FUNDAMENTAL GROUP SCHEMES OF SOME QUOT SCHEMES ON A SMOOTH PROJECTIVE CURVE

CHANDRANANDAN GANGOPADHYAY AND RONNIE SEBASTIAN

ABSTRACT. Let k be an algebraically closed field of characteristic $p > 0$. Let C be an irreducible smooth projective curve over k. Let E be a locally free sheaf on C of rank ≥ 2. Fix an integer $d \geq 2$. Let $\mathcal{Q}$ denote the Quot scheme parameterizing torsion quotients of E of degree d. In this article we compute the S-fundamental group scheme of $\mathcal{Q}$.

1. INTRODUCTION

Let X be a connected, reduced and complete scheme over a perfect field k and let $x \in X$ be a k-rational point. In [Nor76], Nori introduced a k-group scheme $\pi^N(X, x)$ associated to essentially finite vector bundles on X. In [Nor82], Nori extends the definition of $\pi^N(X, x)$ to connected and reduced k-schemes. In [BPS06], Biswas, Parameswaran and Subramanian defined the notion of S-*fundamental group scheme* $\pi^S(X, x)$ for X a smooth projective curve over any algebraically closed field k. This is generalized to higher dimensional connected smooth projective k-schemes and studied extensively by Langer in [Lan11, Lan12].

Let C be a connected smooth projective curve defined over an algebraically closed field k of characteristic $p > 0$. Fix a locally free sheaf E on C of rank ≥ 2 and an integer $d \geq 2$. Let $\mathcal{Q}$ denote the Quot scheme parameterizing torsion quotients of E of degree d. It is a smooth and projective variety over k. In this article we shall compute the S-fundamental group scheme of $\mathcal{Q}$. We mention some of the earlier results where fundamental group schemes were computed. In [BH15] it is proved that for a smooth projective surface X, the etale fundamental group $\pi^{\text{ét}}(Hilb_X^n, nx)$ of the Hilbert scheme of n points ($n \geq 2$), is isomorphic to $\pi^{\text{ét}}(X, x)_{\text{ab}}$. The main result in [PS19b] is to generalize this to the S-fundamental group scheme. In [Lan12] it is proved that $\pi^S(\text{Alb}(C), 0)$ is isomorphic to $\pi^S(C, c)_{\text{ab}}$. Let S_d ($d \geq 2$) be the permutation group of d symbols and denote $S^d C := C^d / S_d$. In [PS19a] the authors prove that the $\pi^S(S^d C, d[c])$ is isomorphic to $\pi^S(C, c)_{\text{ab}}$. Once we have such a result for the S-fundamental group scheme, one deduces similar results for the Nori and etale fundamental group schemes.

There is a Hilbert-Chow map $\phi : \mathcal{Q} \to S^d C$. The main result we prove in this article is the following.

Theorem (Theorem 7.4). *For any closed point $q \in \mathcal{Q}$, there is an isomorphism of affine k-group schemes*

$$\phi_*^S : \pi^S(\mathcal{Q}, q) \xrightarrow{\sim} \pi^S(S^d C, \phi(q)).$$

From this the following corollary follows easily.

Corollary (Corollary 7.5). *For any closed point $q \in \mathcal{Q}$, there are isomorphism of affine k-group schemes $\phi_*^N : \pi^N(\mathcal{Q}, q) \xrightarrow{\sim} \pi^N(S^d C, \phi(q))$ and $\phi_*^{\text{ét}} : \pi^{\text{ét}}(\mathcal{Q}, q) \xrightarrow{\sim} \pi^{\text{ét}}(S^d C, \phi(q))$.*

In view of [PS19a] it follows that

Corollary 1.1. $\pi^?(\mathcal{Q}, q) \cong \pi^?(C, c)_{\text{ab}}$ *for* $? = S, N, \text{ét}$.

The key ingredient in the proof of the above theorem is the following. Let D be a divisor corresponding to a closed point of $S^d C$. Let $\mathcal{Q}_D$ denote the scheme theoretic fiber over the point D. Then we have the following result, which is of independent interest.

Corollary (Corollary 6.6). *The fiber $\mathcal{Q}_D$ is reduced, irreducible and normal.*

Further, there is a smooth projective rational variety S_d and a birational map $g_d : S_d \to \mathcal{Q}_D$, see Proposition 5.13. This allows us to conclude easily, using Grauert's theorem, that every numerically flat bundle is the pullback of a numerically flat along ϕ.

In the rest of this article, E will be a locally free sheaf of rank ≥ 2 and d will be an integer ≥ 2.

2. HILBERT-CHOW MORPHISM

In this section we recall the definition of the Hilbert Chow morphism $\phi : \mathcal{Q} \to S^d C$. Let $p_1 : C \times \mathcal{Q} \to C$ and $p_2 : C \times \mathcal{Q} \to \mathcal{Q}$ denote the projections and let

$$(2.1) \qquad\qquad 0 \to K \to p_1^* E \to B \to 0$$

denote the universal quotient on $C \times \mathcal{Q}$. Since B and $p_1^* E$ are flat over $\mathcal{Q}$, it follows that K is a flat $\mathcal{Q}$ sheaf. Let $q \in \mathcal{Q}$ denote a closed point. Restricting this quotient to $C \times q$ we get the exact sequence

$$(2.2) \qquad\qquad 0 \to K|_{C \times q} \to E \to B|_{C \times q} \to 0 \,.$$

It follows that $K|_{C \times q}$ is a locally free sheaf on C. From Nakayama's lemma it follows that K is a locally free $C \times \mathcal{Q}$ sheaf of rank $r := \text{rank } E$. Taking determinant of the inclusion in (2.1) we get an exact sequence

$$0 \to \det(K) \to \det(p_1^* E) \to \mathcal{F} \to 0 \,.$$

To show that $\mathcal{F}$ is flat over $\mathcal{Q}$ it suffice to show that the restriction of this sequence to $C \times q$ remains exact on the left. But this is clear as the restriction of this sequence to $C \times q$ is precisely the sequence obtained by taking determinant of the inclusion in (2.2), which remains exact on the left. Thus, on $C \times \mathcal{Q}$ we get a quotient

$$0 \to \det(K) \otimes \det(p_1^* E)^{-1} \to \mathcal{O} \to \mathcal{F} \otimes \det(p_1^* E)^{-1} \to 0 \,.$$

This defines a morphism

(2.3) $$\phi : \mathcal{Q} \to S^d C \,.$$

In the following sections we will study the fibers of this morphism.

3. LOCUS WHERE ϕ IS SMOOTH

Consider the map $\phi : \mathcal{Q} \to S^d C$. Let D denote the divisor $\sum_{i=1}^k d_i [c_i]$ and consider a quotient q

(3.1) $$E \xrightarrow{q} \mathcal{O}_D \to 0 \,.$$

Lemma 3.2. *Given a quotient q as above, there is a line bundle L and a surjection $E \to L \to 0$ such that q factors as*

$$E \to L \to \mathcal{O}_D \,.$$

Proof. Let L' be any line bundle on C. Then we have the exact sequence

$$0 \to L'(-D) \to L' \to L'|_D \to 0 \,.$$

Applying the functor $\mathrm{Hom}(E,)$ to the above exact sequence, we get

$$0 \to \mathrm{Hom}(E, L'(-D)) \to \mathrm{Hom}(E, L') \to \mathrm{Hom}(E, L'|_D) \to \mathrm{Ext}^1(E, L(-D))$$

Now $\mathrm{Ext}^1(E, L'(-D)) \cong H^1(E^\vee \bigotimes L'(-D))$, so for L' of sufficiently high degree we get $\mathrm{Ext}^1(E, L'(-D)) = 0$, that is, we have an exact sequence when $\deg L' \gg 0$

$$0 \to \mathrm{Hom}(E, L'(-D)) \to \mathrm{Hom}(E, L') \to \mathrm{Hom}(E, L'|_D) \to 0 \,.$$

In other words, for any homomorphism $E \to L'|_D$, we have a morphism $E \to L'$ such that the following diagram commutes:

In particular, taking the quotient $q : E \to \mathcal{O}_D$, there is a line bundle L' such that q factors as $E \to L' \to \mathcal{O}_D$. Let L be the image of E in L'. Then, we have a surjection

$$E \to L \to \mathcal{O}_D \,,$$

which proves the lemma. $\qquad \square$

Lemma 3.3. *The map $\phi : \mathcal{Q} \to S^d C$ is smooth at q (corresponding to the quotient in equation (3.1)).*

Proof. We will show that the map of Zariski tangent spaces $T_q\mathcal{Q} \to T_{\phi(q)}S^dC$ is surjective. Let $T := \operatorname{Spec} k[\epsilon]/(\epsilon^2)$. We will show that for any map

$$T \xrightarrow{v} S^dC$$

such that the image of the closed point of T is $\phi(q)$, there is a map $T \xrightarrow{v'} \mathcal{Q}$, such that the closed point maps to q and the following diagram commutes

$$
\begin{array}{ccc}
 & & \mathcal{Q} \\
 & \nearrow^{v'} & \downarrow{\phi} \\
T & \xrightarrow{v} & S^dC
\end{array}
$$

By universal property of S^dC, the morphism v corresponds to a quotient over $C \times T$ given by

$$\mathcal{O}_{C\times T} \xrightarrow{J_v} \mathcal{O}_{\mathcal{D}} \to 0$$

such that $\mathcal{O}_{\mathcal{D}}$ is T-flat and the restriction of J_v to $C \times \operatorname{Spec} k$ is equivalent to the quotient $[\mathcal{O}_C \to \mathcal{O}_D]$ (which corresponds to the point $\phi(q)$). Note that $\mathcal{O}_{\mathcal{D}}$ is an Artinian ring.

Let $f_1 : C \times T \to C$ and $f_2 : C \times T \to T$ denote the projections. We fix a line bundle L over C as in Lemma 3.2. We have $1 \otimes J_v : f_1^*L \to f_1^*L \otimes \mathcal{O}_{\mathcal{D}}$. Define a quotient $J_{v'}$ over $C \times T$ as the composition

$$(3.4) \qquad J_{v'} : f_1^*E \to f_1^*L \xrightarrow{1 \otimes J_v} f_1^*L \otimes \mathcal{O}_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}\,.$$

Clearly, $f_1^*L \otimes \mathcal{O}_{\mathcal{D}}$ is T-flat and $J_{v'}|_{C\times\{\operatorname{Spec} k\}}$ is equivalent to q by Lemma 3.2. Hence, $J_{v'}$ induces a morphism $v' : T \to \mathcal{Q}$. Next we show that $\phi \circ v' = v$. Let us denote the kernel of $J_{v'}$ by $E_{v'}$. Thus, we have an exact sequence

$$0 \to E_{v'} \xrightarrow{\iota} f_1^*E \xrightarrow{J_{v'}} \mathcal{O}_{\mathcal{D}} \to 0\,.$$

Let t_0 denote the closed point of T. From the T-flatness of $\mathcal{O}_{\mathcal{D}}$ we conclude that $E_{v'}|_{C\times t_0}$ is locally free. Now using Nakayama's lemma we conclude that $E_{v'}$ is locally free sheaf on $C \times T$. Recall from the definition of ϕ, that the map $\phi \circ v'$ is given by the following quotient on $C \times T$

$$0 \to \det(E_{v'}) \otimes \det(f_1^*E)^{-1} \xrightarrow{\det(\iota)} \mathcal{O}_{C\times T} \to \mathcal{F} \to 0\,.$$

Passing to the local rings at (c, t_0) and using equation (3.4), it is easily checked that $\mathcal{F} \cong \mathcal{O}_{\mathcal{D}}$ and that the above quotient is exactly $J_v : \mathcal{O}_{C\times T} \to \mathcal{O}_{\mathcal{D}}$. This completes the proof of the lemma. $\qquad\square$

Proposition 3.5. *In every fiber of ϕ there is a point at which ϕ is a smooth morphism.*

Proof. Let D be the divisor corresponding to a point $x \in S^dC$. Fix a line bundle L which is a surjective quotient of E. Then the composite $q : E \to L \to L \otimes \mathcal{O}_D$ is a quotient such that $\phi(q) = x$. The proposition now follows from lemma 3.3. $\qquad\square$

4. The space S_d

Let the rank of the vector bundle E be r. We will inductively define (S_d, A_d), where A_d is a vector bundle of rank r defined over $C \times S_d$.

Define $S_0 = \operatorname{Spec} k$ and $A_0 = E$. To define (S_j, A_j) we assume that we have defined (S_{j-1}, A_{j-1}). Let $i_{j-1} : \{c\} \times S_{j-1} \hookrightarrow C \times S_{j-1}$ be the natural closed immersion. Define $S_j := \mathbb{P}(i_{j-1}^* A_{j-1})$ and let $f_{j,j-1} : S_j \to S_{j-1}$ be the structure morphism. Finally let $F_{j,j-1} := id_C \times f_{j,j-1} : C \times S_j \to C \times S_{j-1}$. Let $p_{1,j}$ and $p_{2,j}$ be the projections from $C \times S_j$ to C and S_j, respectively.

For each j, we have the following diagram

$$
\begin{array}{ccccc}
\{c\} \times S_j & \xrightarrow{\;i_j\;} & C \times S_j & \xrightarrow{\;F_{j,j-1}\;} & C \times S_{j-1} \\
& {\scriptstyle =} \searrow & \downarrow{\scriptstyle p_{2,j}} & & \downarrow{\scriptstyle p_{2,j-1}} \\
& & S_j & \xrightarrow{\;f_{j,j-1}\;} & S_{j-1}
\end{array}
\Big)_{i_{j-1}}
$$

Let $\mathcal{O}_j(1)$ the universal line bundle over S_j. Then over $C \times S_j$ we have the quotient

$$
\begin{aligned}
F_{j,j-1}^* A_{j-1} &\to (i_j)_* i_j^* F_{j,j-1}^* A_{j-1} \\
&= (i_j)_* f_{j,j-1}^* i_{j-1}^* A_{j-1} \\
&\to (i_j)_* \mathcal{O}_j(1)
\end{aligned}
$$

Define A_j to be the kernel of the above quotient. Since $(i_j)_* \mathcal{O}_j(1)$ is flat over S_j, restricting the exact sequence

$$(4.1) \qquad 0 \to A_j \to F_{j,j-1}^* A_{j-1} \to (i_j)_* \mathcal{O}_j(1) \to 0$$

to $C \times s$ we see that $A_j|_{C \times s}$ is torsion free and so is locally free. It follows from Nakayama's lemma that A_j is locally free on $C \times S_j$. Thus, we have defined (S_j, A_j). It is clear that closed points of S_d are in 1-1 correspondence with filtrations

$$(4.2) \qquad E_d \subset E_{d-1} \subset E_{d-2} \subset \cdots \subset E_0 = E$$

where each E_j is a locally free sheaf of rank r on C and E_j/E_{j+1} is a skyscraper sheaf of rank one supported at $c \in C$.

5. Birationality of S_d and $\mathcal{Q}_{d[c]}$

Define the following morphisms for $j > i$:

$$
\begin{aligned}
f_{j,i} &= f_{j,j-1} \circ \ldots \circ f_{i+1,i} : S_j \to S_i, \\
F_{j,i} &= F_{j,j-1} \circ \ldots \circ F_{i+1,i} : C \times S_j \to C \times S_i .
\end{aligned}
$$

Note that both of these morphisms are flat. Let $V \subset \mathcal{Q}_{d[c]}$ be the open subset whose points parameterize quotients of the type $E \to \mathcal{O}_C / \mathfrak{m}_{C,c}^d$.

Lemma 5.1. *There exists a morphism $g_d : S_d \to \mathcal{Q}_{d[c]}$ such that*

(i) g_d is surjective on closed points,

(ii) $g_d^{-1}(V) \to V$ is a bijection,

(iii) $S_d \setminus g_d^{-1}(V) \to \mathcal{Q}_{d[c]} \setminus V$ has positive dimensional fibers.

Proof. We will define a quotient of p_1^*E on $C \times S_d$. Using the flatness of $F_{d,i}$, we have inclusions (recall the definition of A_j from (4.1))

(5.2) $\qquad A_d \subset F_{d,d-1}^*A_{d-1} \subset F_{d,d-2}^*A_{d-2} \subset \ldots \subset F_{d,1}^*A_1 \subseteq p_1^*E.$

Define

$$B_j^d := p_1^*E/F_{d,j}^*A_j$$
$$\cong F_{d,j}^*(p_1^*E/A_j) \cong F_{d,j}^*B_j^j.$$

For each j there is an exact sequence on $C \times S_d$

(5.3) $\qquad 0 \to F_{d,j-1}^*A_{j-1}/F_{d,j}^*A_j \to B_j^d \to B_{j-1}^d \to 0.$

On $C \times S_j$ we have the quotient (4.1)

$$0 \to A_j \to F_{j,j-1}^*A_{j-1} \to F_{j,j-1}^*A_{j-1}/A_j \cong (i_j)_*(\mathcal{O}_j(1)) \to 0.$$

As $F_{j,j-1}^*A_{j-1}/A_j$ is S_j-flat, the pullback along $F_{d,j}$, that is, $F_{d,j-1}^*A_{j-1}/F_{d,j}^*A_j$ is S_d-flat. When restricted to $C \times s$ for $s \in S_d$, it is a degree one torsion sheaf supported at c. By induction on j, using equation (5.3), one sees that B_j^d is S_d-flat and the restriction of B_j^d to $C \times s$ is a torsion sheaf of degree j supported at c. In particular,

(5.4) $\qquad 0 \to A_d \to p_1^*E \to B_d^d \to 0$

is a quotient such that B_d^d is S_d flat and for each $s \in S_d$, $B_d^d|_{C \times \{s\}}$ is a torsion sheaf of degree d supported at c. By the universal property of $\mathcal{Q}$, we have a morphism

$$S_d \to \mathcal{Q},$$

such that the set theoretic image of the composition

$$S_d \to \mathcal{Q} \xrightarrow{\phi} S^dC$$

is the point $d[c]$. Since S_d is reduced, the scheme theoretic image is the scheme $\{d[c]\} \hookrightarrow S^dC$. In other words, we get that the above morphism factors as

$$S_d \xrightarrow{\ g_d\ } \mathcal{Q}_{d[c]} \longleftrightarrow \mathcal{Q}.$$

A closed point of S_d corresponds to a filtration as in (4.2). Under g_d this point maps to the quotient $E \to E/E_d$. Conversely, given a closed point of $\mathcal{Q}_{d[c]}$ it is clear that we can find a closed point of S_d which maps to it. This proves (i). Suppose $E_d \subset E$ is such that $E/E_d \cong \mathcal{O}/\mathfrak{m}_{C,c}^d$, then for every $0 \le j \le d$ there is a unique E_j such that $E_d \subset E_j \subset E$ and $E/E_j \cong \mathcal{O}/\mathfrak{m}_{C,c}^j$. From this one easily concludes (ii).

For a closed point in $\mathcal{Q}_{d[c]} \setminus V$, corresponding to a quotient $E \to \mathcal{F}_d$, we have $\mathrm{rank}(\mathcal{F}_d \otimes \mathcal{O}/\mathfrak{m}_{C,c}) \ge 2$. We can construct infinitely many chains $\mathcal{F}_d \to \mathcal{F}_{d-1} \to \ldots \to \mathcal{F}_1$. Therefore, the fiber over such a closed point is positive dimensional. This proves (iii). $\qquad \square$

Corollary 5.5. *The fiber $\mathcal{Q}_{d[c]}$ is irreducible of dimension $d(r-1)$.*

Proof. Since S_d is irreducible, it is clear that $\mathcal{Q}_{d[c]}$ is irreducible. It is clear that the dimension of S_d is $d(r-1)$. Thus, the dimension of $\mathcal{Q}_{d[c]}$ is at most $d(r-1)$. On the other hand, the dimension of the fiber of ϕ over a general point is $d(r-1)$. This shows that the dimension of $\mathcal{Q}_{d[c]}$ is at least $d(r-1)$. $\square$

Corollary 5.6. *The codimension of $\mathcal{Q}_{d[c]} \setminus V$ in $\mathcal{Q}_{d[c]}$ is ≥ 2.*

Proof. As S_d and $\mathcal{Q}_{d[c]}$ have the same dimension and S_d is irreducible, this follows easily using (iii) in lemma 5.1. $\square$

Corollary 5.7. *The fiber $\mathcal{Q}_{d[c]}$ satisfies Serre's condition (R_1).*

Proof. Since the map ϕ is smooth at points $v \in V$, it follows that $\mathcal{O}_{\mathcal{Q}_{d[c]},v}$ is a regular local ring for all $v \in V$. Further, from the preceding corollary V contains all prime ideals of height 1. The corollary follows. $\square$

Next we will show that g_d is birational. Let

$$p_1^* E \to B'$$

be the restriction of the universal quotient B over $C \times \mathcal{Q}$ to the subscheme $C \times \mathcal{Q}_{d[c]}$. Let us define the inclusion

$$i : \mathrm{Spec}\,(\mathcal{O}_{C,c}/\mathfrak{m}_{C,c}^d) \times \mathcal{Q}_{d[c]} \hookrightarrow C \times \mathcal{Q}_{d[c]}\,.$$

Lemma 5.8. *There is a coherent sheaf F_d over $\mathrm{Spec}\,(\mathcal{O}_{C,c}/\mathfrak{m}_{C,c}^d) \times \mathcal{Q}_{d[c]}$ such that $B' = i_* F_d$.*

Proof. It is enough to show that the $p_1^*(E \otimes \mathcal{O}(-dc))$ is contained in the kernel of $p_1^* E \to B'$. Denote the kernel by A'. Let $0 \to E' \xrightarrow{h} E$ be locally free sheaves of the same rank on a scheme Y. Let $\mathcal{I}$ denote the ideal sheaf determined by $\det(h)$. Then it is easy to see that $\mathcal{I}E \subset h(E') \subset E$. Thus, it suffices to find the ideal sheaf corresponding to the following exact sequence

(5.9) $$0 \to A' \xrightarrow{h} p_1^* E \to B' \to 0$$

on $C \times \mathcal{Q}_{d[c]}$. By the definition of ϕ, the map $\mathcal{Q}_{d[c]} \to \mathcal{Q} \xrightarrow{\phi} S^d C$ is given by the quotient

$$0 \to \det(A') \xrightarrow{\det(h)} \det(p_1^* E) \to \mathcal{F} \to 0$$

on $C \times \mathcal{Q}_{d[c]}$. But since the image of $\mathcal{Q}_{d[c]}$ under this morphism is precisely $d[c]$, it follows that this quotient is isomorphic to the quotient

$$p_1^* \mathcal{O}_C \to p_1^*(\mathcal{O}_C/\mathfrak{m}_{C,c}^d)\,.$$

It is clear that the ideal sheaf $\mathcal{I}$ corresponding to the exact sequence (5.9) is $p_1^*(\mathcal{O}_C(-dc))$. The lemma now follows. $\square$

Define subschemes

$$D_j := \mathrm{Spec}\,(\mathcal{O}_{C,c}/\mathfrak{m}_{C,c}^j) \times V \overset{\alpha_j}{\hookrightarrow} C \times V\,.$$

By the previous lemma there is a sheaf F_d on D_d such that $B'|_{C\times V} \cong (\alpha_d)_*(F_d)$. Clearly F_d is flat over V since B' is.

Lemma 5.10. F_d *is a line bundle over* D_d.

Proof. By definition of V, for each $v \in V$, $(F_d)_v \cong \mathcal{O}_C/\mathfrak{m}_{C,c}^d$. Using F_d is V-flat and Nakayama's lemma we see that F_d is a line bundle. $\qquad\square$

Corollary 5.11. *The restriction* $F_j := F|_{D_j}$ *is a line bundle on* D_j.

Remark 5.12. We will use the following fact in the proof of the next theorem. Let A and B be rings and let M be an $A \otimes_k B$ module. Let $B \to C$ be a ring homomorphism. Then

$$M \otimes_{A\otimes_k B} (A \otimes_k C) \cong M \otimes_{A\otimes_k B} (A \otimes_k B) \otimes_B C \cong M \otimes_B C\,.$$

In particular, if $0 \to N' \to N \to M \to 0$ is a short exact sequence of $A \otimes_k B$ modules, where M is flat as a B-module, then it remains exact when we apply the functor $- \otimes_{A\otimes_k B} (A \otimes_k C)$.

Proposition 5.13. *The restriction* $g_d : g_d^{-1}(V) \to V$ *is an isomorphism.*

Proof. We will use induction on j to define maps $V \to S_j$. Define A'_0 on $C \times V$ to be $p_1^* E$. For $j \geq 1$ define sheaves A'_j on $C \times V$ as follows

$$(5.14) \qquad\qquad 0 \to A'_j \to p_1^* E \to (\alpha_j)_*(F_j) \to 0\,.$$

Observe that we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A'_j & \longrightarrow & p_1^* E & \longrightarrow & (\alpha_j)_*(F_j) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & A'_{j-1} & \longrightarrow & p_1^* E & \longrightarrow & (\alpha_{j-1})_*(F_{j-1}) & \longrightarrow & 0
\end{array}
$$

The kernel of the right vertical arrow is $(\alpha_1)_*(F_1)$. Thus, there is an exact sequence of sheaves on $C \times V$,

$$(5.15) \qquad\qquad 0 \to A'_j \to A'_{j-1} \xrightarrow{\delta_{j-1}} (\alpha_1)_*(F_1) \to 0\,.$$

Note that $S_1 = \mathbb{P}(E_c)$. Thus, to give a map from $V \to S_1$ we need to give a line bundle quotient of $E_c \otimes O_V$. Restricting the universal quotient $p_1^* E \to (\alpha_d)_*(F_d)$ on $C \times V$ to $c \times V$ we get the quotient $E_c \otimes \mathcal{O}_V \to F_1$. This defines a morphism $h_1 : V \to S_1$. On $C \times S_1$ one has the exact sequence (4.1). Using remark 5.12 we see that the pullback of this along $id_C \times h_1$ gives the following exact sequence on $C \times V$,

$$0 \to (id_C \times h_1)^* A_1 \to p_1^* E \xrightarrow{\delta_0} (\alpha_1)_*(F_1) \to 0\,.$$

We see that $A'_1 = (id_C \times h_1)^* A_1$. Let us assume that we have constructed maps $h_{j-1} : V \to S_{j-1}$ such that the pullback of (4.1) along $id_C \times h_{j-1}$ yields the exact sequence

(5.16) $$0 \to A'_{j-1} \to A'_{j-2} \xrightarrow{\delta_{j-2}} (\alpha_1)_*(F_1) \to 0 \,.$$

Consider the diagram

$$\begin{array}{ccccc}
\{c\} \times V & \xhookrightarrow{\alpha_1} & C \times V & \xrightarrow{id_C \times h_{j-1}} & C \times S_{j-1} \\
 & \searrow{\scriptstyle =} & \downarrow & & \downarrow \\
 & & V & \xrightarrow{h_{j-1}} & S_{j-1}
\end{array} \Big) i_{j-1}$$

To give a map $V \to S_j$ we need to give a line bundle quotient of

$$h^*_{j-1} i^*_{j-1} A_{j-1} \cong (\alpha_1)^*(id_C \times h_{j-1})^* A_{j-1} \cong (\alpha_1)^* A'_{j-1},$$

where the last isomorphism follows from (5.16). Restricting (5.15) to $c \times V$, we get a line bundle quotient

$$(\alpha_1)^* A'_{j-1} \to F_1 \,.$$

This defines a morphism $h_j : V \to S_j$. Pulling back (4.1) along $id_C \times h_j$, using 5.12 and (5.16), we get the following short exact sequence on $C \times V$

$$0 \to (id_C \times h_j)^* A_j \to A'_{j-1} \xrightarrow{\delta_{j-1}} (\alpha_1)_*(F_1) \to 0 \,.$$

Now equation (5.15) shows that $A'_j \cong (id_C \times h_j)^* A_j$ and there is an exact sequence

$$0 \to A'_j \to A'_{j-1} \xrightarrow{\delta_{j-1}} (\alpha_1)_*(F_1) \to 0 \,.$$

Thus, inductively we have constructed a map $h_d : V \to S_d$.

To show that the composite $V \xrightarrow{h_d} S_d \xrightarrow{g_d} \mathcal{Q}_{d[c]}$ is an isomorphism onto V, it suffices to show that the pullback of the universal quotient on $C \times \mathcal{Q}_{d[c]}$ along $id_C \times (g_d \circ h_d)$ is the restriction of the universal quotient to $C \times V$. Recall from (5.14) the universal quotient on $C \times V$ is

$$0 \to A'_d \to p_1^* E \to B' \to 0 \,.$$

Pulling this back along $id_C \times g_d$ is the quotient (recall from (5.4))

$$0 \to A_d \to p_1^* E \to B_d^d \to 0,$$

by the definition of the map g_d. From the definition of h_d, one easily checks that the pullback along $id_C \times h_d$ of the filtration in (5.2) is the following filtration on $C \times V$,

$$A'_d \subset A'_{d-1} \subset \cdots \subset p_1^* E \,.$$

Thus, it follows that the pullback along $id_C \times h_d$ of (5.4) is

$$0 \to A'_d \to p_1^* E \to B' \to 0 \,,$$

which is the universal quotient on $C \times V$. This proves that $g_d \circ h_d$ is the identity on V. By lemma 3.3 the morphism ϕ is smooth at a point $v \in V$. This shows that the local ring $\mathcal{O}_{V,v}$ is a domain. Thus, we have maps

$\mathcal{O}_{V,v} \to \mathcal{O}_{S_d, h_d(v)} \to \mathcal{O}_{V,v}$ such that the composite is the identity. Since both rings have the same dimension, the kernel of $\mathcal{O}_{S_d, h_d(v)} \to \mathcal{O}_{V,v}$ is forced to be 0, which shows that the local rings are isomorphic. This proves the proposition. $\square$

6. Normality of all fibers

For a point $D = \sum d_i[c_i] \in S^d C$, denote by $\mathcal{Q}_D$ the scheme theoretic fiber of ϕ over the closed point corresponding to D.

Proposition 6.1. *The fiber $\mathcal{Q}_D$ is irreducible of dimension $d(r-1)$.*

Proof. We define a morphism $\prod \mathcal{Q}_{d_i[c_i]} \to \mathcal{Q}_D$ as follows. Let p_j be the projections $C \times \prod \mathcal{Q}_{d_i[c_i]} \to C \times \mathcal{Q}_{d_j[c_j]}$ and p be the projection $C \times \prod \mathcal{Q}_{d_i[c_i]} \to C$. Let $B_{d_i[c_i]}$ denote the universal quotient over $C \times \mathcal{Q}_{d_i[c_i]}$. Then over $C \times \prod \mathcal{Q}_{d_i[c_i]}$, we define a quotient

$$p^* E \to \bigoplus p_i^* B_{d_i[c_i]}$$

Clearly, $\bigoplus p_i^* B_{d_i[c_i]}$ is flat, and hence induces a morphism

(6.2) $\theta_D : \prod \mathcal{Q}_{d_i[c_i]} \to \mathcal{Q}$

which is bijective onto the closed points of $\mathcal{Q}_D$. Therefore, $\mathcal{Q}_D$ is irreducible. Since the dimension of the general fiber of ϕ is $d(r-1)$, we get

$$d(r-1) \le \dim \mathcal{Q}_D \le \bigoplus \dim \mathcal{Q}_{d_i[c_i]} = \sum d_i(r-1) = d(r-1)$$

This proves the corollary. $\square$

Corollary 6.3. *The map ϕ is flat.*

Proof. This follows using [Har77, Chapter III, Exercise 10.9] $\square$

Corollary 6.4. *The fiber $\mathcal{Q}_{d[c]}$ is reduced, irreducible and normal. In particular, it is integral.*

Proof. Since ϕ is flat and $\mathcal{Q}$ is smooth, it follows from [Stk, Tag 045J] (or see Corollary to [Mat86, Theorem 23.3]) that the fiber $\mathcal{Q}_{d[c]}$ is Cohen-Macaulay. Thus, the fiber satisfies Serre's condition (S_2). Now from corollary 5.7 it follows that the fiber satisfies (R_0) and (S_1) and so it is reduced. Since it satisfies (R_1) and (S_2) it is normal. $\square$

Lemma 6.5. $\mathcal{Q}_D \cong \prod \mathcal{Q}_{d_i[c_i]}$.

Proof. The map θ_D in (6.2) sits in a commutative diagram

$$
\begin{array}{ccc}
\prod \mathcal{Q}_{d_i[c_i]} & \xrightarrow{\theta_D} & \mathcal{Q} \\
\downarrow & & \downarrow \\
\prod S^{d_i} C & \longrightarrow & S^d C
\end{array}
$$

From the above diagram it is clear that θ_D factors to give a map

$$\tilde{\theta}_D : \prod \mathcal{Q}_{d_i[c_i]} \to \mathcal{Q}_D\,.$$

We want to give a map in the other direction. Let $p_D : C \times \mathcal{Q}_D \to C$ be the first projection. Let us denote the restriction of the universal quotient to $C \times \mathcal{Q}_D$ by

$$p_D^* E \to B_D\,.$$

There are integers e_i such that the quotient B_D is supported on the following closed subscheme of $C \times \mathcal{Q}_D$

$$\bigsqcup_i \mathrm{Spec}\,(\mathcal{O}_C/\mathfrak{m}_{C,c_i}^{e_i}) \times \mathcal{Q}_D\,.$$

Let $\mathfrak{j}_i : \mathrm{Spec}\,(\mathcal{O}_C/\mathfrak{m}_{C,c_i}^{e_i}) \times \mathcal{Q}_D \hookrightarrow C \times \mathcal{Q}_D$ denote the closed immersion. Let

$$B_{d_i[c_i]} := \mathfrak{j}_{i*}\Big(B_D\big|_{\mathrm{Spec}\,(\mathcal{O}_C/\mathfrak{m}_{C,c_i}^{e_i})\times\mathcal{Q}_D}\Big)\,.$$

Clearly, since B_D is flat over $\mathcal{Q}_D$, the $B_{d_i[c_i]}$ is also flat over $\mathcal{Q}_D$. We define the quotients

$$p^* E \to B_D \to B_{d_i[c_i]}\,,$$

which defines a morphism $\mathcal{Q}_D \to \mathcal{Q}_{d_i[c_i]}$. This defines a morphism $\gamma_D : \mathcal{Q}_D \to \prod \mathcal{Q}_{d_i[c_i]}$. One easily checks that the pullback along $id_C \times (\theta_D \circ \gamma_D)$ of the universal quotient is $p_1^* E \to B_D$. This shows that $\tilde{\theta}_D \circ \gamma_D$ is the identity. Arguing as in the last para of the proof of proposition 5.13, the lemma is proved. □

Corollary 6.6. *The fiber $\mathcal{Q}_D$ is reduced, irreducible and normal.*

7. MAIN THEOREM

Definition 7.1. *Let X be a connected, projective and reduced k-scheme. Let $\mathcal{C}^{\mathrm{nf}}(X)$ denote the full subcategory of coherent sheaves whose objects are coherent sheaves E on X satisfying the following two conditions:*

(1) E is locally free, and
(2) for any smooth projective curve C over k and any morphism $f : C \longrightarrow X$, the vector bundle $f^* E$ is semistable of degree 0.

We call the objects of the category $\mathcal{C}^{\mathrm{nf}}(X)$ *numerically flat vector bundles* on X. Fix a k-valued point $x \in X$. Let Vect_k be the category of finite dimensional k-vector spaces. Let $T_x : \mathcal{C}^{\mathrm{nf}}(X) \longrightarrow \mathrm{Vect}_k$ be the fiber functor defined by sending an object E of $\mathcal{C}^{\mathrm{nf}}(X)$ to its fiber $E_x \in \mathrm{Vect}_k$ at x. Then $(\mathcal{C}^{\mathrm{nf}}(X), \otimes, T_x, \mathcal{O}_X)$ is a neutral Tannaka category [Lan11, Proposition 5.5, p. 2096]. The affine k-group scheme $\pi^S(X, x)$ Tannaka dual to this category is called the *S-fundamental group scheme* of X with base point x [Lan11, Definition 6.1, p. 2097].

A vector bundle E is said to be *finite* if there are distinct non-zero polynomials $f, g \in \mathbb{Z}[t]$ with non-negative coefficients such that $f(E) \cong g(E)$.

Definition 7.2. *A vector bundle E on X is said to be essentially finite if there exist two numerically flat vector bundles V_1, V_2 and finitely many finite vector bundles $F_1, \ldots, F_n$ on X with $V_2 \subseteq V_1 \subseteq \bigoplus_{i=1}^{n} F_i$ such that $E \cong V_1/V_2$.*

Let $\mathrm{EF}(X)$ be the full subcategory of coherent sheaves whose objects are essentially finite vector bundles on X. Fix a closed point $x \in X$ and let $T_x : \mathrm{EF}(X) \longrightarrow \mathrm{Vect}_k$ be the fiber functor defined by sending an object $E \in \mathrm{EF}(X)$ to its fiber E_x at x. Then the quadruple $(\mathrm{EF}(X), \bigotimes, T_x, \mathcal{O}_X)$ is a neutral Tannakian category. The affine k-group scheme $\pi^N(X, x)$ Tannaka dual to this category is referred to as the *Nori-fundamental group scheme* of X with base point x, see [Nor76] for more details.

In [Lan11, Proposition 8.2] it is proved that the S-fundamental group of projective space is trivial. In [HM11] it is proved that the S-fundamental group scheme is a birational invariant of smooth projective varieties.

Let S_{d_i, c_i} be the space defined in section 4 by taking $d = d_i$ and $c = c_i$. In view of the discussion in section 5 there is a birational map

$$\eta_D := (\tilde{\theta}_D \circ \prod g_{d_i}) : \prod S_{d_i, c_i} \to \prod \mathcal{Q}_{d_i[c_i]} \to \mathcal{Q}_D \,.$$

Proposition 7.3. *A numerically flat bundle on $\mathcal{Q}_D$ is trivial.*

Proof. Let W be a numerically flat bundle on $\mathcal{Q}_D$. As $\mathcal{Q}_D$ is normal, η_D is birational, $\prod S_{d_i, c_i}$ is a smooth rational variety, we have

$$W \cong \eta_{D*}\eta_D^* W$$
$$\cong \eta_{D*}(\mathcal{O})^{\oplus r}$$
$$\cong \mathcal{O}_{\mathcal{Q}_D}^{\oplus r}$$

In the above we have used the result of [HM11]. This proves the proposition.
$\square$

Theorem 7.4. *The induced map $\phi_*^S : \pi^S(\mathcal{Q}, q) \to \pi^S(S^d C, \phi(q))$ is an isomorphism.*

Proof. Since the fibers of ϕ are projective integral varieties, and ϕ is flat, it follows that $\phi_*(\mathcal{O}_{\mathcal{Q}_D}) = \mathcal{O}_{S^d C}$. Now applying [Lan11, Lemma 8.1] we see that ϕ_*^S is faithfully flat. To prove ϕ_*^S is a closed immersion we will use [DMOS82, Proposition 2.21(b)]. By Grauert's theorem [Har77, Corollary 12.9] and the previous proposition, it follows that if W is a numerically flat bundle on $\mathcal{Q}$ then $\phi_*(W)$ is a locally free sheaf on $S^d C$ and the natural map $\phi^*\phi_*(W) \to W$ is an isomorphism. It follows easily that $\phi_*(W)$ is numerically flat. This proves that ϕ_*^S is a closed immersion. $\square$

From the S-fundamental group scheme we recover the Nori fundamental group scheme as the inverse limit of finite quotients. Similarly, the etale fundamental group scheme can be recovered as the inverse limit of finite and reduced quotients. Thus, we get the following corollary. (See §5.5 in [PS19b] for more details.)

Corollary 7.5. *The induced map $\phi_*^N : \pi^N(\mathcal{Q}, q) \to \pi^N(S^d C, \phi(q))$ is an isomorphism. The induced map $\phi_*^{\text{ét}} : \pi^{\text{ét}}(\mathcal{Q}, q) \to \pi^{\text{ét}}(S^d C, \phi(q))$ is an isomorphism.*

REFERENCES

[BH15]   Indranil Biswas and Amit Hogadi. On the fundamental group of a variety with quotient singularities. *Int. Math. Res. Not. IMRN*, (5):1421–1444, 2015. doi:10.1093/imrn/rnt261.

[BPS06]   Indranil Biswas, A. J. Parameswaran, and S. Subramanian. Monodromy group for a strongly semistable principal bundle over a curve. *Duke Math. J.*, 132(1):1–48, 2006. doi:10.1215/S0012-7094-06-13211-8.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, chapter 2, pages 101–228. Springer-Verlag, Berlin-New York, 1982. doi:10.1007/978-3-540-38955-2_4.

[Har77]   Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HM11]   Amit Hogadi and Vikram Mehta. Birational invariance of the S-fundamental group scheme. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1361–1369, 2011.

[Lan11]   Adrian Langer. On the S-fundamental group scheme. *Ann. Inst. Fourier (Grenoble)*, 61(5):2077–2119 (2012), 2011. doi:10.5802/aif.2667.

[Lan12]   Adrian Langer. On the S-fundamental group scheme. II. *J. Inst. Math. Jussieu*, 11(4):835–854, 2012. doi:10.1017/S1474748012000011.

[Mat86]   Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[Nor76]   Madhav V. Nori. On the representations of the fundamental group. *Compositio Math.*, 33(1):29–41, 1976. URL http://www.numdam.org/item?id=CM_1976__33_1_29_0.

[Nor82]   Madhav V. Nori. The fundamental group-scheme. *Proc. Indian Acad. Sci. Math. Sci.*, 91(2):73–122, 1982. doi:10.1007/BF02967978.

[PS19a]   Arjun Paul and Ronnie Sebastian. Fundamental group schemes of n-fold symmetric product of a smooth projective curve, 2019, arXiv:1907.09388.

[PS19b]   Arjun Paul and Ronnie Sebastian. Fundamental group schemes of Hilbert scheme of n points on a smooth projective surface, 2019, arXiv:1907.04290.

[Stk]   The Stack Project. https://stacks.math.columbia.edu.

INSTITUTE OF MATHEMATICAL SCIENCES, CHENNAI 600113, TAMIL NADU, INDIA.
E-mail address: chandrang@imsc.res.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI 400076, MAHARASHTRA, INDIA.
E-mail address: ronnie@math.iitb.ac.in