.

Now we need to find an upper bound of $\dim(S_{p,i}^{[l'',l',l]})$. We have three cases : $l'' = l' - 2$, $l'' = l' - 1$ and $l'' = l'$. We first consider the cases $l'' = l' - 2$ or $l' - 1$. There is a natural map

$$S_{p,i}^{[l'',l',l]} \longrightarrow S_{p,i}^{[l]} \times S_p^{[l'',l']}$$

which is an inclusion on closed points. If $l'' = l' - 2$ then we use [BE16, Corollary 7.5], and if $l'' = l' - 1$ then we use [BE16, Corollary 5.9], to conclude $\dim(S_p^{[l'',l']}) = l' - 1$. Also from (3.6), we get $\dim(S_{p,i}^{[l]}) \leqslant l + 1 - i$. So it follows that

$$\dim S_{p,i}^{[l'',l',l]} \leqslant (l + 1 - i) + (l' - 1) \leqslant 2l - i\,.$$

This gives

$$\dim(p_1^{-1}(p)) \leqslant 2(m - l) + 2l - i = 2m - i\,.$$

Thus we get

$$\dim(W_{i,[n,n+2,m],l'',l',l}) \leqslant 2m + 2 - i\,.$$

Next we consider the case $l'' = l'$. In this case $S_{p,i}^{[l'',l',l]}$ is same as $S_{p,i}^{[l',l]}$ which has dimension at most $2l - i$ by (3.7). Thus again we get

$$\dim(p_1^{-1}(p)) \leqslant 2(m - l) + 2l - i = 2m - i$$

and hence

$$\dim(W_{i,[n,n+2,m],l'',l',l}) \leqslant 2m + 2 - i\,.$$

This proves the lemma. □

Theorem 5.2. *Let n and m be two positive integers such that $n+2 < m$. Then $S^{[n,n+2,m,m+1]}$ and $S^{[n,n+2,m]}$ are irreducible.*

Proof. We follow the same method as we used in proof of Theorem 3.8. Let $\mathcal{A}$ be the set of pairs of integers (a,b) with $1 \leqslant a$, $a+2 < b$ and $S^{[a,a+2,b]}$ reducible. We prove that $\mathcal{A}$ is empty. Taking $(n,m) = (a,a+2)$ in Theorem 4.7 shows that $S^{[a,a+1,a+2,a+3]}$ is irreducible and so it follows that $S^{[a,a+2,a+3]}$ is irreducible. Thus, it follows that for every $a \geqslant 1$ the pair $(a,a+3) \notin \mathcal{A}$. Consider the projection map to the first coordinate $\mathcal{A} \longrightarrow \mathbb{Z}_{\geqslant 1}$. Let n be the smallest integer such in the image of this map. Among the set of integers b such that $(n,b) \in \mathcal{A}$, let b_0 be the smallest. Clearly, $b_0 > n+3$. Let $m = b_0 - 1$. Then $m \geqslant n+3$. We conclude that for all pairs of integers (a,b) with $1 \leqslant a$, $a+2 < b$, if $a < n$ then $S^{[a,a+2,b]}$ is irreducible, and $S^{[n,n+2,b]}$ is irreducible if $b \leqslant m$. Further $S^{[n,n+2,m+1]}$ is reducible. A similar argument as in the proof of Theorem 3.8, after replacing Lemma 3.3 with Lemma 5.1, concludes the proof of the Theorem. □

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