MEHTA-RAMANATHAN RESTRICTION THEOREMS

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Contents

These notes are based on two lectures at IIT-Bombay by Arjun Paul, on the Restriction Theorems of Mehta-Ramanathan. We follow the treatment in [HL10], which is the main reference for us.

1. Introduction

1.1.Pure sheaves. Let X be a scheme of finite-type over an algebraically closed field k. Let E be a coherent sheaf on X. We define $\dim(E) = \dim(\text{Supp } E)$.

Definition 1.2. *E is called pure if for any non trivial coherent subsheaf $0 \neq F \subset E$, we have $\dim(F) = \dim(E)$. Given a coherent sheaf E, we define $T(E)$ as the maximal subsheaf of E of dimension $\dim(E) - 1$. Then $E/T(E)$ is pure of dimension $\dim(E)$.*

Remark 1.3.

(1) Recall that a coherent sheaf is torsion-free if $\forall\, x \in X$, $E \otimes \mathcal{O}_{X,x}$ is a torsion free $\mathcal{O}_{X,x}$ module. If X is integral, we have that E is torsion free iff E is pure of dimension $\dim(X)$.

(2) The pure sheaves we will encounter in these notes will always be of dimension $\dim X$. So from here onwards, by "pure sheaves" we mean "pure sheaves of $\dim(X)$". Also whenever the underlying scheme is integral, we will use "torsion free" sheaves and "pure sheaves" interchangeably.

1.4.Semistability. Let X be a projective scheme over an algebraically closed field k of dimension n. Fix a very ample line bundle $\mathcal{O}_X(1)$ on X. For a coherent sheaf E on X we denote by $P(E,t)$ the Hilbert polynomial of E with respect to $\mathcal{O}_X(1)$. Let $P(E,t) = \sum_i \alpha_i(E)\frac{t^i}{i!}$ with $\alpha_i(E) \in \mathbb{Q}$. Recall that if $d = \dim(\mathrm{Supp}(E))$ then deg $P(E,t) = d$. In other words, $\alpha_i(E) = 0 \ \forall\ i > d$ and $\alpha_d(E) \neq 0$.

Definition 1.5. *If $d = \dim(\mathrm{Supp}\ E)$, then the reduced Hilbert polynomial of E is defined as $p(E,t) := \dfrac{P(E,t)}{\alpha_d(E)}$.*

For two polynomials $p_1 = \sum_i \alpha_i \frac{t^i}{i!}$ and $p_2 = \sum_i \beta_i \frac{t^i}{i!}$ in $\mathbb{Q}[t]$, we say $p_1 > p_2$ if there exists $j \geq 0$ such that for all $i > j$ we have $\alpha_i = \beta_i$ and $\alpha_j > \beta_j$. This is same as saying that $p_1(n) > p_2(n)$ for $n \gg 0$.

Definition 1.6. *A coherent sheaf E on X is called semistable (respectively, stable) with respect to $\mathcal{O}_X(1)$ if*

(1) E is pure of dimension X.
(2) for all proper subsheaves $0 \neq F \subset E$ we have $p(F) \leq p(E)$ (respectively, $p(F) < p(E)$).

Theorem 1.7. [HL10, Theorem 1.3.4] *Let E be pure of dimension X. Then there exists a fitration of E called the Harder-Narasimhan filtration(or HN filtration)*

$$0 = HN_0(E) \subsetneq HN_1(E) \subsetneq \ldots \subsetneq HN_{e-1}(E) \subsetneq HN_e(E) = E$$

satisfing the following two properties:

(1) Each $HN_i(E)/HN_{i-1}(E)$ is semistable,
(2) Let $p_i(E) = p(HN_i(E)/HN_{i-1}(E))$. Then $p_1(E) > p_2(E) > \ldots > p_e(E)$.

Moreover the above filtration is uniquely determined by the above two properties.

Next we state a relative version of Theorem 1.7. First we fix some notations. Let $f : Y \to S$ be a projective morphism of finite type k-schemes. For any morphism $g : T \to S$ we will denote $Y \times_S T$ by Y_T. If $\mathcal{E}$ is a sheaf over Y then the sheaf $(f \times_S g)^*\mathcal{E}$ on Y_T will be denoted by $\mathcal{E}_T$.

Theorem 1.8. [HL10, Theorem 2.3.2] *Let S be an integral finite type scheme over k. Let $f : Y \to S$ be a projective morphism with a f-very ample line bundle $\mathcal{O}_Y(1)$. Let $\mathcal{E}$ be a coherent sheaf on Y which is flat over S. Assume that there is a closed point $s \in S$ such that $\mathcal{E}_s$ is pure of dimension Y_s. Then there is a non-empty open set $U \subset S$ and a filtration over Y_U*

$$0 = HN_0(\mathcal{E}_U) \subsetneq HN_1(\mathcal{E}_U) \subsetneq \ldots \subsetneq HN_{e-1}(\mathcal{E}_U) \subsetneq HN_e(\mathcal{E}_U) = \mathcal{E}_U$$

such that

(1) $HN_i(\mathcal{E})/HN_{i-1}(\mathcal{E}_U)$ *is flat over* U.
(2) $\forall\ s \in U$, $\mathcal{E}_s$ *is pure of dimension* Y_s
(3) The filtration

$$0 = HN_0(\mathcal{E}_U)_s \subsetneq HN_1(\mathcal{E}_U)_s \subsetneq \ldots \subsetneq HN_{e-1}(\mathcal{E}_U)_s \subsetneq HN_e(\mathcal{E}_U)_s = \mathcal{E}_s$$

is the HN-filtration of $\mathcal{E}_s$.

1.9.μ-semistability. We have the following invariants associated to E.

Definition 1.10. *We define*

(1) the rank of E *as* $\mathrm{rk}(E) := \dfrac{\alpha_n(E)}{\alpha_n(\mathcal{O}_X)}$.
(2) the degree of E *as* $\deg(E) := \alpha_{n-1}(E) - rk(E).\alpha_{n-1}(\mathcal{O}_X)$.
(3) if $\mathrm{rk}\ E \neq 0$, *the slope of* E *as* $\mu(E) := \dfrac{\deg\ E}{\mathrm{rk}\ E}$.

Remark 1.11.

(1) If X is integral then $\mathrm{rk}(E)$ is nothing but the rank of the vector space E_η over $k(\eta)$, where η is the generic point of X.
(2) If X is smooth outside a closed subset of codimension ≥ 2 then $\deg(E) = \deg(\det(E)) = ([\det(E)] \cdot [\mathcal{O}_X(1)]^{n-1})$.

Definition 1.12. *Let* E *be a coherent sheaf on* X. *We say that it is* μ-*semistable (respectively, stable) with respect to* $\mathcal{O}_X(1)$ *if*

(1) It is pure of dimension X,
(2) For any subsheaf $0 \neq F \subset E$ *with* $rk(F) < rk(E)$, *we have* $\mu(F) \leq \mu(E)$ *(respectively,* $\mu(F) < \mu(E)$*).*

Lemma 1.13. E *is* μ-*semistable iff for any pure quotient* $E \to G \to 0$ *we have* $\mu(E) \leq \mu(G)$.

Proof. Let us consider an exact sequence

$$0 \to F \to E \to G \to 0\,.$$

Suppose $\mathrm{rk}\ F, \mathrm{rk}\ G > 0$. Since deg and rk are additive, we have

$$\mathrm{rk}(G)(\mu(E) - \mu(G)) = \mathrm{rk}(F)(\mu(F) - \mu(E))\,. \tag{1.14}$$

If E is μ-semistable, then the RHS is ≤ 0 and so it follows that $\mu(E) \leq \mu(G)$.

Suppose that for any pure quotient $E \to G \to 0$ we have $\mu(E) \leq \mu(G)$. Let $F \subset E$ and its quotient be G. If $\mathrm{rk}(G) = 0$, it is immediate from the definition of degree that $\deg(G) \geq 0$. Therefore, since $\mathrm{rk}(F) = \mathrm{rk}(E)$ and $\deg(F) = \deg(E) - \deg(G) \leq \deg(E)$, it follows that $\mu(F) \leq \mu(E)$. Now suppose $\mathrm{rk}(G) > 0$. Consider the surjection $E \to G \to G/T(G)$, where $T(G)$ is the torsion subsheaf of G. Then $\mu(E) \leq \mu(G/T(G))$. Since $\deg(T(G)) \geq 0$ we have $\mu(E) \leq \mu(G/T(G)) \leq \mu(G)$. Again using 1.14 we have $\mu(F) \leq \mu(E)$. □

Lemma 1.15. *Let* E *and* G *are* μ-*semistable sheaves with* $\mu(E) > \mu(G)$. *Then* $Hom(E, G) = 0$.

Proof. Let $f : E \to G$ be a non-trivial morphism. Then by Lemma 1.13, $\mu(E) \leq \mu(f(E)) \leq \mu(G)$. Hence we arrive at a contradiction. □

The following Lemma will be needed in the proof of Theorem 1.17.

Lemma 1.16. *Let E be μ-semistable. Then the slopes $\mu(HN_i(E)/HN_{i-1}(E))$ occurring in the HN filtration of E are all equal to $\mu(E)$. The slopes $\mu(HN_i(E))$ are also equal to the slope $\mu(E)$.*

Proof. Since E is μ-semistable, from the inclusion $HN_1(E) \subset E$ we get that $\mu(HN_1(E)) \leq \mu(E)$. Similarly, from the quotient $E \to E/HN_{e-1}(E)$ we get $\mu(E) \leq \mu(E/HN_{e-1}(E))$. This shows that $\mu(HN_1(E)) \leq \mu(E/HN_{e-1}(E))$. If we had $\mu(HN_1(E)) < \mu(E/HN_{e-1}(E))$ then this would mean that $p_1(E) < p_e(E)$, which is not possible. Thus, $\mu(HN_1(E)) = \mu(E/HN_{e-1}(E))$. From the exact sequence $0 \to HN_{e-1}(E) \to E \to E/HN_{e-1}(E) \to 0$ it follows that $HN_{e-1}(E)$ is pure of dimension X and μ-semistable with slope $\mu(E)$. We proceed in the same way replacing E with $HN_{e-1}(E)$. □

Just as in Theorem 1.7, we have the μ-Harder Narasimhan filtration:

Theorem 1.17. *Let E be a pure sheaf on X. Then there exists a fitration of E called the μ-Harder-Narasimhan filtration (or μ-HN filtration)*

$$0 = \mu\text{-}HN_0(E) \subsetneq \mu\text{-}HN_1(E) \subsetneq \ldots \subsetneq \mu\text{-}HN_{l-1}(E) \subsetneq \mu\text{-}HN_l(E) = E$$

satisfing the following two properties:

(1) Each μ-$HN_i(E)/\mu$-$HN_{i-1}(E)$ is μ-semistable
(2) Let $\mu_i(E) = \mu(\mu\text{-}HN_i(E)/\mu\text{-}HN_{i-1}(E))$. Then $\mu_1(E) > \mu_2(E) > \ldots > \mu_l(E)$.

Moreover the above filtration is uniquely determined by the above two properties.

Proof. Let

$$0 = HN_0(E) \subsetneq HN_1(E) \subsetneq \ldots \subsetneq HN_{l-1}(E) \subsetneq HN_e(E) = E$$

be the HN-filtration of E. Let ν_i denote the slope of $HN_i(E)/HN_{i-1}(E)$. Then we have

$$\nu_1 \geq \nu_2 \geq \ldots \geq \nu_e\,.$$

Let $i_1 > i_2 > \ldots$ be the indices where there is a strict drop in the slope, that is,

$$\nu_{i_j} > \nu_{i_j+1}\,.$$

Define the μ-HN filtration of E as

$$\mu\text{-}HN_j(E) := HN_{i_j}(E)\,.$$

Then μ-HN_j/μ-HN_{j-1} is μ-semistable with $\mu(\mu\text{-}HN_j/\mu\text{-}HN_{j-1}) = \nu_{i_j}$. This follows easily using the HN-filtration and the fact that if $0 \to E_1 \to E \to E_2 \to 0$ is a short exact sequence of μ-semistable sheaves such that E_1 and E_2 have the same slope, then E is also μ-semistable of same slope.

Suppose we are given a filtration $F_1 \subset F_2 \subset \ldots \subset F_r = E$ which satisfies

(1) F_i/F_{i-1} is μ-semistable, and
(2) $\mu(F_1) > \mu(F_2/F_1) > \mu(F_3/F_2) > \dots$

Consider the HN filtration of F_i and F_{i-1}. Let the length of the HN filtration of F_i and F_{i-1} be f_i and f_{i-1} respectively. By the previous lemma we have

$$\mu(F_{i-1}/HN_{f_{i-1}-1}(F_{i-1})) = \mu(F_{i-1}/F_{i-2}) > \mu(F_i/F_{i-1}) = \mu(HN_1(F_i/F_{i-1})).$$

This implies that $p(F_{i-1}/HN_{f_{i-1}-1}(F_{i-1})) > p(HN_1(F_i/F_{i-1}))$. This shows that we can "put together" the HN filtrations of F_i/F_{i-1}. Let $g_i : F_i \to F_i/F_{i-1}$ and let $HN_\bullet(F_i/F_{i-1})$ denote the HN filtration of F_i/F_{i-1}. Then we have a filtration

$$F_{i-1} \subset g_i^{-1}(HN_1(F_i/F_{i-1})) \subset g_i^{-1}(HN_2(F_i/F_{i-1})) \subset \dots \subset F_i$$

Putting these together, it is clear that the graded pieces are semistable and that the reduced Hilbert polynomials satisfy the strictly decreasing condition. Thus, this is the HN filtration of E. Now it is clear that the F_i are precisely the places where the slope strictly drops. □

Theorem 1.18. *Let S be an integral finite type scheme over k. Let $f : Y \to S$ be a projective morphism with a f-very ample line bundle $\mathcal{O}_Y(1)$. Let $\mathcal{F}$ be a coherent sheaf on Y which is flat over S. Assume that there is a closed point $s \in S$ such that $\mathcal{F}_s$ is pure of dimension Y_s. Then exists an non-empty open set $U \subset S$ and a filtration over $f^{-1}(U)$*

$$0 = \mu\text{-}HN_0(\mathcal{F}_U) \subsetneq \mu\text{-}HN_1(\mathcal{F}_U) \subsetneq \dots \subsetneq \mu\text{-}HN_{l-1}(\mathcal{F}_U) \subsetneq \mu\text{-}HN_l(\mathcal{F}_U) = \mathcal{F}_U$$

such that

(1) $\mu\text{-}HN_i(\mathcal{F})/\mu\text{-}HN_{i-1}(\mathcal{F}_U)$ is flat over U.
(2) $\forall\ s \in U$, $\mathcal{F}_s$ is torsion free.
(3) The filtration

$$0 = \mu\text{-}HN_0(\mathcal{F}_U)_s \subsetneq \mu\text{-}HN_1(\mathcal{F}_U)_s \subsetneq \dots \subsetneq \mu\text{-}HN_{l-1}(\mathcal{F}_U)_s \subsetneq \mu\text{-}HN_l(\mathcal{F}_U)_s = \mathcal{F}_s$$

is the HN-filtration of $\mathcal{F}_s$.

Proof. As we saw in the proof of 1.17, the HN filtration of a torsion-free coherent sheaf E is a refinement of the μ-HN filtration of E. Hence the statement follows immediately from Theorem 1.8. □

1.19.μ-Minimal destabilising quotient.

We define $E/\mu\text{-}HN_{l(E)-1}E$ to be the **μ-minimal destabilising quotient** of E. Note that $E/\mu\text{-}HN_{l(E)-1}E$ is μ-semistable and $\mu(E/\mu\text{-}HN_{l(E)-1}E) \leq \mu(E)$. We define $\mu_{\min}(E) = \mu_{l(E)}$ and $\mu_{\max} = \mu_1(E)$.

Lemma 1.20. *Let E, G be two pure sheaves. Let $\mu_{\min}(E) > \mu_{\max}(G)$. Then $Hom(E, G) = 0$.*

Proof. Let $f : E \to G$ be a non-trivial morphism. Let i be such that $f(\mu\text{-}HN_i(E)) = 0$ and $f(\mu\text{-}HN_{i+1}(E)) \neq 0$. Let j be such that $f(\mu\text{-}HN_{i+1}(E)) \nsubseteq$

μ-$HN_j(G)$ and $f(\mu\text{-}HN_{i+1}(E)) \subseteq \mu\text{-}HN_{j+1}(G)$. Then we have a non-trivial map

$$\mu\text{-}HN_{i+1}(E)/\mu\text{-}HN_i(E) \to \mu\text{-}HN_{j+1}(G)/\mu\text{-}HN_j(G)$$

Now both μ-$HN_{i+1}(E)/\mu$-$HN_i(E)$ and μ-$HN_{j+1}(G)/\mu$-$HN_j(G)$ are μ-semistable sheaves of slope $\mu_{i+1}(E)$ and $\mu_{j+1}(G)$ respectively and by assumption we have

$$\mu_{i+1}(E) \geq \mu_{\min}(E) > \mu_{\max}(G) \geq \mu_{j+1}(G)\,.$$

By Lemma 1.15 this morphism is zero and we arrive at a contradiction. □

Theorem 1.21. *Let $E \to G \to 0$ be a quotient such that G is pure. Then $\mu_{\min}(E) \leq \mu(G)$. If $\mu(G) = \mu_{\min}(E)$ then $E \to G$ factors as*

$$E \to E/\mu\text{-}HN_{l(E)-1}(E) \to G\,.$$

Proof. Let us suppose the contrary, that is,

$$\mu(G) < \mu_{\min}(E) = \mu(E/\mu\text{-}HN_{l(E)-1}(E))\,.$$

Replacing G by $G/\mu - HN_{l(G)-1}(G)$, (since $\mu(G/\mu\text{-}HN_{l(G)-1}G) \leq \mu(G)$) we may assume G is μ-semistable. Consider the composition

$$\mu\text{-}HN_{l(E)-1}(E) \to E \to G\,.$$

Note that

$$\mu_{\min}(\mu\text{-}HN_{l(E)-1}(E)) = \mu_{l-1}(E) > \mu_l(E) = \mu_{\min}(E) > \mu(G) = \mu_{\max}(G)\,.$$

By Lemma 1.20 this composition is zero. Therefore there is a surjection

$$E/\mu\text{-}HN_{l(E)-1}(E) \to G \to 0,$$

which implies that $\mu_{\min}(E) \leq \mu(G)$. This is a contradiction.

Now suppose $\mu_{\min}(E) = \mu(G)$. If $G \to G'$ is any quotient then applying the first part of this theorem we have $\mu(G') \geq \mu_{\min}(E) = \mu(G)$. This implies G is μ-semistable. Now consider the composition

$$\mu\text{-}HN_{l(E)-1}(E) \to E \to G\,.$$

Note that

$$\mu_{\min}(\mu\text{-}HN_{l(E)-1}(E)) = \mu_{l(E)-1} > \mu_{l(E)} = \mu_{\min}(E) = \mu(G) = \mu_{\max}(G)\,.$$

By Lemma 1.20 we have that the above compostion is zero. Hence $E \to G$ factors as

$$E \to E/\mu\text{-}HN_{l(E)-1}(E) \to G\,.$$

□

Corollary 1.22. *Assume X is smooth. Let $E \to E_1$ be the μ-minimal destabilising quotient. Let $E \to G \to 0$ be such that $\mu_{\min}(E) = \mu(G)$ and* rk $G =$ rk E_1. *Then $E_1 \cong G$ outside a closed subset of codimension ≥ 2.*

Proof. Let us consider the surjection $E \to G \to G/T(G)$. Applying the above theorem, we have that $\mu(G/T(G)) \geq \mu(E_1)$. On the other hand, we get that $\mu(G/T(G)) \leq \mu(G)$ since $T(G)$ is torsion. Hence we have

$$\mu(E_1) = \mu(G) = \mu(G/T(G)) \,.$$

In particular, this implies that codimension of $\mathrm{Supp}(T(G)) \geq 2$. Again applying previous theorem, we have

$$E \to G/T(G)$$

factors as $E \twoheadrightarrow E_1 \twoheadrightarrow G/T(G)$. Since both E_1 and $G/T(G)$ are torsion free sheaves of same rank and X is smooth, we have that this is an isomorphism outside a closed subset of codimension ≥ 2 . □

2. A degeneration argument

Define $\Pi_a := \mathbb{P}(H^0(X, \mathcal{O}_X(a))^\vee)$. Recall that associated to Π_a we have a closed subscheme $Z_a \hookrightarrow X \times \Pi_a$ called the incidence variety which has the following property: if the $p : Z_a \to \Pi_a$ and $q : Z_a \to X$ are the projections, then the fibre $p^{-1}([D]) = D \hookrightarrow X$. To define it in a more formal manner, let $p_1 : X \times \Pi_a \to \Pi_a$ and $q_1 : X \times \Pi_a \to X$ be the two projections. Then Z_a is defined as the zero scheme of the composition

$$p_1^*\mathcal{O}_{\Pi_a}(-1) \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_{X\times\Pi_a} \to q_1^*\mathcal{O}_X(a) \tag{2.1}$$

It is clear that Z_a has the above property. Define $K := \ker(H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_X \to \mathcal{O}_X(a))$.

Lemma 2.2. *$Z_a \xrightarrow{q} X$ is isomorphic to the projective bundle $\mathbb{P}(K^\vee) \to X$. In particular it is smooth and integral.*

Proof. Let $Y := \mathbb{P}(K^\vee)$. Note that $K \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_X$ induces a closed immersion $Y \hookrightarrow X \times \Pi_a$. By the definition of this closed immersion the morphism

$$p_1^*\mathcal{O}_{\Pi_a}(-1)|_Y \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_{X\times\Pi_a}|_Y$$

factors as

$$p_1^*\mathcal{O}_{\Pi_a}(-1)|_Y \hookrightarrow q_1^*K|_Y \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_{X\times\Pi_a}|_Y$$

Hence the composition 2.1 restricted to Y is zero. Hence $Y \hookrightarrow Z_a$.

By definition the following composition is zero.

$$p_1^*\mathcal{O}_{\Pi_a}(-1)|_{Z_a} \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_{Z_a} \to q_1^*\mathcal{O}_X(a)|_{Z_a}$$

Hence $p_1^*\mathcal{O}_{\Pi_a}(-1)|_{Z_a} \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_{Z_a}$ factors as

$$p_1^*\mathcal{O}_{\Pi_a}(-1)|_{Z_a} \hookrightarrow q_1^*K \hookrightarrow H^0(X, \mathcal{O}_X(a)) \otimes \mathcal{O}_{Z_a}$$

Hence we have a surjection $q_1^*K^\vee|_{Z_a} \to p_1^*\mathcal{O}_{\Pi_a}(1)|_{Z_a} \to 0$ and this defines a map $Z_a \to Y$ over X. It is easy to see that this is the inverse of $Y \to Z_a$. □

Definition 2.3 (Conormal sheaf)**.** *Let $W \hookrightarrow Z$ be closed immersion of finite type schemes over k. Then we define the conormal sheaf $\mathcal{C}_{W/Z} := \mathcal{I}_{W/Z}/\mathcal{I}^2_{W/Z}$, where $\mathcal{I}_{W/Z}$ is the ideal sheaf of W in Z.*

Remark 2.4. The cotangent sheaf Ω_W of W and the conormal sheaf $\mathcal{C}_{W/Z}$ are related by the right exact sequence (Tag 01UZ):

$$\mathcal{C}_{W/Z} \to \Omega_Z|_W \to \Omega_W \to 0$$

It is standard that if Z is smooth then W is smooth iff the sequence is left exact. (One may deduce from this the following more general statement which we will not need. If W is smooth then the above sequence is left exact (Tag 01UZ).) In particular, if $i : C \hookrightarrow \Pi_a$ is a smooth closed curve, then the above sequence is left exact. From the exact sequence it also follows that $\mathcal{C}_{C/\Pi_a}$ is locally free. Let Z_C denote the scheme theoretic inverse image of C. We wish to conclude something about the smooth locus of Z_C.

Lemma 2.5. *Let $C \subset U \subset \Pi_a$ be a closed immersion of a smooth curve into an open subset of Π_a. Let $z \in Z_C$ be a closed point. Then Z_C is smooth at z iff the composition*

$$\mathcal{C}_{C/U}|_{p(z)} \to \Omega_U|_{p(z)} \to \Omega_Z|_z$$

is injective.

Proof. Recall that Z_C is smooth at z iff $\dim \Omega_{Z_C}|_z = \dim(\mathfrak{m}_{Z_C,z}/\mathfrak{m}^2_{Z_C,z}) = \dim \mathcal{O}_{Z_C,z}$. Since p is flat, Z_C is equidimensional of dimension n. Hence $\dim \mathcal{O}_{Z_C,z} = n$ and therefore Z_C is smooth at z iff $\dim \Omega_{Z_C}(z) = n$. Recall that we have we have a commutative diagram whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & p^*\mathcal{C}_{C/U} \arrow[r] \arrow[d, "\sim"] & p^*\Omega_U \arrow[r] \arrow[d] & p^*\Omega_C \arrow[r] \arrow[d] & 0 \\
 & \mathcal{C}_{Z_C/Z} \arrow[r] & \Omega_Z \arrow[r] & \Omega_{Z_C} \arrow[r] & 0
\end{tikzcd}$$

Since p is flat we have $\mathcal{C}_{Z_C/Z} \cong p^*\mathcal{C}_{C/U}$. Restricting to z (and using the top row is a sequence of locally free sheaves), we get

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_{C/U}|_{p(z)} \arrow[r] \arrow[d, "="] & \Omega_U|_{p(z)} \arrow[r] \arrow[d] & \Omega_C|_{p(z)} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{C}_{C/U}|_z \arrow[r] & \Omega_Z|_z \arrow[r] & \Omega_{Z_C}|_z \arrow[r] & 0
\end{tikzcd}$$

Comparing the dimensions, we get that Z_C is smooth at z iff under the map $\Omega_U|_{p(z)} \to \Omega_Z|_z$ the subspace $\mathcal{C}_{C/U}|_{p(z)}$ maps injectively. $\square$

Corollary 2.6. *Assume the hypothesis of Lemma 2.5. If z is a smooth point of the fibre $p^{-1}(p(z))$, then Z_C is smooth at z.*

Proof. Let $c = p(z) \in C$ and let Z_c denote the fiber over c. We have the exact sequence:

$$\Omega_U|_{p(z)} \to \Omega_Z|_z \to \Omega_{Z_c}|_z \to 0\,.$$

Since Z_c is equidimensional of dimension $n-1$ and smooth at z,

$$\dim\, \Omega_{Z_c}|_z = n-1\,.$$

Hence, the morphism $\Omega_U|_{p(z)} \to \Omega|_z$ itself is injective, so the statement follows from Lemma 2.5. □

For this section, let $U_a \subset \Pi_a$ be a non-empty subset such that for each point $[D] \in U_a$, the divisor D is smooth. By [Har77, Chapter 3, Corollary 7.9] each such divisor is connected, and hence integral.

Lemma 2.7. *Let $D_1 \in U_{a_1}, D_2 \in U_{a_2}$ be such that $D := D_1 + D_2$ is a SNC divisor of degree $a := a_1 + a_2$. Then $\exists$ a smooth (non-proper) curve $C \hookrightarrow \Pi_a$ such that $[D] \in C$, $C \setminus [D] \subset U_a$ and* $\operatorname{codim}(\operatorname{Sing}(Z_C), Z_C) \geq 3$.

Proof. We will in fact show that C can be chosen to be in an open subset of lines passing through $[D]$. There is a bijection between the set of lines through $[D] \in \Pi_a$ with the one dimensional subspaces in the tangent space $T_{\Pi_a}|_{[D]}$, which is in bijection with the hyperplanes in $\Omega_{\Pi_a}|_{[D]}$, which is in bijection with the closed points in $\mathbb{P}(\Omega_{\Pi_a}|_{[D]})$. Moreover, the set of hyperplanes H such that the corresponding line through $[D]$ intersects U_a, corresponds to a non-empty open subset of $\mathbb{P}(\Omega_{\Pi_a}|_{[D]})$.

Now let L be any line in Π_a passing through $[D]$ and intersecting U_a. We will denote the corresponding hyperplane in $\Omega_{\Pi_a}|_{[D]}$ by H. Define $C := (L \cap U_a) \cup \{[D]\}$. For $c \in C$, choosing a small neighbourhood U around c and applying Corollary 2.6 to the restriction $p : Z_U \setminus (D_1 \cap D_2) \to U$, we get that if $z \in Z_C$ and $z \notin D_1 \cap D_2$, then Z_C is smooth at z. So $\operatorname{Sing}(Z_C) \subset D_1 \cap D_2$.

Let $z \in D_1 \cap D_2$. Then $p(z) = [D]$ and z is not a smooth point of D and hence the following exact sequence is not left exact:

$$\Omega_{\Pi_a}|_{[D]} \to \Omega_Z|_z \to \Omega_D|_z \to 0\,. \tag{2.8}$$

However, $\Omega_D|_z$ being a quotient of $\Omega_X|_z$ has dimension n or $n-1$. Since D is not smooth at z we have $\dim\, \Omega_D|_z = n$. Therefore

$$\dim\, \operatorname{Im}(\Omega_{\Pi_a}|_{[D]} \to \Omega_Z|_z) = \dim\, Z - n = \dim\, \Pi_a - 1 \tag{2.9}$$

hence we get that the kernel of the morphism $\Omega_{\Pi_a}|_{[D]} \to \Omega_Z|_z$ has dimension 1. Observe that $\mathcal{C}_{C/U}|_{[D]} \subset \Omega_{\Pi_a}|_{[D]}$ is precisely the subspace H. By Lemma 2.5, it is enough to find a hyperplane $H \subset \Omega_{\Pi_a}|_{[D]}$ and points z in each component of $D_1 \cap D_2$ for which H does not contain the kernel of $\Omega_{\Pi_a}|_{[D]} \to \Omega_Z|_z$. The corresponding line $L \subset \Pi_a$ will satisfy the required property.

Now consider the (set theoretically given) morphism

$$D_1 \cap D_2 \to \mathbb{P}(\Omega_{\Pi_a}|_{[D]}^{\vee})$$

with

$$z \mapsto \operatorname{Ker}(\Omega_{\Pi_a}|_{[D]} \to \Omega_Z|_z)\,.$$

More precisely, consider equation (2.8) on $D_1 \cap D_2$. We get an exact sequence

$$0 \to \mathcal{K} \to \Omega_{\Pi_a}|_{[D]} \otimes \mathcal{O}_{D_1 \cap D_2} \to \Omega_Z|_{D_1 \cap D_2} \to \Omega_D|_{D_1 \cap D_2} \to 0\,.$$

Since $D_1 \cap D_2$ is smooth and the rank of $\Omega_D|_{D_1 \cap D_2}$ is constant at all closed points, it follows that this is locally free. It follows that all sheaves in the above are locally free and that $\mathcal{K}$ is a line bundle on $D_1 \cap D_2$. Taking dual we get a line bundle quotient of $\Omega_{\Pi_a}|_{[D]}^{\vee}$ which defines the above morphism.

Since $\mathcal{O}_X(1)$ is very ample, $\dim\, \Pi_a \geq n+1$. Therefore

$$\dim\, \mathbb{P}(\Omega_{\Pi_a}([D])^{\vee}) \geq n > n-2 = \dim\, (D_1 \cap D_2)$$

For each irreducible component of $D_1 \cap D_2$, fix a closed point. Hence, we can find a hyperplane $H \subset \mathbb{P}(\Omega_{\Pi_a}([D])^{\vee})$ such that H does not contain the images of these points. Finding such a hyperplane in $\mathbb{P}(\Omega_{\Pi_a}([D])^{\vee})$ is equivalent to finding a hyperplane in $\Omega_{\Pi_a}([D])$ having the required property.

This proves that $\mathrm{Sing}(Z_C)$ does not contain any irreducible component of $D_1 \cap D_2$, which shows that $\mathrm{codim}(\mathrm{Sing}(Z_C), Z_C) \geq 3$. □

Although this is not required in the proof of the restriction theorems, we mention some properties of the scheme Z_C.

Lemma 2.10. *The scheme Z_C is irreducible, Cohen-Macaulay, integral and normal.*

Proof. Since $Z_C \to C$ is flat and proper, every irreducible component of Z_C will map surjectively onto C. Since the general fiber of this map is irreducible, it follows that Z_C is irreducible. By Corollary to [Mat86, Theorem 23.3] we have that Z_C is Cohen-Macauley. Thus, it satisfies Serre's condition S_2. Also since $\mathrm{codim}(\mathrm{Sing}(Z_C), Z_C) \geq 3$ it satisfies Serre's condition R_1. Hence, Z_C is an integral and normal scheme. □

For the next two lemmas we fix a smooth curve $C \subset \Pi_a$ as in Lemma 2.7.

Lemma 2.11. *Let $q^*E|_{Z_{C\setminus[D]}} \to G_{C\setminus[D]} \to 0$ be a quotient over $Z_{C\setminus[D]}$ such that $G_{C\setminus[D]}$ is flat over $C \setminus [D]$. Then this quotient extends uniquely to a quotient $q^*E|_{Z_C} \to G_C \to 0$ over Z_C such that G_C is flat over C.*

Proof. The quotient

$$q^*E|_{Z_{C\setminus[D]}} \to G_{C\setminus[D]} \to 0$$

induces a map $C \setminus [D] \to \mathrm{Quot}_{Z_a/\Pi_a}(q^*E, P)$ where P is the Hilbert polynomial of $G_{C\setminus[D]}|_{D'}$ for $[D'] \in U_a$. Since C is smooth and $\mathrm{Quot}_{Z_a/\Pi_a}(q^*E, P)$ is proper, this map extends and we get a flat quotient $q^*E|_C \to G_C$ over C. □

Let G_C be a coherent sheaf over Z_C which is flat over C. Since G_C is flat, we have that $\forall [D'] \in C$ the polynomial $P(G_C|_{D'})$ is independent of D'. In particular, the rank and degree are independent of D'. We denote this rank by r. Define $G := G_C|_D$, $\bar{G} := G/T(G)$ and $\bar{G}_{D_i} := \bar{G}|_{D_i}$. Then we have the following lemma.

Lemma 2.12. *Assume $r \neq 0$. Then we have*

(1) $r = \mathrm{rk}(\bar{G}) = \mathrm{rk}(\bar{G}_{D_1}/T(\bar{G}_{D_1})) = \mathrm{rk}(\bar{G}_{D_2}/T(\bar{G}_{D_2}))$
(2) $\mu(G) \geq \mu(\bar{G}) \geq \mu(\bar{G}_{D_1}/T(\bar{G}_{D_1})) + \mu(\bar{G}_{D_2}/T(\bar{G}_{D_2}))$
(3) Assume that $\mu(G) = \mu(\bar{G}_{D_1}/T(\bar{G}_{D_1})) + \mu(\bar{G}_{D_2}/T(\bar{G}_{D_2}))$. *Let* $U \subset Z_C^{\mathrm{reg}}$ *denote the open subset over which* G_C *is locally free of rank* r. *There is a closed subset* $D_s \subset D$ *such that* $\mathrm{codim}(D_s, D) \geq 2$ *and* $D \setminus D_s \subset U$.

Proof. Proof of (1). By definition $\dim(T(G)) \leq n-2$. Therefore $\alpha_{n-1}(G) = \alpha_{n-1}(\bar{G})$. Hence

$$r = \mathrm{rk}(G) = \frac{\alpha_{n-1}(G)}{\alpha_{n-1}(\mathcal{O}_D)} = \frac{\alpha_{n-1}(\bar{G})}{\alpha_{n-1}(\mathcal{O}_D)} \tag{2.13}$$

and $\mu = \mu(G) \geq \mu(\bar{G})$. We first relate the rank and degree of $\bar{G}$ with the rank and degree of $\bar{G}_{D_1}$ and $\bar{G}_{D_2}$. Consider the exact sequence:

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \to \mathcal{O}_{D_1 \cap D_2} \to 0$$

Tensoring with $\bar{G}$ we get

$$0 \to \bar{K} \to \bar{G} \to \bar{G}_{D_1} \oplus \bar{G}_{D_2} \to \bar{G}_{D_1 \cap D_2} \to 0\,.$$

Notice that when we restrict this to the open subset $D_1 \setminus D_2$ we see that $\bar{K}|_{D_1 \setminus D_2} = 0$. Similarly, for the other open set $D_2 \setminus D_1$. This shows that $\bar{K}$ is supported on a closed subset of dimension $\leq n-2$. But since $\bar{G}$ is pure of dimension $n-1$, it follows that $\bar{K} = 0$ and the following sequence is exact

$$0 \to \bar{G} \to \bar{G}_{D_1} \oplus \bar{G}_{D_2} \to \bar{G}_{D_1 \cap D_2} \to 0$$

is exact on D.

Therefore we get that

$$P(\bar{G}) = P(\bar{G}_{D_1}) + P(\bar{G}_{D_2}) - P(\bar{G}_{D_1 \cap D_2})\,.$$

From this we get that

$$\alpha_{n-1}(\bar{G}) = \alpha_{n-1}(\bar{G}_{D_1}) + \alpha_{n-1}(\bar{G}_{D_2})\,. \tag{2.14}$$

Also we have

$$\alpha_{n-1}(\mathcal{O}_D) = \alpha_{n-1}(\mathcal{O}_{D_1}) + \alpha_{n-1}(\mathcal{O}_{D_2})\,. \tag{2.15}$$

We have already seen that Z_C is integral. Therefore $\dim_k(G_C|_z) = r$ for a general closed point $z \in Z_C$, and $\dim_k G_C|_{z'} \geq r$ for any closed point $z' \in Z_C$. Since D_i is integral, we have that for a general point $z_i' \in D_i$

$$\mathrm{rk}(\bar{G}_{D_i}) = \dim_k(\bar{G}_{D_i}|_{z'})\,.$$

Since $\dim(T(G)) \leq n-2$, G and $\bar{G}$ are equal over a non-empty open subset of D_i. Hence for a general $z' \in D_i$

$$\begin{aligned} \mathrm{rk}(\bar{G}_{D_i}) = \mathrm{rk}(G_{D_i}) &= \dim_k(G_{D_i}|_{z'}) \\ &= \dim_k(G_C|_{z'}) \geq r\,. \end{aligned}$$

In other words, (using equation (2.13))

$$r = \frac{\alpha_{n-1}(G)}{\alpha_{n-1}(\mathcal{O}_D)} = \frac{\alpha_{n-1}(\bar{G})}{\alpha_{n-1}(\mathcal{O}_D)} \leq \frac{\alpha_{n-1}(\bar{G}_{D_1})}{\alpha_{n-1}(\mathcal{O}_{D_1})}, \frac{\alpha_{n-1}(\bar{G}_{D_2})}{\alpha_{n-1}(\mathcal{O}_{D_2})}.$$

If

$$\frac{\alpha_{n-1}(\bar{G}_{D_1})}{\alpha_{n-1}(\mathcal{O}_{D_1})} \neq \frac{\alpha_{n-1}(\bar{G}_{D_2})}{\alpha_{n-1}(\mathcal{O}_{D_2})},$$

then dividing (2.14) by (2.15) we get that

$$r = \frac{\alpha_{n-1}(\bar{G})}{\alpha_{n-1}(\mathcal{O}_D)} > \min\left\{\frac{\alpha_{n-1}(\bar{G}_{D_1})}{\alpha_{n-1}(\mathcal{O}_{D_1})}, \frac{\alpha_{n-1}(\bar{G}_{D_2})}{\alpha_{n-1}(\mathcal{O}_{D_2})}\right\}$$

which gives a contradiction. Thus, we get that

$$r = \frac{\alpha_{n-1}(G)}{\alpha_{n-1}(\mathcal{O}_D)} = \frac{\alpha_{n-1}(\bar{G})}{\alpha_{n-1}(\mathcal{O}_D)} = \frac{\alpha_{n-1}(\bar{G}_{D_1})}{\alpha_{n-1}(\mathcal{O}_{D_1})} = \frac{\alpha_{n-1}(\bar{G}_{D_2})}{\alpha_{n-1}(\mathcal{O}_{D_2})}. \tag{2.16}$$

Hence we get

$$r = \mathrm{rk}(G) = \mathrm{rk}(\bar{G}) = \mathrm{rk}(\bar{G}_{D_1}) = \mathrm{rk}(\bar{G}_{D_2}).$$

From this (1) follows.

Proof of (2). Now we look at the slope.

$$\begin{aligned}
\mu(\bar{G}) &= \frac{\deg(\bar{G})}{r} \\
&= \frac{\alpha_{n-2}(\bar{G})}{r} - \alpha_{n-2}(\mathcal{O}_D) \\
&= \frac{\alpha_{n-2}(\bar{G})}{r} - (\alpha_{n-2}(\mathcal{O}_{D_1}) + \alpha_{n-2}(\mathcal{O}_{D_2}) - \alpha_{n-2}(\mathcal{O}_{D_1\cap D_2})) \\
&= \frac{\deg(\bar{G}_{D_1})}{r} + \frac{\deg(\bar{G}_{D_2})}{r} - \frac{\alpha_{n-2}(\mathcal{O}_{D_1\cap D_2})}{r}(\mathrm{rk}(\bar{G}_{D_1\cap D_2}) - r) \\
&= \mu(\bar{G}_{D_1}) + \mu(\bar{G}_{D_2}) - \frac{\alpha_{n-2}(\mathcal{O}_{D_1\cap D_2})}{r}(\mathrm{rk}(\bar{G}_{D_1\cap D_2}) - r) \qquad (2.17)
\end{aligned}$$

Define a filtration

$$T'(\bar{G}_{D_i}) \subset T(\bar{G}_{D_i}) \subset \bar{G}_{D_i}$$

on $\bar{G}_{D_i}$ as follows. Let $T(\bar{G}_{D_i})$ be the largest torsion subsheaf and $T'(\bar{G}_{D_i})$ is the torsion subsheaf which is supported on $D_1 \cap D_2$.

Define $G_i := \bar{G}_{D_i}/T'(\bar{G}_{D_i})$. We will relate the degree (or slope) of $\bar{G}_{D_i}$ and G_i and then use (2.17) to compare the degrees of G_i and $\bar{G}$. Consider the short exact sequence

$$0 \to T'(\bar{G}_{D_i}) \to \bar{G}_{D_i} \to G_i \to 0. \tag{2.18}$$

Thus, we have

$$\alpha_{n-2}(\bar{G}_{D_i}) = \alpha_{n-2}(G_i) + \alpha_{n-2}(T'(\bar{G}_{D_i})).$$

Dividing by r we get

$$\mu(\bar{G}_{D_i}) = \mu(G_i) + \frac{\alpha_{n-2}(T'(\bar{G}_{D_i}))}{r}\,. \tag{2.19}$$

Restricting equation (2.18) to $D_1 \cap D_2$ we get an exact sequence

$$T'(\bar{G}_{D_i})|_{D_1\cap D_2} \to \bar{G}_{D_i}|_{D_1\cap D_2} \to G_i|_{D_1\cap D_2} \to 0\,. \tag{2.20}$$

If η is the generic point of $D_1 \cap D_2$ then $\mathcal{O}_{D_i,\eta}$ is a discrete valuation ring. Therefore the localisation $G_{i,\eta}$ of G_i at η being a torsion free module is in fact free. This shows two things, first that

$$\mathrm{rk}(G_i|_{D_1\cap D_2}) = \dim_{k(\eta)} G_i \otimes k(\eta) = \mathrm{rank}_{\mathcal{O}_{D_i,\eta}} G_{i,\eta} = \mathrm{rk}(G_i) = r\,.$$

Second that the sequence (2.20) is left exact when we tensor with $\mathcal{O}_{D_1\cap D_2,\eta}$. Thus, we conclude that

$$\begin{aligned}\mathrm{rk}(T'(\bar{G}_{D_i})|_{D_1\cap D_2}) &= \mathrm{rk}(\bar{G}_{D_i}|_{D_1\cap D_2}) - \mathrm{rk}(G_i|_{D_1\cap D_2})\\ &= \mathrm{rk}(\bar{G}_{D_i}|_{D_1\cap D_2}) - \mathrm{rk}(G_i)\\ &= \mathrm{rk}(\bar{G}_{D_i}|_{D_1\cap D_2}) - r\,.\end{aligned}$$

Therefore we have

$$\mathrm{rk}(T'(\bar{G}_{D_i})|_{D_1\cap D_2}) = \frac{\alpha_{n-2}(T'(\bar{G}_{D_i}))}{\alpha_{n-2}(\mathcal{O}_{D_1\cap D_2})} = \mathrm{rk}(\bar{G}_{D_i}|_{D_1\cap D_2}) - r\,.$$

Thus, using this we rewrite equation (2.19) as

$$\mu(G_i) = \mu(\bar{G}_{D_i}) - \alpha_{n-2}(\mathcal{O}_{D_1\cap D_2})\frac{\mathrm{rk}(\bar{G}_{D_i}|_{D_1\cap D_2}) - r}{r}\,.$$

Substituting this into equation (2.17) we get

$$\mu(\bar{G}) = \mu(G_1) + \mu(G_2) + \alpha_{n-2}(\mathcal{O}_{D_1\cap D_2})\frac{\mathrm{rk}(\bar{G}_{D_1\cap D_2}) - r}{r}\,. \tag{2.21}$$

Since $\mathrm{rk}(\bar{G}_{D_1}) = r$, for any closed point $z \in D_1$, $\dim_k(\bar{G}_{D_1} \otimes k(z)) \geq r$. Since $D_1 \cap D_2$ is integral, $\mathrm{rk}(G_{D_1\cap D_2}) \geq r$. Hence we have

$$\mu(\bar{G}) \geq \mu(G_1) + \mu(G_2) \geq \mu(\bar{G}_{D_1}/T(\bar{G}_{D_1})) + \mu(\bar{G}_{D_2}/T(\bar{G}_{D_2}))$$

This completes the proof of (2).

Proof of (3). We continue with notation as above. Let us assume that we have equality

$$\mu(G) = \mu(\bar{G}) = \mu(G_1) + \mu(G_2) = \mu(\bar{G}_{D_1}/T(\bar{G}_{D_1})) + \mu(\bar{G}_{D_2}/T(\bar{G}_{D_2}))$$

(a) Since $\mu(G) = \mu(G/T(G))$, we get that $\mathrm{Supp}(T(G)) \subset D$ is a closed subset whose codimension in D is ≥ 2.
(b) Using (2.21) we get $\mathrm{rk}(\bar{G}_{D_1\cap D_2}) = r$. Using (2.17), we get $\mu(\bar{G}) = \mu(\bar{G}_{D_1}) + \mu(\bar{G}_{D_2})$. Since $\mu(\bar{G}) = \mu(\bar{G}_{D_i}/T(\bar{G}_{D_1})) + \mu(\bar{G}_{D_i}/T(\bar{G}_{D_i}))$ and $\mu(\bar{G}_{D_i}) \geq \mu(\bar{G}_{D_i}/T(\bar{G}_{D_i}))$, we get $\mu(\bar{G}_{D_i}) = \mu(\bar{G}_{D_i}/T(\bar{G}_{D_i}))$. It follows that $\mathrm{Supp}(T(\bar{G}_{D_i})) \subset D_i$ is a closed subset whose codimension in D_i is ≥ 2.

(c) Since $\bar{G}_{D_i}/T(\bar{G}_{D_i})$ is torsion free on D_i, it is locally free on an open subset whose complement in D_i has codimension ≥ 2. From this and the previous point we conclude that $\bar{G}_{D_i}$ is locally free on an open subset whose complement in D_i has codimension ≥ 2.

(d) Since $\operatorname{codim}(Z_C \setminus Z_C^{\mathrm{reg}}, Z_C) \geq 3$, it follows that $\operatorname{codim}(D \setminus Z_C^{\mathrm{reg}}, D) \geq 2$.

Thus, we have obtained some closed subsets of D, each of which has codimension ≥ 2 in D. Let D_s be the union of all these. Let $z \in D \setminus D_s$ be a closed point, and assume $z \in D_1$ (the same argument holds for $z \in D_2$). We claim that G_C is locally free in a neighbourhood of z. We already know that $r = \operatorname{rk}(G_C) = \operatorname{rk}(\bar{G}_{D_i})$, see equation (2.16). Then

$$\begin{aligned}
\dim_k(G_C \otimes k(z)) &= \dim_k(G_C|_D \otimes k(z)) \\
&\text{(using definition of } G \text{ we get)} \\
&= \dim_k(G \otimes k(z)) \\
&\text{(since in a nbd of } z\text{, using (a), we have } G = \bar{G}) \\
&= \dim_k(\bar{G} \otimes k(z)) \\
&\text{(since } z \in D_1 \text{ we get)} \\
&= \dim_k(\bar{G}_{D_1} \otimes k(z)) \\
&= r\,.
\end{aligned}$$

The local ring $\mathcal{O}_{Z_C^{\mathrm{reg}},z}$ is integral. If η denotes the generic point, then the above shows that

$$\dim_{k(\eta)} G_C \otimes k(\eta) = \dim_k(G_C \otimes k(z)) = r\,.$$

It follows that G_C is locally free in a neighbourhood of z. This proves that $D \setminus D_s \subset U$. This completes the proof of (3). □

3. μ-Semistable Restriction Theorem

In this section we will prove the μ-semistable restriction theorem [MR82, Theorem 6.1].

Theorem 3.1. *Let X be a smooth projective variety of dimension $n \geq 2$ over an algebraically closed field k. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let E be a μ-semistable sheaf on X. Then there is an integer a_0 such that for all $a \geq a_0$ there is a non-empty open set $U_a \subset \mathbb{P}(H^0(X, \mathcal{O}_X(a))^\vee)$ such that for all $[D] \in U_a$ the divisor D is smooth and $E|_D$ is μ-semistable with respect to $\mathcal{O}_X(1)|_D$.*

Lemma 3.2. *For each $a \geq 1$ there exists an open set $U_a \subset \Pi_a$ and a quotient $q^*E|_{Z_{U_a}} \to F_a \to 0$ over $Z_{U_a} := p^{-1}(U_a)$ such that*

(1) each $[D] \in U_a$ is smooth and integral.
(2) F_a is U_a-flat.
(3) $E|_D$ is torsion-free.

(4) *For $D \in U_a$, $E|_D \to F_a|_D$ is the μ-minimal destabilising sheaf of $E|_D$.*

Proof. The first assertion is just Bertini's theorem. The rest of the claims will follow from Theorem 1.18 once we show that q^*E is flat over Π_a and $E|_D$ is torsion-free for atleast one $D \in \Pi_a$. The latter fact follows from [HL10, Corollary 1.1.14(ii)] and [HL10, Lemma 1.1.12]. Let $[D] \in \Pi_a$. Note that the map $E(-a) \xrightarrow{\otimes D} E$ is injective since locally it is given by multiplication of a non-zero element and E is torsion free. Hence we have an exact sequence

$$0 \to E(-a) \to E \to E|_D \to 0$$

Therefore $P(E|_D) = P(E) - P(E(-a))$, that is, the Hilbert polynomial of q^*E restricted to any closed fibre of p is constant. Hence q^*E is flat over Π_a. □

Since $\mathcal{F}_a$ is flat over U_a, both $\mathrm{rk}(F_a|_D)$ and $\deg(F_a|_D)$ are independent of $[D] \in U_a$. We define $r(a) := \mathrm{rk}(F_a|_D)$ and $\mu(a) := \mu(F_a|_D)$.

Since Z_a is smooth, we have the line bundle $\det(F_a)$ over Z_{U_a} and it can be extended it to a line bundle $\mathcal{Q}$ over Z_a. Now $Z_a \cong \mathbb{P}(K^\vee)$ and it follows from the proof of Lemma 2.2 that under this isomorphism $p^*\mathcal{O}_{\Pi_a}(1) \cong \mathcal{O}_{\mathbb{P}(K^\vee)}(1)$. Hence we can decompose $\mathcal{Q}$ uniquely as $\mathcal{Q} = q^*\mathcal{L}_a \otimes p^*\mathcal{M}$ with $\mathcal{L}_a \in \mathrm{Pic}\ X$, $\mathcal{M} \in \mathrm{Pic}\ \Pi_a$. By Lemma 3.3 if $a \geq 3$ then $\mathcal{L}_a$ does not depend on the choice of the extension $\mathcal{Q}$.

Lemma 3.3. *Let $a \geq 3$. Let $L', L'' \in$ Pic X such that $L'|_D \cong L''|_D$ for $[D]$ in a non-empty open set in Π_a. Then $L' \cong L''$.*

Proof. Define $L := L' \otimes (L'')^{-1}$. Then $q^*L|_D \cong \mathcal{O}_D\ \forall\ D \in U$, where U is an open set in Π_a. In particular, we get $h^0(D, q^*L|_D) = h^0(D, q^*L|_D) = 1\ \forall\ D \in U$. Now the proof of Lemma 3.2 shows that q^*L is flat over Π_a. Applying semicontinuity theorem, we get $h^0(D, q^*L|_D) = h^0(D, q^*L|_D) = 1\ \forall\ D \in \Pi_a$. If D is integral, this implies that $q^*L|_D \cong \mathcal{O}_D$. By [MR82, Lemma 2.1.3(ii)] the set of integral divisors is open in Π_a. Let us denote this open set by B_a.

Let B'_a be the open set in Π_a parametrizing smooth divisors. By [Har77, Chapter 3, Corollary 7.9] each such divisor is connected, and hence integral. Thus, $\emptyset \subsetneq B'_a \subset B_a$. From the proof of [Har77, Chapter 2, Theorem 8.18] it follows that $\Pi_a \setminus B'_a$ is irreducible. By [MR82, Lemma 2.1.3(ii)] this inclusion is strict. Therefore we get that $\mathrm{codim}(\Pi_a \setminus B_a, \Pi_a) \geq 2$.

Now consider the sheaf p_*q^*L. By Grauert's theorem $p_*q^*L|_{B_a}$ is a line bundle on B_a. Let $\mathcal{N} \in$ Pic Π_a be such that $\mathcal{N}|_{B_a} \cong p_*q^*L|_{B_a}$. This induces an isomorphism $p^*\mathcal{N} \to q^*L$ on $p^{-1}(B_a)$. Since p is flat, $\mathrm{codim}(Z_a \setminus p^{-1}(B_a), Z_a) \geq 2$. Therefore $p^*\mathcal{N} \cong q^*L$. But by Lemma 2.2 we have Pic $Z_a = p^*$Pic $\Pi_a \oplus q^*$Pic X. This implies $\mathcal{N} = \mathcal{O}_{\Pi_a}$ and $L = \mathcal{O}_X$. □

Next we will prove the following two statements in the form of various lemmas:

(1) $\exists\ 0 < r \leq \mathrm{rk}(E)$ such that for all $a \gg 0$ we have $r(a) = r$.

(2) $\exists\, \mathcal{L} \in \operatorname{Pic} X$ such that for all $a \gg 0$ we have $\mathcal{L} \cong \mathcal{L}_a$.

As the first step towards proving these statements, we will prove a Lemma which shows how the numbers $\mu(a_1), \mu(a_2), \mu(a_1+a_2), r(a_1), r(a_2), r(a_1+a_2)$ are related.

Lemma 3.4. *Let $a = a_1 + a_2$. Then $\mu(a) \geq \mu(a_1) + \mu(a_2)$. In case of equality, we have $r(a) \leq \min\{r(a_1), r(a_2)\}$.*

Proof. Fix $D_1 \in U_{a_1}$. By Bertini's theorem there exists $D_2 \in U_{a_2}$ such that $D = D_1 + D_2$ is a simple normal crossing divisor. By Lemma 2.7 choose a $C \subset \Pi_a$ such that $[D] \in C$ and $C \setminus [D] \subset U_a$ and consider the quotient $q^*E|_{Z_{C\setminus[D]}} \to F_a|_{Z_{C\setminus[D]}} \to 0$. By Lemma 2.11 this extends to a flat quotient $q^*E|_{Z_C} \to F_C \to 0$. Let $F := F_C|_D$. Then $\mu(F) = \mu(a)$. Let $\bar{F} := F/T(F)$. Applying Lemma 2.12 we have

$$\mu(a) \geq \mu(\bar{F}_{D_1}/T(\bar{F}_{D_1})) + \mu(\bar{F}_{D_2}/T(\bar{F}_{D_2}))\,.$$

Since $\bar{F}_{D_i}/T(\bar{F}_{D_i})$ is a torsion free quotient of $E|_{D_i}$, by Theorem 1.21 we have $\mu(a_i) \leq \mu(\bar{F}_{D_1}/T(\bar{F}_{D_1}))$ and the first statement follows.

If equality happens then we get $\mu(\bar{F}_{D_i}/T(\bar{F}_{D_i})) = \mu(a_i) = \mu_{\min}(E|_{D_i})$. Now we apply Theorem 1.21, which shows that $r(a) = \operatorname{rk}(\bar{F}_{D_i}/T(\bar{F}_{D_i})) \leq r(a_i)$. □

Corollary 3.5. *$r(a)$ and $\dfrac{\mu(a)}{a}$ are constant for $a \gg 0$.*

Proof. Since

$$\frac{\mu(a)}{a} = \frac{\deg(F_a|_D)}{a \cdot r(a)} = \frac{\deg(\mathcal{L}_a|_D)}{a \cdot r(a)} = \frac{\deg(\mathcal{L}_a)}{r(a)} \in \frac{\mathbb{Z}}{\mathrm{rk(E)!}},$$

it follows it belongs to a discrete set. Here by $\deg(\mathcal{L}_a)$ we mean the degree of the line bundle $\mathcal{L}_a$ on X computed with respect to $\mathcal{O}_X(1)$. Moreover, $\mu(E_D) = a\mu(E)$. Since $\mu(E_D) \geq \mu(a)$ it follows that $\mu(a)/a$ is bounded above by $\mu(E)$. Thus, it attains a maximum at some b_0. That is,

$$\frac{\mu(b_0)}{b_0} = \max\{\frac{\mu(b)}{b} \,|\, b \geq 2\}\,.$$

Now consider the second set

$$\frac{\mu(b_1)}{b_1} = \max\{\frac{\mu(b)}{b} \,|\, b \geq 2,\ (b, b_0) = 1\}\,.$$

Clearly, $\mu(b_1)/b_1 \leq \mu(b_0)/b_0$ and b_1 is coprime to b_0. Let $b = \beta_1 b_1 + \beta_0 b_0$ be such that b is coprime to b_0 and $\beta_i \geq 1$. Then by Lemma 3.4

$$\begin{aligned}
\mu(b) &\geq \beta_1\mu(b_1) + \beta_0\mu(b_0) \\
&= \beta_1 b_1 \frac{\mu(b_1)}{b_1} + \beta_0 b_0 \frac{\mu(b_0)}{b_0} \\
&\geq \beta_1 b_1 \frac{\mu(b_1)}{b_1} + \beta_0 b_0 \frac{\mu(b_1)}{b_1}
\end{aligned}$$

This shows that $\mu(b)/b \geq \mu(b_1)/b_1$. But since b is coprime to b_0, it follows that $\mu(b)/b = \mu(b_1)/b_1$. But this shows that $\mu(b_0)/b_0 = \mu(b_1)/b_1$. Since every b sufficiently large can be written as a positive linear combination of b_0 and b_1, it follows that $\mu(a)/a$ is constant for $a \geq a_0$. Let $\lambda = \mu(a)/a$. Then

$$\frac{\mu(a_1) + \mu(a_2)}{a_1 + a_2} = \frac{\lambda(a_1 + a_2)}{a_1 + a_2} = \frac{\mu(a)}{a}$$

Since $a = a_1 + a_2$ it follows that

$$\mu(a) = \mu(a_1) + \mu(a_2)\,. \tag{3.6}$$

Now from Lemma 3.4 it follows that $r(a) \leq \min\{r(a_1), r(a_2)\}$. So if we take $a \geq 2a_0$, we see that $r(a) \leq \min\{r(a_0), r(a - a_0)\} \leq r(a_0)$. Let $a_1 \geq 2a_0$ be the number at which the minimum is attained. Let $a \geq 2a_1$. Then we get that $r(a) \leq \min\{r(a_1), r(a - a_1)\} \leq r(a_1)$. This shows that $r(a) = r(a_1)$. This proves that $r(a)$ is eventually constant. □

Lemma 3.7. *$\exists\ \mathcal{L} \in \operatorname{Pic} X$ such that $\mathcal{L}_a \cong \mathcal{L}$ for $a \gg 0$.*

Proof. Let us choose $d_0 \geq 3$ such that for $a \geq d_0$ both $r(a)$ and $\mu(a)/a$ are constant. Define $r := r(a)$. Let $a \geq 2d_0$. Choose $a_1 = d_0$ and $a_2 = a - d_0 \geq d_0$. Let the notation be as in Lemma 3.4. Let $U \subset Z_C^{\mathrm{reg}}$ be the locus of points where F_C is locally free. Then using Lemma 2.12 (3) we see that $\operatorname{codim}(D \setminus U, D) \geq 2$.

By intersecting U with fibers of the map $p : Z_C^{\mathrm{reg}} \to C$ one checks that $\operatorname{codim}(Z_C^{\mathrm{reg}} \setminus U, Z_C^{\mathrm{reg}}) \geq 2$. Consider the line bundle $\mathcal{A} := \det(F_C|_U)$ on U. Since Z_C^{reg} is smooth, this extends uniquely to a line bundle on Z_C^{reg}. By [Har77, Chapter II, Exc. 5.15] we can extend $\mathcal{A}$ by a coherent sheaf $\tilde{\mathcal{A}}$ over Z_C. Notice that we can assume $\tilde{\mathcal{A}}$ is torsion free (if not, replace $\tilde{\mathcal{A}}$ by $\tilde{\mathcal{A}}/T(\tilde{\mathcal{A}})$. Since $\mathcal{A}$ is torsion-free, $\tilde{\mathcal{A}}|_{Z_C^{\mathrm{reg}}} = \mathcal{A}$.) Thus, it is C-flat.

Alternatively, we can do the following. Recall that Z_C is normal and integral. Now let $j : Z_C^{\mathrm{reg}} \to Z_C$ be the inclusion. Define $\tilde{\mathcal{A}} := j_*\mathcal{A}$. If $\operatorname{Spec} R \subset Z_C$ is an affine open subset, then $\tilde{\mathcal{A}}(\operatorname{Spec} R) = \mathcal{A}(U \cap \operatorname{Spec} R)$. Now $(U \cap \operatorname{Spec} R) \subset U$ is an open set whose complement has codimension ≥ 2 in $\operatorname{Spec} R$. Since R is normal, we have $\mathcal{O}(U \cap \operatorname{Spec} R) = R$. Now since $\mathcal{A}$ is coherent over Z_C^{reg}, $\mathcal{A}(U \cap \operatorname{Spec} R)$ is finitely generated over $\mathcal{O}(U \cap \operatorname{Spec} R) = R$. Hence $\tilde{\mathcal{A}}$ is a coherent sheaf over Z_C. The above argument also shows that $\tilde{\mathcal{A}}$ is in fact torsion free. Thus, it is C-flat.

We need to make an observation about restricting $\tilde{\mathcal{A}}$ to $D_i \setminus D_s$. Since $D_i \setminus D_s \subset U$ (using F_C is locally free on U), it follows that

$$\begin{aligned} \tilde{\mathcal{A}}|_{D_i \setminus D_s} &= \mathcal{A}|_{D_i \setminus D_s} \\ &= \det(\bar{F}_{D_i}|_{D_i \setminus D_s}) \\ &= \det((\bar{F}_{D_i}/T(\bar{F}_{D_i}))|_{D_i \setminus D_s}) \end{aligned} \tag{3.8}$$

Now $\bar{F}_{D_i}/T(\bar{F}_{D_i})$ is a quotient of $E|_{D_i}$ with

$$\mu(\bar{F}_{D_i}/T(\bar{F}_{D_i})) = \mu(a_i) = \mu_{\min}(E|_{D_i})\,,$$

and $\mathrm{rk}(\bar{F}_{D_i}/T(\bar{F}_{D_i})) = r = r(a) = r(a_i)$. Now we apply Corollary 1.22, which shows that $F_{a_i}|_{D_i}$ and $\bar{F}_{D_i}/T(\bar{F}_{D_i})$ agree on an open subset of D_i whose complement has codimension ≥ 2. In particular, they have the same determinant. This proves that

$$\det((\bar{F}_{D_i}/T(\bar{F}_{D_i}))|_{D_i \setminus D_s}) = \mathcal{L}_{a_i}\,.$$

Thus, we get

$$\tilde{\mathcal{A}}|_{D_i \setminus D_s} \cong \mathcal{L}_{a_i}|_{D_i \setminus D_s}\,.$$

It is clear that for a point $[D'] \in C \setminus [D]$

$$\tilde{\mathcal{A}}|_{D'} \cong \mathcal{L}_a|_{D'}\,.$$

Recall $p : Z_C \to C$. Consider $p_*(\mathcal{L}_a^\vee \otimes \tilde{\mathcal{A}})$. Since $h^0(D', \mathcal{L}_a^\vee \otimes \tilde{\mathcal{A}}|_{D'}) = 1$, it follows by semi-continuity that $h^0(D, \mathcal{L}_a^\vee \otimes \tilde{\mathcal{A}}|_D) \geq 1$. Let $\phi : \mathcal{L}_a|_D \to \tilde{\mathcal{A}}|_D$ be a non-zero map. It has to be non-zero restricted to one of the D_i, say D_1. So we have a non-zero map $\phi : \mathcal{L}_a|_{D_1} \to \tilde{\mathcal{A}}|_{D_1}$. But we have seen above that $\tilde{\mathcal{A}}|_{D_1 \setminus D_s} \cong \mathcal{L}_{a_1}|_{D_1 \setminus D_s}$. Thus, we have a non-zero map $\phi : \mathcal{L}_a|_{D_1 \setminus D_s} \to \mathcal{L}_{a_1}|_{D_1 \setminus D_s}$. We claim that both $\mathcal{L}_a$ and $\mathcal{L}_{a_1}$ have the same degree on X and so they have the same degree on D_1. Note that

$$\mu(a) = \mu(F_a|_{D'}) = \frac{\deg(\mathcal{L}_a|_{D'})}{r(a)} = \frac{a\deg(\mathcal{L}_a)}{r(a)}\,.$$

Thus,

$$\deg(\mathcal{L}_a) = \frac{r(a)\mu(a)}{a} = \frac{r(a_1)\mu(a_1)}{a_1} = \deg(\mathcal{L}_{a_1})\,.$$

Since $D_1 \setminus D_s$ is an open subset whose complement has codimension ≥ 2 in D, this proves that $\phi : \mathcal{L}_a|_{D_1} \to \mathcal{L}_{a_1}|_{D_1}$ is an isomorphism. Restrict this isomorphism to a point $z \in (D_1 \cap D_2) \setminus D_s$. This shows that the restriction of $\phi : \mathcal{L}_a|_{D_2} \to \mathcal{L}_{a_2}|_{D_2}$ is non-zero and so this is also an isomorphism by the same reason.

We can fix D_2 and vary D_1 in an open set and apply the above argument. Then this shows that $\mathcal{L}_a|_{D_1} \cong \mathcal{L}_{a_1}|_{D_1}$ for D_1 varying in an open subset of $|\mathcal{O}(a_1)|$. Applying Lemma 3.3 we see that $\mathcal{L}_a \cong \mathcal{L}_{a_1}$. This proves that all the $\mathcal{L}_a$ are isomorphic for $a \geq 2d_0$. □

To summarize, we have proved the following. For each $a \gg 0$, we have a non-empty open set $U_a \subset \mathbb{P}(H^0(X, \mathcal{O}_X(a))^\vee)$ such that each $[D] \in U_a$ is smooth and integral and $E|_D$ is torsion free on D. Over Z_{U_a} we have a quotient $q^*E \to F_a \to 0$ such that

(1) F_a is U_a-flat.
(2) For $D \in U_a$, $F_a|_D$ is the μ-minimal destabilising quotient of $E|_D$. In particular, it is torsion free and $\mu(F_a|_D) \leq \mu(E|_D)$.
(3) We have an integer $0 < r \leq \mathrm{rk}\ E$ such that $\mathrm{rk}\ F_a = r$ for $a \gg 0$.

(4) There exists $\mathcal{L} \in Pic(X)$ such that $(\det F_a)|_D \cong \mathcal{L}|_D$ for $a \gg 0$. In particular,

$$\deg \mathcal{L} = \frac{\deg \mathcal{L}|_D}{a} = \frac{\deg(F_a|_D)}{a} = \frac{r\mu(F_a|_D)}{a} = \frac{r\mu(a)}{a} \tag{3.9}$$

Lemma 3.10. *Suppose we are given an infinite set $N \subset \mathbb{N}$ such that for each $a \in N$, we have a non-empty open set $W_a \subset \mathbb{P}(H^0(X, \mathcal{O}_X(a))^\vee)$ with each $[D'] \in W_a$ is smooth and integral. Moreover, also assume that over Z_{W_a} we have a quotient $q^*E \to G_a \to 0$ such that*

(1) $E|_{D'}$ is torsion free $\forall [D'] \in W_a$.
(2) G_a is W_a-flat.
(3) $G_a|_{D'}$ is torsion-free and $\mathrm{rk}(G_a|_{D'}) = r \ \forall\ a \in N,\ \forall [D'] \in W_a$.
(4) There exists $\mathcal{L} \in Pic(X)$ such that $\det(G_a)|_{D'} \cong \mathcal{L}|_{D'}$ for $\forall\ a \in N$.

Then $\exists$ an open set $X' \subset X$ with $\mathrm{codim}(X \setminus X', X) \geq 2$ and quotient $E|_{X'} \to F_{X'} \to 0$ over X' with $\det(F_{X'}) = \mathcal{L}$.

Proof. Fix $a \in N$ and $[D'] \in W_a$. Then we have the quotient $E|_{D'} \to G_a|_{D'}$. Let $U' \subset D'$ be the largest open set over which $E|_{D'}$ and $G_a|_{D'}$ are locally free. Since both are assumed to be torsion-free and D' is smooth, we have $\mathrm{codim}(D' \setminus U', D') \geq 2$. We have a surjection over U' :

$$E|_{U'} \longrightarrow G_a|_{U'} \longrightarrow 0 \tag{3.11}$$

We will show that there is an $a \gg 0$ in N such that the quotient 3.11 extends to a quotient $E|_{X'} \to F_{X'}$ of locally free sheaves over a large open set $U' \subset X' \subset X$ and $\det(F_{X'}) = \mathcal{L}$.

Note that a quotient as in 3.11 induces a morphism

$$U' \longrightarrow \mathrm{Gr}(E_{U'}, r)\,.$$

Consider the composite of the above with the Plucker embedding $U' \to \mathrm{Gr}(E_{U'}, r) \to \mathbb{P}(\bigwedge^r E_{U'})$. By definition it is defined by taking r-th exterior power of 3.11:

$$\textstyle\bigwedge^r (E|_{U'}) \longrightarrow \bigwedge^r (G_a|_{U'}) = \det(G_a)|_{U'} = \mathcal{L}|_{U'}$$

Since D' is smooth, $\mathrm{codim}(D' \setminus U', D') \geq 2$ and $\mathcal{L}|_{D'}$ is locally free, this extends to a homomorphism $\sigma_{D'} : \bigwedge^r (E|_{D'}) \to \mathcal{L}|_{D'}$. This is clear if $E|_{D'}$ is locally free. If not, then we may take a resolution of the type $\mathcal{O}_{D'}(-\beta)^{\oplus s} \to \mathcal{O}_{D'}(-\alpha)^{\oplus t} \to \mathcal{F} \to 0$, and use the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{D'}(\mathcal{F}, \mathcal{L}) & \longrightarrow & \mathrm{Hom}_{D'}(\mathcal{O}_{D'}(-\alpha)^{\oplus t}, \mathcal{L}) & \longrightarrow & \mathrm{Hom}_{D'}(\mathcal{O}_{D'}(-\beta)^{\oplus s}, \mathcal{L}) \\
 & \downarrow & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & \mathrm{Hom}_{U'}(\mathcal{F}, \mathcal{L}) & \longrightarrow & \mathrm{Hom}_{U'}(\mathcal{O}_{D'}(-\alpha)^{\oplus t}, \mathcal{L}) & \longrightarrow & \mathrm{Hom}_{U'}(\mathcal{O}_{D'}(-\beta)^{\oplus s}, \mathcal{L})
\end{array}$$

Consider the exact sequence

$$0 \to \mathcal{L}(-a) \to \mathcal{L} \to \mathcal{L}|_{D'} \to 0\,.$$

Applying $\mathrm{Hom}(\bigwedge^r E,\ )$ we get:

$$\begin{aligned} 0 \to \mathrm{Hom}(\bigwedge^r E, \mathcal{L}(-a)) \to \mathrm{Hom}(\bigwedge^r E, \mathcal{L}) \to & \mathrm{Hom}(\bigwedge^r E,\ \mathcal{L}_{D'}) \\ & \to \mathrm{Ext}^1(\bigwedge^r E, \mathcal{L}(-a))\,. \end{aligned}$$

Now

$$\begin{aligned} \mathrm{Ext}^1(\bigwedge^r E, \mathcal{L}(-a)) =& H^{n-1}(X, \bigwedge^r E \otimes \omega_X \otimes \mathcal{L}(a))^\vee \\ =& 0 \qquad \text{for} \qquad a \gg 0 \end{aligned}$$

Hence we get that $\sigma_{D'}$ extends to $\sigma : \bigwedge^r E \to \mathcal{L}$.

Define $X' \subset X$ as the open set where E is locally free and σ is a surjection. Since X is smooth, E is torsion free, it follows that if $E_{D'}$ is locally free at a point $z \in D'$, then E is locally free at $z \in X$. This shows $U' \subset X'$. The restriction $\sigma_{X'} := \sigma|_{X'}$ defines a map $X' \to \mathbb{P}(\bigwedge^r E_{X'})$ which extends the map $U' \to \mathbb{P}(\bigwedge^r E_{U'})$.

Next we show that $\mathrm{codim}(X \setminus X', X) \geq 2$. If not, let $Z' \subset X \setminus X'$ be a divisor. Clearly, since D' is ample, the intersection $D' \cap Z'$ is a divisor in D'. Moreover, $\bigwedge^r E \to \mathcal{L}$ is not surjective on $D' \cap Z'$. But this will imply that $\mathrm{codim}(D' \setminus U', D') = 1$, which is a contradiction.

Moreover, replacing X' by the open set $X' \setminus (D' \setminus U')$ we can assume that $X' \cap D' = U'$. Since $D' \setminus U'$ has codimension ≥ 2 in D', the complement of this modified X' again has codimension ≥ 2 in X.

Next we want to show that for $a \gg 0$ the morphism $X' \to \mathbb{P}(\bigwedge^r E_{X'})$ factors as $X' \to \mathrm{Gr}(E_{X'}, r) \hookrightarrow \mathbb{P}(\bigwedge^r E_{X'})$, that is, that we have a commutative diagram

$$\begin{tikzcd} U' \arrow[rr] \arrow[dr] & & X' \arrow[dl, dotted] \\ & \mathrm{Gr}(E_{X'}, r) & \end{tikzcd}$$

Recall that $\mathbb{P}(\bigwedge^r E_{X'})$ is the relative Proj associated to the graded sheaf of $\mathcal{O}_{X'}$-algebra $S^\bullet(\bigwedge^r E_{X'})$. Let $\mathcal{I} \subset S^\bullet(\bigwedge^r E_{X'})$ be the graded sheaf of ideals associated to the closed subscheme $\mathrm{Gr}(E_{X'}, r) \hookrightarrow \mathbb{P}(\bigwedge^r E_{X'})$. Since $\mathcal{I}$ is finitely generated and graded, we can assume that $\mathcal{I}_\nu := \mathcal{I} \cap S^\nu(\bigwedge^r E_{X'})$, for $\nu \leq \nu_0$, generate $\mathcal{I}$ as an $S^\bullet(\bigwedge^r E_{X'})$-module.

The map $\sigma_{X'}$ is induced by the following homomorphism of $\mathcal{O}_{X'}$-algebras.

$$S^\bullet(\bigwedge^r E|_{X'}) = \bigoplus_{\nu \geq 0} S^\nu(\bigwedge^r E|_{X'}) \xrightarrow{\oplus_\nu \psi_\nu} \bigoplus_{\nu \geq 0} \mathcal{L}^\nu|_{X'} \tag{3.12}$$

Thus, $\sigma_{X'} : X' \to \mathbb{P}(\bigwedge^r E_{X'})$ may be written as

$$\begin{array}{ccc} \mathrm{Proj}(\bigoplus_{\nu\geq 0} L^\nu|_{X'}) & \longrightarrow & \mathrm{Proj}(S^\bullet(\bigwedge^r E|_{X'})) \\ \downarrow\sim & & \downarrow\sim \\ X' & \longrightarrow & \mathbb{P}(\bigwedge^r E|_{X'}) \end{array}$$

Thus, $\sigma_{X'}$ factors through $\mathrm{Gr}(E_{X'}, r)$ iff the image of $\mathcal{I}|_{X'}$ under the homomorphism 3.12 is zero. Since $\mathcal{I}|_{X'}$ is generated by $\mathcal{I}_\nu|_{X'}$ for $\nu \leq \nu_0$, it is enough to show that the maps $\psi_\nu : \mathcal{I}_\nu|_{X'} \to \mathcal{L}^\nu|_{X'}$ are zero for $\nu \leq \nu_0$. Since U' already factors through $\mathrm{Gr}(E_{U'}, r)$, we have $\psi_\nu|_{U'} = 0$. Now consider the exact sequence over X:

$$0 \to \mathcal{L}^\nu(-a) \to \mathcal{L}^\nu \to \mathcal{L}^\nu|_{D'} \to 0$$

Restricting this exact sequence to X' and using the fact that $U' = X' \cap D'$ we have an exact sequence over X':

$$0 \to \mathcal{L}^\nu(-a)|_{X'} \to \mathcal{L}^\nu|_{X'} \to \mathcal{L}^\nu|_{U'} \to 0$$

Applying $\mathrm{Hom}(\mathcal{I}_\nu|_{X'},\ )$ we get the left exact sequence

$$0 \to \mathrm{Hom}(\mathcal{I}_\nu|_{X'}, \mathcal{L}^\nu(-a)|_{X'}) \to \mathrm{Hom}(\mathcal{I}_\nu|_{X'}, \mathcal{L}^\nu|_{X'}) \to \mathrm{Hom}(\mathcal{I}_\nu|_{X'}, \mathcal{L}^\nu|_{U'})$$

since $\psi_\nu|_{U'}$ is zero, we get $\psi \in \mathrm{Hom}(\mathcal{I}_\nu|_{X'}, \mathcal{L}^\nu(-a)|_{X'})$. Taking a surjection of the type $\mathcal{O}_{X'}(-t)^{\oplus s} \twoheadrightarrow \mathcal{I}_\nu|_{X'}$, we easily see that, $\mathrm{Hom}(\mathcal{I}_\nu|_{X'}, \mathcal{L}^\nu(-a)|_{X'}) = 0$ for $a \gg 0$.

Therefore, by choosing $a \gg 0$ in N, we see that the morphism $X' \to \mathbb{P}(\bigwedge^r E_{X'})$ factors through a morphism $X' \to \mathrm{Gr}(E_{X'}, r)$. In other words, we have a quotient of locally free sheaves

$$E|_{X'} \to F_{X'} \to 0$$

such that its r-th exterior power is the morphism $\bigwedge^r E|_{X'} \to \mathcal{L}|_{X'} \to 0$ □

Proof of Theorem 3.1. Choose $a_0 \gg 0$ as in Lemma 3.7. We claim that the restriction of E to a general hypersurface D' of degree $\geq a_0$ is μ-semistable. Assume that this is not the case. Then the slope

$$\mu(a) = \mu(F_a|_{D'}) < \mu(E|_{D'})\,.$$

Taking $N = \mathbb{N}_{\geq a_0}$, $W_a = U_a$, $G_a = F_a$ and $\mathcal{L} = \mathcal{L}_a$ in Lemma 3.10, we get a locally free quotient $E_{X'} \to F_{X'} \to 0$ where $X' \subset X$ is an open set such that $\mathrm{codim}_X(X \setminus X') \geq 2$, $\det(F_{X'}) = \mathcal{L}$ and $\mathrm{rk}(F_{X'}) = r$. Since E is μ-semistable we get that (see equation (3.9))

$$\mu(E) \leq \mu(F_{X'}) = \frac{\deg(\mathcal{L})}{r} = \frac{\mu(a)}{a}\,.$$

Hence $\mu(a) \geq a.\mu(E) = \mu(E|_{D'})$ for $[D'] \in U_a$. This gives

$$\mu(a) < \mu(E|_{D'}) \leq \mu(a),$$

a contradiction. □

4. Fields of definitions

Lemma 4.1. *Let $k \subset K$ be a Galois extension and let $G = \mathrm{Gal}(K/k)$. Let V be a K-vector space and let $W_K \subset V \otimes K$ be a subspace which is invariant under G. Then there is a subspace $W \subset V$ such that $W_K = W \otimes K$.*

Proof. The inclusion $W_K \subset V \otimes K$ is a G-equivariant map of K vector spaces. Denote the quotient by Q. Then the natural map $V \otimes K \xrightarrow{\pi} Q$ is G-equivariant and a map of K vector spaces. Let e_i, for $i \in I$, be a k basis for V. Then the K span of the images $\pi(e_i)$ is equal to Q. Thus, we may find a subset $J \subset I$ such that $\pi(e_j)$, for $j \in J$, is a basis for Q as a K-vector space. Now consider the map

$$\bigoplus_{j \in J} e_j \otimes K \to Q\,.$$

This map is G-equivariant and an isomorphism of K-vector spaces. Thus, we have found a subspace $V' \subset V$ (the k span of e_j for $j \in J$) such that

$$V \otimes K \cong V' \otimes K \bigoplus W_K\,.$$

The isomorphism is G equivariant. Taking G invariants on both sides we get

$$V \cong V' \otimes (W_K)^G\,.$$

Now it follows easily that $(W_K)^G \otimes K \to W_K$ is an isomorphism. This completes the proof of the lemma. □

Definition 4.2. *Let $k \subset E$ be fields. A k-derivation of E is a k-linear map $D : E \to E$ which satisfies the Leibniz rule, that is, $D(ef) = eD(f) + fD(e)$.*

Since $D(1.1) = D(1) + D(1)$ it follows that $D(1) = 0$ and so $D(k) = 0$. The set of derivations is denoted $\mathrm{Der}_k(E)$. It is a Lie algebra under the Lie bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Lemma 4.3. *Let k be a field of char $p > 0$ and let $k \subset E$ be a purely inseparable extension of degree p. If $\beta \in E$ is such that $D(\beta) = 0$ for all $D \in \mathrm{Der}_k(E)$ then $\beta \in k$.*

Proof. Clearly $E = k[T]/(T^p - \alpha)$ for some $\alpha \in k$. It is easily checked that the map defined as

$$D : k[T] \to k[T] \qquad\qquad D(T^i) := iT^{i-1} \qquad i > 0$$

and extended k-linearly, descends to E and defines a k-derivation on E. Let $\beta \in E$ be such that $D(\beta) = 0$. Let $f(T) \in k[T]$ be a lift of β such that the degree of $f(T) < p$. Since $D(\beta) = 0$, it follows that $D(f(T)) = (T^p - \alpha)g(T)$. Looking at the degree we see that $D(f(T)) = 0$. This forces that $f(T) = h(T^p)$, but again looking at the degree we see that $f(T)$ is a constant. Thus, $\beta \in k$. □

Lemma 4.4. *Let $k \subset K$ be a purely inseparable extension. Let V be a sheaf over X_k and let $W_K \subset V \otimes K$ be a subsheaf. Assume that $\mathrm{Hom}_{X_K}(W_K, (V_K/W_K)) = 0$. Then there is a subsheaf $W \subset V$ such that $W_K = W \otimes K$.*

Proof. We will first show that for every $D \in \mathrm{Der}_k(K)$ we have $D(W_K) \subset W_K$. For this consider the map

$$\psi : W_K \subset V_K \xrightarrow{D} V_K \to V_K/W_K\,.$$

This map is K-linear since

$$\begin{aligned}\psi(\lambda w) &= D(\lambda w) \qquad \mathrm{mod}\, W_K \\ &= D(\lambda)w + \lambda D(w) \qquad \mathrm{mod}\, W_K \\ &= \lambda D(w) \qquad \mathrm{mod}\, W_K \\ &= \lambda\psi(w)\end{aligned}$$

In the above we have used that $D(\lambda) \in K$ and so $D(\lambda)w \in W_K$. Since $\mathrm{Hom}_{X_K}(W_K, (V_K/W_K)) = 0$ it follows that $D(W_K) \subset W_K$. We have a short exact sequence

$$0 \to W_K \to V \otimes K \to Q \to 0$$

where Q is a K-vector space and also a $\mathrm{Der}_k(K)$-module. Let e_i, for $i \in I$, be a k basis for V. Then the K span of the images $\pi(e_i)$ is equal to Q. Thus, we may find a subset $J \subset I$ such that $\pi(e_j)$, for $j \in J$, is a basis for Q as a K-vector space. Now consider the map

$$\bigoplus_{j \in J} e_j \otimes K \to Q\,.$$

This is an isomorphism of K-vector spaces such that the diagram

$$\begin{array}{ccc} \bigoplus_{j\in J} e_j \otimes K & \longrightarrow & Q \\ \Big\downarrow D & & \Big\downarrow D \\ \bigoplus_{j\in J} e_j \otimes K & \longrightarrow & Q \end{array}$$

commutes for all $D \in \mathrm{Der}_k(K)$. Thus, we have found a subspace $V' \subset V$ (the k span of e_j for $j \in J$) such that

$$V \otimes K \cong V' \otimes K \bigoplus W_K\,.$$

This isomorphism respects the action of $\mathrm{Der}_k(K)$ on both sides. There is a field K_1 such that $k \subset K_1 \subset K$ and $[K : K_1] = p$. Let A denote the algebra $\mathrm{Der}_{K_1}(K)$. Then $A \subset \mathrm{Der}_k(K)$. Taking the elements on both sides which are annihilated by A we get that

$$V \otimes_k K_1 \cong V' \otimes_k K_1 \bigoplus W_1$$

Here we have used the previous Lemma. Clearly, $W_1 \otimes_{K_1} K \to W_K$ is an isomorphism. We have thus descended W_K to K_1. Since $\mathrm{Hom}_{X_K}(W_K, (V_K/W_K)) =$

0 it follows that $\mathrm{Hom}_{X_{K_1}}(W_1, (V_{K_1}/W_1)) = 0$. Proceeding in this fashion we descend W_K to $W \subset V$. □

5. Socle and extended socle for semistable sheaves

All the results in this section can be found in [HL10, Section 1.5]. However, we mention them to motivate the results in the next section.

Definition 5.1. *Let E be a semistable sheaf on X. The socle is defined to be the largest polystable sheaf (defined over X) which is contained in E and has the same reduced Hilbert polynomial as E. We denote the socle by* $\mathrm{Soc}(E)$. *(See* [HL10, Lemma 1.5.5]*)*

Definition 5.2. *Let E be a semistable sheaf on X. The extended socle is the largest subsheaf $F \subset E$ with the same reduced Hilbert polynomial as E such that graded pieces appearing in a Jordan Holder filtration of F are the same as the graded pieces appearing in the socle.*

Lemma 5.3. *The socle and extended socle are invariant under automorphisms of X and E. (See* [HL10, Lemma 1.5.9]*)*

Lemma 5.4. *The extended socle satisfies* $\mathrm{Hom}_X(F, E/F) = 0$.

Proof. If $F = E$ then there is nothing to prove. Let us assume that there is a non-zero homomorphism $\varphi : F \to E/F$, and let $W \subset E/F$ denote the image. Let W_1 denote the preimage of W in E. One easily checks that the reduced Hilbert polynomial of W_1 is the same as that of E and that W_1 is semistable. Moreover, $F \subsetneqq W_1$ and every graded piece of W_1 in a Jordan Holder filtration is a graded piece appearing in a Jordan Holder filtration of F or in a Jordan Holder filtration of W. But since W is a quotient of F, the graded pieces appearing in a Jordan Holder filtration of W_1 already appear as graded pieces in the Jordan Holder filtration of F. Thus, $F \subsetneqq W_1$ is semistable, with the same graded pieces in a Jordan Holder filtration as that of $\mathrm{Soc}(E)$ and has the same reduced Hilbert polynomial as E. This contradicts the maximality of F. □

Lemma 5.5. *Let E be a stable sheaf on X_k. Let $k \subset K$ be an algebraic extension. Then the extended socle of E_K is equal to E_K.*

Proof. Suppose the extended socle is $F_K \subsetneqq E_K$. By Lemma 5.4 we know that $\mathrm{Hom}_{X_K}(F_K, E_K/F_K) = 0$. Since the extended socle is invariant under automorphisms of X_K and E_K it follows that it is invariant under $\mathrm{Gal}(K/k)$. By Lemma 4.1 and Lemma 4.4 it follows that there is a subsheaf $F \subsetneqq E$ such that $F_K = F \otimes K$. But this contradicts the stability of E since we get a destabilizing sheaf. □

Lemma 5.6. *If E is simple, semistable and equals its extended socle then E is stable.*

Proof. Let $E \to F$ be a quotient such that F is stable and $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$. Since E is its own extended socle, it follows that F appears in $\mathrm{Soc}(E)$. Thus, there is a map $E \to F \subset E$. Since E is simple, it follows that $E \cong F$, which is a contradiction. □

Lemma 5.7. *Let E be a semistable and simple sheaf. Then E is stable iff E is geometrically stable.*

Proof. Let $k \subset K$ denote the algebraic closure. Assume E_K is stable. Then it is clear that E is stable. Conversely, assume that E is stable. Since E_K is simple and semistable, by the previous lemma it suffices to show that E_K equals its own extended socle. But this has been proved in Lemma 5.5. □

6. Socle and extended socle for μ-semistable sheaves

We need to modify the discussion in the preceding section for the proof of the μ-stable restriction theorem. We briefly discuss this. Most of this section is contained in [HL10, Section 1.6].

Definition 6.1. *Let E be a μ-semistable sheaf. A μ-Jordan Holder filtration for E is a filtration $0 \subsetneqq E_1 \subsetneqq E_2 \subsetneqq \ldots \subsetneqq E_r = E$ such that $\mu(E_{i+1}/E_i) = \mu(E)$.*

We remark that we do not require that the sheaves E_{i+1}/E_i are torsion free. However, it is easily checked that the torsion in E_{i+1}/E_i will be in codimension ≥ 2. It is also easily checked that given two Jordan Holder filtrations E_i and E'_j, there is an open subset U with $\mathrm{codim}(X \setminus U, X) \geq 2$, such that when restricted to U, the sheaves $\oplus_i E_{i+1}/E_i$ and $\oplus_i E'_{i+1}/E'_i$ are isomorphic. Let $S := \{F \,|\, F \subset E\,, \mu(F) = \mu(E)\,, F \text{ is } \mu\text{-stable }\}$. Let F_0 be a subsheaf of E, of largest possible rank, such that F_0 is a direct sum of sheaves in S. Let $\tilde{F}_0$ be the saturation of F_0, that is, the kernel of the map $E \to E/F_0 \to (E/F_0)/T(E/F_0)$. One easily checks that if $F \subset E$ is any μ-stable sheaf with $\mu(F) = \mu(E)$, then $F \subset \tilde{F}_0$. Thus, we may also characterize $\tilde{F}_0$ as the saturation of the sum of all μ-stable subsheaves of E with slope $\mu(E)$. Define the socle of E to be $\mathrm{Soc}(E) := \tilde{F}_0$.

It is clear that if K/k is a Galois extension and E_K is μ-semistable on X_K, then $\mathrm{Soc}(E_K)$ is invariant under $\mathrm{Gal}(K/k)$.

Next we define the extended socle for a μ-semistable sheaf E. Consider the collection of sheaves $F \subset E$ which satisfy the following conditions

(1) $\mathrm{Soc}(E) \subset F$
(2) Let F_i be a μ-Jordan Holder filtration for F. Then each F_{i+1}/F_i agrees with a graded piece in the μ-Jordan Holder filtration of $\mathrm{Soc}(E)$ on some open subset U such that $\mathrm{codim}(X \setminus U, X) \geq 2$.

Let F be a maximal sheaf in this collection. If F_1 and F_2 are two such maximal sheaves, then one easily proves that $\mathrm{Hom}(F_1, E/F_2) = 0$. This shows that there is a unique maximal sheaf which satisfies these properties. Define this to be the extended socle of E. It is clear that F satisfies

$\mathrm{Hom}(F, E/F) = 0$. It is also clear that if K/k is a Galois extension and E_K is μ-semistable on X_K, then the extended socle is invariant under $\mathrm{Gal}(K/k)$.

Lemma 6.2. *Let E be a μ-stable sheaf on X_k. Let $k \subset K$ be the algebraic closure. Then the extended socle of E_K is equal to E_K.*

Proof. Suppose the extended socle is $F_K \subsetneqq E_K$. From the above discussion we know that $\mathrm{Hom}_{X_K}(F_K, E_K/F_K) = 0$. Let L be the separable closure of k in K. By Lemma 4.4 it follows that there is a subsheaf $F_L \subsetneqq E_L$ such that $F_K = F_L \otimes_L K$.

Tha Galois group $\mathrm{Gal}(L/k)$ acts on E_L. Let us check that $g(F_L) = F_L$. Note that $g(F_L) \otimes_L K = g(F_L \otimes_L K) = g(F_K) = F_K$. This forces that $g(F_L) = F_L$. By Lemma 4.1 it follows that there is a subsheaf $F \subsetneqq E$ such that $F_L = F \otimes_k L$. But this contradicts the stability of E since we get a destabilizing sheaf. □

Lemma 6.3. *Let X be a normal and integral scheme. If E is reflexive, simple, μ-semistable and equals its extended socle then E is μ-stable.*

Proof. Let $E \to F$ be a quotient such that F is μ-stable and $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$. Since E is its own extended socle, it follows that there is a large open subset U such that F_U appears in $\mathrm{Soc}(E)_U$. Thus, there is a map $E_U \to F_U \subset E_U$. Since E is reflexive on a normal and integral scheme it satisfies Serre's condition S_2. Using [Har77, Chapter III, Ex. 2.3, Ex 3.4] it follows that $\mathrm{Hom}(E, E) \to \mathrm{Hom}_U(E, E)$ is surjective. Since E_U is simple, we get a contradiction. □

Remark 6.4. Since E is torsion free, it follows that $\mathcal{H}om(E, E)$ is torsion free. Again applying [Har77, Chapter III, Ex. 2.3, Ex 3.4] we have shown in the above lemma that the sheaf $\mathcal{H}om(E, E)$ is reflexive if E is reflexive on a normal and integral scheme.

Lemma 6.5. *Let X be a normal and integral scheme. Let E be a reflexive, μ-semistable and simple sheaf. Then E is μ-stable iff E is geometrically μ-stable.*

Proof. Let $k \subset K$ denote the algebraic closure. Assume E_K is μ-stable. Then it is clear that E is μ-stable. Conversely, assume that E is μ-stable. Since E_K is reflexive, simple and μ-semistable, by the previous lemma it suffices to show that E_K equals its own extended socle. But this has been proved in Lemma 6.2. □

7. Openness of certain loci

Lemma 7.1. *Let k be a field. Let Y be a projective k-scheme with a fixed very ample line bundle $\mathcal{O}_Y(1)$. Let $\mathcal{F}$ be a coherent sheaf on Y such that $\mathrm{Supp}(\mathcal{F}) = Y$. Let $\mathrm{reg}(\mathcal{F}) \leq \rho$ and $\dim \mathcal{F} = \dim Y = d$. Let V be a vector space of dimension $P(\mathcal{F}, \rho)$ and define $\mathcal{G} := V \otimes \mathcal{O}_{\mathbb{P}^d_k}(-\rho)$ on $\mathbb{P}^d_k$. Then we*

have an inclusion of sets of polynomials

$$\{P(F) \,|\, \mathcal{F} \to F \to 0,\ F \text{ is pure},\ \widehat{\mu}(P(F)) \leq \lambda\} \subset \\ \{P(G) \,|\, \mathcal{G} \to G \to 0,\ G \text{ is pure},\ \widehat{\mu}(P(G)) \leq \lambda\}\,.$$

Proof. Let $q : \mathcal{F} \to F \to 0$ be a pure quotient on Y. We want to construct a quotient $q' : \mathcal{G} \to G \to 0$ on $\mathbb{P}^d$ such that G is pure and $P(G) = P(F)$. We have the closed immersion $Y \hookrightarrow \mathbb{P}^N_k$ given by $\mathcal{O}_Y(1)$. Choose a linear subspace $L \subset \mathbb{P}^N_k$ of dimension $N-d-1$ which is disjoint from Y. Then we have the projection $\mathbb{P}^N_k \setminus L \to \mathbb{P}^d_k$. Denote the composition $Y \hookrightarrow \mathbb{P}^N \setminus L \to \mathbb{P}^d$ by π. Then π is finite and $\pi^*\mathcal{O}_{\mathbb{P}^d}(1) = \mathcal{O}_Y(1)$. Hence, $\pi_* q$ is surjective. Therefore, by projection formula and finiteness of π, we have

$$P(\pi_*\mathcal{F}) = P(\mathcal{F}),\ P(\pi_* F) = P(F),\ \operatorname{reg}(\pi_*\mathcal{F}) = \operatorname{reg}(\mathcal{F}) \leq \rho\,.$$

Using this last equality we get a surjection $H^0(\mathbb{P}^d, \pi_*\mathcal{F}(\rho)) \otimes \mathcal{O}_{\mathbb{P}^d}(-\rho) \to \pi_*(\mathcal{F})$. Again using $\operatorname{reg}(\pi_*\mathcal{F}) \leq \rho$ we have $H^i(\mathbb{P}^d, \pi_*\mathcal{F}(\rho)) = 0\ \forall\ i > 0$. Hence $H^0(\mathbb{P}^d, \pi_*\mathcal{F}(\rho)) = P(\pi_*\mathcal{F}, \rho) = P(\mathcal{F}, \rho)$. Therefore, we get a quotient

$$H^0(\mathbb{P}^d, \pi_*\mathcal{F}(\rho)) \otimes \mathcal{O}_{\mathbb{P}^d}(-\rho) \to \pi_*(\mathcal{F}) \to \pi_* F$$

It is clear that $\pi_* F$ is pure. Since $P(\pi_* F) = P(F)$ it follows that $\widehat{\mu}(P(\pi_* F)) = \widehat{\mu}(P(F)) \leq \lambda$. □

Proposition 7.2. [HL10, Proposition 2.3.1] *Let $f : Z \to S$ be a projective morphism of k-schemes of finite type. Let $\mathcal{F}$ be a coherent sheaf on Z which is flat over S. Further assume that* $\operatorname{Supp}(\mathcal{F}) = Z$. *Then the following subsets of S are open*

(1) $U_{\text{sim}} = \{s \in S \,|\, \mathcal{F}_{k(s)}$ *is simple on* $Z_{k(s)}\}$
(2) $U_{\text{pr}} = \{s \in S \,|\, \mathcal{F}_{k(s)}$ *is pure on* $Z_{k(s)}\}$
(3) $U_{\text{st}} = \{s \in S \,|\, \mathcal{F}_{\overline{k(s)}}$ *is stable on* $Z_{\overline{k(s)}}\}$
(4) $U_{\text{ss}} = \{s \in S \,|\, \mathcal{F}_{k(s)}$ *is semistable on* $Z_{k(s)}\}$
(5) $U_{\mu-\text{ss}} = \{s \in S \,|\, \mathcal{F}_{k(s)}$ *is μ-semistable on* $Z_{k(s)}\}$
(6) $U_{\mu-\text{st}} = \{s \in S \,|\, \mathcal{F}_{\overline{k(s)}}$ *is μ-stable on* $Z_{\overline{k(s)}}\}$

Proof. The statement (1) in the proposition is a consequence of semi-continuity for relative Ext sheaves.

Let $Z \hookrightarrow S \times P^m_k$ be an embedding and consider the pullback of $\mathcal{O}(1)$ to Z. The Hilbert polynomial of $\mathcal{F}_{k(s)}$, with respect to $\mathcal{O}(1)$ is independent of $s \in S$. We denote this Hilbert polynomial by P and the reduced Hilbert polynomial by p. Recall that we defined $\alpha_i(P)$ as the coefficient of $\dfrac{t^i}{i!}$ in P i.e. $P(t) = \sum \alpha_i(P)\dfrac{t^i}{i!}$. If $P(t)$ is a degree d polynomial, then $\widehat{\mu}(P) := \dfrac{\alpha_{d-1}(P)}{\alpha_d(P)}$.

Define A to be the set of polynomials $P(F', t)$, where F' is a sheaf satisfying the following three conditions

(a) There is a point $s \in S$ such that $\mathcal{F}_{\overline{k(s)}} \to F'$ is a quotient on $Z_{\overline{k(s)}}$
(b) F' is pure of dimension $d = \dim(Z_{\overline{k(s)}})$

(c) $\widehat{\mu}(P(F')) \leq \widehat{\mu}(P)$

We will first show that A is finite. Since the set $\{\mathcal{F}_{k(s)} | s \in S\}$ is bounded, by [HL10, Lemma 1.7.6] there is ρ such that $\mathrm{reg}(\mathcal{F}_{k(s)}) \leq \rho$. This shows that for every $s \in S$, the regularity of the sheaf $\mathcal{F}_{\overline{k(s)}}$ on $Z_{\overline{k(s)}}$ is $\leq \rho$. It is clear that $\mathrm{Supp}(\mathcal{F}_{\overline{k(s)}}) = Z_{\overline{k(s)}}$. Hence by Lemma 7.1 we have

$$\{P(F') \,|\, \mathcal{F}_{\overline{k(s)}} \to F' \to 0,\ F' \text{ is pure on } Z_{\overline{k(s)}},\ \widehat{\mu}(P(F')) \leq \lambda\} \subset \\ \{P(G) \,|\, \mathcal{G}_{\overline{k(s)}} \to G \to 0,\ G \text{ is pure on } \mathbb{P}^d_{\overline{k(s)}},\ \widehat{\mu}(P(G)) \leq \lambda\}\,.$$

But every polynomial in the latter set already occurs in the set

$$\{P(G) \,|\, \mathcal{G}_{\bar{k}} \to G \to 0,\ G \text{ is pure over } \mathbb{P}^d_{\bar{k}},\ \widehat{\mu}(P(G)) \leq \lambda\}$$

since every $\overline{k(s)}$ point of a Quot scheme factors through a $\bar{k}$ point. Thus, A is contained in the set

$$\{P(G) \,|\, \mathcal{G}_{\bar{k}} \to G \to 0,\ G \text{ is pure on } \mathbb{P}^d_{\bar{k}},\ \widehat{\mu}(P(G)) \leq \widehat{\mu}(P)\}\,.$$

By [HL10, Lemma 1.7.9] we have that A is finite.

To prove (2)-(6) we will consider each of the following sets:

(2) $A_2 := \{P' \in A \,|\, \alpha_d(P') = \alpha_d(P) \text{ and } P \neq P'\}$
(3) $A_3 := \{P' \in A \,|\, p' \leq p \text{ and } \alpha_d(P') < \alpha_d(P)\}$
(4) $A_4 := \{P' \in A \,|\, p' < p \text{ and } \alpha_d(P') < \alpha_d(P)\}$
(5) $A_5 := \{P' \in A \,|\, \widehat{\mu}(P') < \widehat{\mu}(P) \text{ and } \alpha_d(P') < \alpha_d(P)\}$
(6) $A_6 := \{P' \in A \,|\, \widehat{\mu}(P') \leq \widehat{\mu}(P) \text{ and } \alpha_d(P') < \alpha_d(P)\}$

Each of the above is a finite set since A is finite. For each $2 \leq i \leq 6$ we consider the morphism

$$\mathrm{Q}_i := \bigsqcup_{P' \in A_i} \mathrm{Quot}_{Z/S}(\mathcal{F}, P') \to S$$

Let S_i be the image of this morphism. It is closed since the above morphism is projective. We claim that $S \setminus (S_i \cup S_2)$ is precisely the set in U in assertion (i) in the statement of the proposition. Here we only prove (2), (5) and (6).

Proof of (2). Let $s \in S \setminus U_2$, that is, $T(\mathcal{F}_{k(s)}) \neq 0$. Therefore, we have the quotient $\mathcal{F}_{k(s)} \to \mathcal{F}_{k(s)}/T(\mathcal{F}_{k(s)}) =: F'$ such that F' is pure, $\widehat{\mu}(P(F')) \leq \widehat{\mu}(P)$ and $\alpha_d(F) = \alpha_d(F')$. Thus, $P(F') \in A_2$ and we get a $k(s)$ point of Q_2 whose image is in S_2. This shows that $S \setminus U_{\mathrm{pr}} \subset S_2$.

Conversely, start with $s \in S_2$. This means that there is a $\overline{k(s)}$ point of Q_2 which maps to the given $\overline{k(s)}$ point of S_2. This implies that there is a quotient $\mathcal{F}_{\overline{k(s)}} \to F'$ on $Z_{\overline{k(s)}}$ with deg $P(F') = d$, $\alpha_d(P(F')) = \alpha_d(P)$ and $P(F') \neq P$. Thus, the kernel of $\mathcal{F}_{\overline{k(s)}} \to F'$ is a non-trivial sheaf of dimension $\leq d-1$. Therefore, $\mathcal{F}_{\overline{k(s)}}$ is not pure and so $\mathcal{F}_{k(s)}$ is not pure. This shows that $S_2 \subset S \setminus U_{\mathrm{pr}}$.

Proof of (5). Let $s \in S \setminus U_{\mu-\mathrm{ss}}$, that is, $\mathcal{F}_{k(s)}$ is not μ-semistable. By definition we can have two situations:

(a) $\mathcal{F}_{k(s)}$ is not pure
(b) $\mathcal{F}_{k(s)}$ is pure and $\exists$ a quotient F' of over $Z_{k(s)}$ such that F' is pure of dimension d, $\alpha_d(P(F')) < \alpha_d(P)$ and $\widehat{\mu}(P(F')) < \widehat{\mu}(P)$.

By (2), (a) implies that $s \in S_2$ and (b) implies $P(F') \in A_5$, giving a $k(s)$ point of Q_5, which in turn implies $s \in S_5$. This proves that $S \setminus U_{\mu-\mathrm{ss}} \subset S_2 \cup S_5$.

Now suppose $s \in S_2 \cup S_5$. If $s \in S_2$, then by (2) $\mathcal{F}_{k(s)}$ is not pure and hence not μ-semistable. So assume $s \in S_5 \setminus S_2$. By (2) we have that $\mathcal{F}_{k(s)}$ is pure. There $\exists$ a quotient $\mathcal{F}_{\overline{k(s)}} \to F'$ over $Z_{\overline{k(s)}}$ with deg $P(F') = d$ and $\widehat{\mu}(P(F')) < \widehat{\mu}(P)$. Therefore we get that $\mathcal{F}_{\overline{k(s)}}$ is not μ-semistable. Because of the existence and uniqueness of Harder-Narasimhan filtration, we have that $\mathcal{F}_{k(s)}$ is not μ-semistable. This shows that $S_2 \cup S_5 \subset S \setminus U_{\mu-\mathrm{ss}}$.

Proof of (6). Let $s \in S \setminus U_{\mu-\mathrm{st}}$, that is, $\mathcal{F}_{\overline{k(s)}}$ is not μ-stable. Again we have two cases:

(a) $\mathcal{F}_{\overline{k(s)}}$ is not pure, which implies $\mathcal{F}_{k(s)}$ is not pure
(b) $\mathcal{F}_{k(s)}$ is pure. $\exists$ a pure quotient $\mathcal{F}_{\overline{k(s)}} \to F'$ over $Z_{\overline{k(s)}}$ such that $\widehat{\mu}(P(F')) \leq \widehat{\mu}(P)$ and $0 < \alpha_d(P(F')) < \alpha_d(P)$.

In case (a) $s \in S_2$ and in case (b) $s \in S_6$. This proves that $S \setminus U_{\mu-\mathrm{st}} \subset S_2 \cup S_6$.

Now suppose $s \in S_2 \cup S_6$. If $s \in S_2$ then $\mathcal{F}_{k(s)}$ is not pure and hence $\mathcal{F}_{\overline{k(s)}}$ is not pure and so not μ-stable. Let $s \in S_6 \setminus S_2$. Then $\exists$ a quotient $\mathcal{F}_{\overline{k(s)}} \to F'$ over $Z_{\overline{k(s)}}$ with deg $P(F') = d$, $\alpha_d(P(F')) < \alpha_d(P)$ and $\widehat{\mu}(P(F')) \leq \widehat{\mu}(P)$. Therefore, $\mathcal{F}_{\overline{k(s)}}$ is not μ-stable. This proves that $S_2 \cup S_6 \subset S \setminus U_{\mu-\mathrm{st}}$.

The other cases are dealt with similarly. The proposition is proved since the set $S_i \cup S_2$ is closed. □

8. Stable Restriction Theorem

In this section we will prove the μ-stable restriction theorem [MR84, Theorem 4.3].

Theorem 8.1. *Let X be a smooth projective variety of dimension $n \geq 2$ over an algebraically closed field k. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let E be a μ-stable sheaf on X. Then there is an integer a_0 such that for all $a \geq a_0$ there is a non-empty open set $U_a \subset \Pi_a$ such that for all $[D] \in U_a$ the divisor D is smooth and $E|_D$ is μ-stable with respect to $\mathcal{O}_X(1)|_D$.*

We will prove this theorem by contradiction, that is, we will assume that there are infinitely many a for which $E|_D$ is not μ-stable a general $D \in \Pi_a$. From this we will construct a set $N \subset \mathbb{N}$ with the following properties. For

each $a \in N$, we have a non-empty open set $W_a \subset \mathbb{P}(H^0(X, \mathcal{O}_X(a))^\vee)$ such that

(1) each $[D] \in W_a$ is smooth and integral.
(2) For $D \in W_a$, $E|_D$ is μ-semistable.

Over Z_{W_a} we have a quotient $q^*E \to H_a \to 0$ such that

(1) H_a is W_a-flat.
(2) For $D \in W_a$, $H_a|_D$ is torsion-free and $\mu(H_a|_D) = \mu(E|_D)$.
(3) We have an integer $0 < r < \text{rk } E$ such that $\text{rk } H_a = r \ \forall \ a \in N$.
(4) There exists $\mathcal{L} \in \text{Pic } X$ such that $\forall \ a \in N, \det(H_a)|_D \cong \mathcal{L}|_D$.

Then we will apply Lemma 3.10 to get a contradiction.

Lemma 8.2. *Let F be a reflexive sheaf on a smooth projective variety X over an algebraically closed field. Then $H^1(X, F(-a)) = 0$ for $a \gg 0$.*

Proof. For a reflexive sheaf there is a short exact $0 \to F \to F_0 \to G \to 0$ where F_0 is a direct sum of line bundles $\mathcal{O}_X(b)$. The Lemma follows from the long exact cohomology sequence and Serre duality ($\text{Ext}^i(A, B) = \text{Ext}^{n-i}(B, A \otimes \omega_X)^\vee$). □

Lemma 8.3. *Let E be reflexive. There is an a_0 depending on E such that the following happens. If $a \geq a_0$ is such that $E|_D$ is not μ-stable for a general $D \in \Pi_a$, then we have a non-empty open set $W_a \subset \mathbb{P}(H^0(X, \mathcal{O}_X(a))^\vee)$ such that*

(1) each $[D] \in W_a$ is smooth and integral.
(2) For $D \in W_a$, $E|_D$ is μ-semistable.

and over Z_{W_a} we have a quotient $q^*E \to H_a \to 0$ such that

(1) H_a is W_a-flat.
(2) For $D \in W_a$, $H_a|_D$ is torsion-free and $\mu(H_a|_D) = \mu(E|_D)$.

Proof. Tensoring the following short exact sequence (defined by a general section of $|\mathcal{O}_X(a)|$ and using Lemma [HL10, Lemma 1.1.12]) with $\mathcal{H}om(E, E)$

$$0 \to \mathcal{O}_X(-a) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

taking cohomology and applying Remark 6.4, Lemma 8.2, we see that there is $a_0 \gg 0$ (and larger than the one appearing in Theorem 3.1) such that for $a \geq a_0$ we have $\text{End}(E) \to \text{End}(E|_D)$ is an isomorphism. Thus, the sheaf $E|_D$ is simple.

Now consider the family $Z_a \to \Pi_a$ and recall that q^*E is flat over Π_a. By Theorem 3.1 we know that if $[D] \in U_a$ then $E|_D$ is μ-semistable. From Proposition 7.2 it follows that over the generic point $\eta \in \Pi_a$, $q^*E_{k(\eta)}$ is simple and μ-semistable.

By [HL10, Corollary 1.1.14 (ii)] $E|_D$ is reflexive for general $[D]$. It follows from [Gro64, Theorem 12.2.1(v)] and the criterion that on a normal and integral scheme reflexive is equivalent to S_2, that the set

$$U_{\text{ref}} = \{s \in S \mid q^*E_{k(s)} \text{ is reflexive on } Z_{k(s)}\}$$

is open in S. Thus, $q^*E_{k(\eta)}$ is also reflexive.

Let us assume that $E|_D$ is not μ-stable for general $[D] \in \Pi_a$. This is equivalent to saying that the set $U_{\mu-\mathrm{st}}$ in Proposition 7.2 is empty, that is, $q^*E_{k(\eta)} \otimes \overline{k(\eta)}$ is not μ-stable. Since $q^*E_{k(\eta)}$ is reflexive, simple and μ-semistable it follows, using Lemma 6.5, that $q^*E_{k(\eta)}$ is not μ-stable. Take quotient by the extended socle and extend it to a quotient $q^*E \to H_a \to 0$ [Har77, Chapter II, Exc. 5.15(d)] on Z_a. Going modulo torsion we may assume that H_a is torsion free on Z_a. Using generic flatness and Proposition 7.2, we get that $\exists\, W_a \subset \Pi_a$ over which H_a has the required properties. □

Taking determinant of H_a, as described in the para preceding Lemma 3.3, we get a line bundle $\mathcal{L}_a \in \mathrm{Pic}\, X$.

Lemma 8.4. *Let E be reflexive and let a_0 be as in Lemma 8.3. Let D_1 be a general hypersurface of degree $a_1 \geq a_0$ such that $E|_{D_1}$ is μ-stable. Then for every $a \geq 2a_1$ and general D of degree a, we have that $E|_D$ is μ-stable.*

Proof. Suppose $a \geq 2a_1$ be such that $E|_{D'}$ is not μ-stable for a general $[D'] \in \Pi_a$. By Lemma 8.3, we have a flat quotient $q^*E|_{W_a} \to H_a \to 0$. Fix $D_1 \in \Pi_{a_1}$ and $D_2 \in \Pi_{a-a_1}$ be such that $E|_{D_i}$ is μ-semistable and $D := D_1 + D_2$ is a SNC divisor. Let $C \subset \Pi_a$ such that $[D] \in C$ and $C \setminus [D] \subset W_a$ be as in Lemma 2.7. Restrict $q^*E|_{W_a} \to H_a \to 0$ to $Z_{C\setminus[D]}$ and by Lemma 2.11, this extends to a C-flat quotient H_C over Z_C. Define $H := H_C|_D$ and $\bar{H} := H/T(H)$. By Lemma 2.12, we get

$$\mu(E|_{D'}) = \mu(H) \geq \mu(\bar{H}_{D_1}/T(\bar{H}_{D_1})) + \mu(\bar{H}_{D_2}/T(\bar{H}_{D_2}))$$

Since $E|_{D_1}$ and $E|_{D_2}$ are μ-semistable, we have

$$\mu(\bar{H}_{D_i}/T(\bar{H}_{D_1})) \geq \mu(E|_{D_i})$$

Now since $\mu(E|_D) = \mu(E|_{D_1}) + \mu(E|_{D_2})$ (recall from equation (3.6)), we get that

$$\mu(\bar{H}_{D_i}/T(\bar{H}_{D_i})) = \mu(E|_{D_i})$$

Also by Lemma 2.12, we know that the rank of $\bar{H}_{D_i}/T(\bar{H}_{D_i})$ is equal to $\mathrm{rk}(H_a) < \mathrm{rk}(E)$. Hence this contradicts the assumption that $E|_{D_1}$ is μ-stable. □

We continue our discussion with the additional assumption that E is reflexive. This assumption will be removed in the end. Let us now assume that there is no a_0 such that for general hypersurface D of degree $a \geq a_0$, the restriction $E|_D$ is μ-stable. In view of the previous Lemma, this means that for every $a \geq a_0$, the restriction $E|_D$ is not μ-stable. Thus, we get a quotient H_a and a line bundle $\mathcal{L}_a \in \mathrm{Pic}\, X$. Let $a \geq 2a_0$. We want to understand what happens when we restrict $\mathcal{L}_a$ to the general hypersurface D_0 of degree a_0.

Lemma 8.5. *The restriction of $\mathcal{L}_a$ to D_0 is the determinant of a destabilizing quotient of $E|_{D_0}$.*

Proof. For this we proceed with the construction of the sheaf $\tilde{\mathcal{A}}$ on Z_C as done in the proof of Lemma 3.7. Note that equation (3.8) holds. Now $\bar{H}_{D_i}/T(\bar{H}_{D_i})$ is a quotient of $E|_{D_i}$ with

$$\mu(\bar{H}_{D_i}/T(\bar{H}_{D_i})) = \mu(E|_{D_i}) = a_i\mu(E)\,.$$

Thus, we get that $\tilde{\mathcal{A}}|_{D_i\setminus D_s}$ is the determinant of a destabilizing quotient of $E|_{D_i}$.

It is clear that for a point $[D'] \in C \setminus [D]$

$$\tilde{\mathcal{A}}|_{D'} \cong \mathcal{L}_a|_{D'}\,.$$

Let $p : Z_C \to C$. Consider $p_*(\mathcal{L}_a^\vee \otimes \tilde{\mathcal{A}})$. Since $h^0(D', \mathcal{L}_a^\vee \otimes \tilde{\mathcal{A}}|_{D'}) = 1$, it follows by semi-continuity that $h^0(D, \mathcal{L}_a^\vee \otimes \tilde{\mathcal{A}}|_D) \geq 1$. Let $\phi : \mathcal{L}_a|_D \to \tilde{\mathcal{A}}|_D$ be a non-zero map. It has to be non-zero restricted to one of the D_i, say D_1. So we have a non-zero map $\phi : \mathcal{L}_a|_{D_1} \to \tilde{\mathcal{A}}|_{D_1}$. But we have seen above that $\tilde{\mathcal{A}}|_{D_i\setminus D_s} \cong \det((\bar{H}_{D_i}/T(\bar{H}_{D_i}))|_{D_i\setminus D_s})$. Thus, we have a non-zero map $\phi : \mathcal{L}_a|_{D_1\setminus D_s} \to \det((\bar{H}_{D_1}/T(\bar{H}_{D_1}))|_{D_1\setminus D_s})$. Let us compute degrees of both. Let D' be a general hypersurface of degree a.

$$\begin{aligned}\deg(\mathcal{L}_a|_{D'}) &= \mathrm{rk}(H_a|_{D'})\mu(H_a|_{D'})\\ &= \mathrm{rk}(H_a)\mu(E|_{D'})\\ &= a\,\mathrm{rk}(H_a)\mu(E)\end{aligned}$$

Since $\deg(\mathcal{L}_a|_{D'}) = a\deg(\mathcal{L}_a)$, we get that

$$\deg(\mathcal{L}_a) = \mathrm{rk}(H_a)\mu(E) \tag{8.6}$$

from which we deduce that

$$\deg(\mathcal{L}_a|_{D_1}) = a_1\,\mathrm{rk}(H_a)\mu(E)\,.$$

Similarly,

$$\begin{aligned}\deg((\bar{H}_{D_1}/T(\bar{H}_{D_1}))|_{D_1\setminus D_s}) &= \mathrm{rk}(\bar{H}_{D_1}/T(\bar{H}_{D_1}))\mu(\bar{H}_{D_1}/T(\bar{H}_{D_1}))\\ &= a_1\mathrm{rk}(\bar{H}_{D_1}/T(\bar{H}_{D_1}))\mu(E)\,.\end{aligned}$$

By Lemma 2.12 we have

$$\mathrm{rk}(H_a) = \mathrm{rk}(H_C) = \mathrm{rk}(H_C|_D) = \mathrm{rk}(\bar{H}_{D_1}/T(\bar{H}_{D_1}))\,.$$

It follows that both line bundles have same degree. Thus, the map $\phi : \mathcal{L}_a|_{D_1} \to \det(\bar{H}_{D_1}/T(\bar{H}_{D_1}))$ is an isomorphism.

Take $D_1 = D_0$ and take D_2 to be a general hypersurface of degree $a - a_0$. This shows that when we restrict $\mathcal{L}_a$ to D_0, we get the determinant of one of the destabilizing quotients of $E|_{D_0}$. □

The set consisting of determinants of the destabilizing quotients of $E|_{D_0}$ has cardinality at most $2^{\mathrm{rk}(E)}$. This set will be denoted by T_{D_0}. Let $m = 2^{\mathrm{rk}(E)} + 1$. Suppose we have distinct integers $a_1, a_2, \ldots, a_m \geq 2a_0$. Define the set $W(i,j) \subset W_{a_0}$ as follows. Let $[D'] \in W_{a_0}$. We say $[D'] \in W(i,j)$ if $\mathcal{L}_{a_i}|_{D'} \cong \mathcal{L}_{a_j}|_{D'}$. Since $T_{D'}$ has cardinality $m-1$, it is clear that $[D']$ is in $W(i,j)$ for some pair (i,j) with $i \neq j$. Thus, $W_{a_0} = \cup_{i\neq j} W(i,j)$ and so one

of the $W(i,j)$ is Zariski dense in W_{a_0}. This forces that $\mathcal{L}_{a_i}|_{D'} \cong \mathcal{L}_{a_j}|_{D'}$ for all $[D'] \in W_{a_0}$, by Lemma 3.3.

We put an equivalence relation on $\mathbb{N}_{\geq 2a_0}$ as follows. Define $a \sim b$ if $\mathcal{L}_a|_{D'} \cong \mathcal{L}_b|_{D'}$ for all $[D'] \in W_{a_0}$. Given any subset $S \subset \mathbb{N}_{\geq 2a_0}$ such that $\#S = m$, we get that two of its elements are equivalent. This shows that there are at most $m-1$ equivalence classes, and so at least one equivalence class has infinite cardinality. Call this equivalence class N_1. Then N_1 has the property that for every $a, b \in N_1$, and for $[D'] \in W_{a_0}$, the bundles $\mathcal{L}_a|_{D'} \cong \mathcal{L}_b|_{D'}$, that is, $\mathcal{L}_a \cong \mathcal{L}_b =: \mathcal{L}$.

Further we may find an infinite subset $N \subset N_1$ such that for every $a \in N$, the rank $\mathrm{rk}(H_a)$ is constant. This set N is the set which satisfies the criterion in the para just before Lemma 8.3. Now we may apply Lemma 3.10.

Proof of Theorem 8.1. Let E be a reflexive sheaf. Applying Lemma 3.10 we get a quotient $E|_{X'} \to H_{X'}$ such that $\det(H_{X'}) = \mathcal{L}$ and $\mathrm{rk}(H_{X'}) = \mathrm{rk}(H_a) < \mathrm{rk}(E)$ for $a \in N$. It follows from equation (8.6) that $\mu(H_{X'}) = \mu(E_{X'})$. This contradicts the stability of E. Thus, we have proved the following, there is an integer a_0 such that for a reflexive μ-stable sheaf E, the restriction $E|_D$, to a general hypersurface of degree $a \geq a_0$, is μ-stable.

Now let E be a μ-stable sheaf on X. Then $E^{\vee\vee}$ is a reflexive μ-stable sheaf. Let T denote the cokernel of the map $E \to E^{\vee\vee}$. It is supported on a closed subset of X codimension ≥ 2. Restricting this to a general D we get

$$E|_D \to E^{\vee\vee}|_D \to T|_D \to 0\,.$$

Since D is general, the two sheaves on the left are torsion free and $T|_D$ is supported on a closed set in D of codimension ≥ 2. Thus, $E|_D$ and $E^{\vee\vee}|_D$ are isomorphic on a large open set. Since $E^{\vee\vee}|_D$ is μ-stable, it follows that $E|_D$ is μ-stable. This completes the proof of the theorem. □

9. Narasimhan-Seshadri Theorem in higher dimensions

Throughout this section we assume that $k = \mathbb{C}$. Let X be a algebraic variety over $\mathbb{C}$. Then $X(\mathbb{C})$ has a structure of an analytic variety, which is denoted by X^h. If X is projective, by GAGA, the two categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X^h)$ are equivalent. For a sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ we denote the corresponding sheaf in $\mathrm{Coh}(X^h)$ by $\mathcal{F}^h$.

In [NS65], the following theorem was proved:

Theorem 9.1. [NS65, §12, Corollary 1] *Let X be a smooth projective curve of genus ≥ 2. Then a vector bundle E of degree zero on X is stable if and only if E^h arises from an irreducible unitary representation of the fundamental group $\pi_1(X^h)$.*

Combining [Don85, Thm. 1] with [Kob87, Chapter IV, Propn. 4.13] and [Kob87, Chapter I, Propn. 4.21] the above theorem was extended to the case of smooth projective surfaces.

Theorem 9.2. *Let X be a smooth projective surface over $\mathbb{C}$. Let H be an ample line bundle on X. Let V be a vector bundle with $c_1(V) = 0$ and $c_2(V) = 0$ ($c_i(V) \in H^{2i}(X, \mathbb{C})$). Then V is μ_H-stable iff V^h comes from an irreducible unitary representation of the fundamental group $\pi_1(X^h)$.*

In [MR84, §5], as an easy and remarkable consequence of Theorem 8.1, Theorem 9.1 was extended to any dimension using Theorem 9.1, Theorem 9.2. In this section we sketch how to do this.

Theorem 9.3. *Let X be a projective nonsingular variety over $\mathbb{C}$ of dimension n. Let H be an ample line bundle on X. Let V be a vector bundle on X with $c_1(V) = 0$ and $c_2(V).H^{n-2} = 0$. Then V is μ_H-stable iff V^h comes from an irreducible unitary representation of the fundamental group $\pi_1(X^h)$.*

Proof. Let V^h come from an irreducible unitary representation of the fundamental group $\rho : \pi_1(X^h) \to \mathrm{U_r}$. By Bertini's theorem, the intersection of $n-1$ general members of $|aH|$ for $a \gg 0$ is a smooth projective curve C. By Lefschetz hyperplane theorem for fundamental groups $\pi_1(C^h) \to \pi_1(X^h)$ is surjective. Hence the representation $\pi_1(C^h) \to \pi_1(X^h) \to \mathrm{U}_r$ is irreducible and unitary. Since the restriction $V|_{C^h}$ is associated to this representation, by Theorem 9.1, we get that $V|_C$ is stable. The μ_H-stability of V easily follows from the μ-stability of $V|_C$.

We will prove the converse by induction on dimension of X. The base case is when dimension of X is 2, whence it is Theorem 9.2. Let V be μ_H-stable on X. Let $\mathcal{S}$ denote the set of isomorphism classes of μ_H-stable vector bundles W with $c_i(W) = 0 \ \forall \ i > 0$ and rk $W =$ rk V. By [HL10, Theorem 3.3.7] $\mathcal{S}$ is bounded. Hence the set of isomorphism classes of vector bundles $\omega_X \otimes W^\vee \otimes V$ with W μ_H-stable, $c_i(W) = 0 \ \forall \ i > 0$ and rk $W =$ rk V is bounded. In particular by [HL10, Lemma 1.7.6], the regularity of these bundles are uniformly bounded and by [HL10, Lemma 1.7.2] there exists $l_1 \in \mathbb{N}$ such that $\forall \ l \geq l_1$ and $W \in \mathcal{S}$, $H^{n-1}(\omega_X \otimes W^\vee \otimes V(l)) = 0$. By Serre duality we get $H^1(X, V^\vee \otimes W(-l)) = 0$. Therefore the map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V|_D, W|_D)$ is surjective for D any member of $|lH|$. Fix such a D which is smooth and such that $V|_D$ is μ_H-stable. By induction hypothesis, there is an irreducible representation $\rho : \pi_1(D^h) \to \mathrm{U_r}$ such that the associated bundle is $(V|_D)^h$. Since the natural map $\pi_1(D^h) \to \pi_1(X^h)$ is an isomorphism, it follows that we get a representation $\rho : \pi_1(X^h) \to U_r$. Let V_ρ denote the associated bundle. It follows from Chern-Weil theory that the Chern classes of V_ρ vanish. By the first part V_ρ is μ_H-stable on X. Thus, $V_\rho \in \mathcal{S}$. Since $\mathrm{Hom}(V, V_\rho) \to \mathrm{Hom}(V|_D, V_\rho|_D)$ is surjective, we get a non-trivial homomorphism $V \to V_\rho$. Since both of these are μ_H-stable of slope 0 we get that this homomorphism is infact an isomorphism. □

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