ALGEBRAIC TOPOLOGY

Computations

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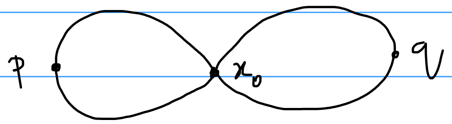
5.6 Cohomology of $\mathbb{P}^k_{\mathbb{R}}$

Fundamental Group

1. Let X_n be the wedge of n circles. Compute the fundamental group of X_n.

$\Rightarrow$ First we comput $\pi_1(X_2)$



let $U = X_2 \setminus q$, $V = X_2 \setminus p$, U, V both deformation retracts to S^1.

clearly $U \cap V$ deformation retracts to the point x_0.

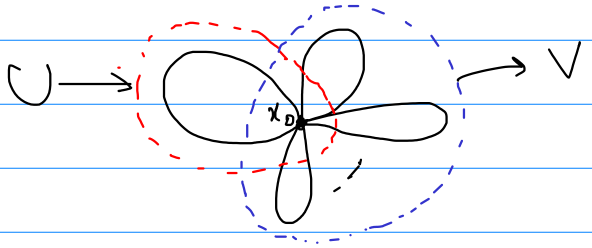
and $X_2 = U \cup V$

So by Van-Kampen theorem we have

$$\pi_1(X_2, x_0) = \pi_1(U, x_0) * \pi_2(V, x_0)$$

$$= \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$$

Now we use induction to find $\pi_1(X_n)$.



U deformation retracts to $S^1 = X_1$

V deformation retracts to X_{n-1}

$U \cap V$ deformation retracts to x_0

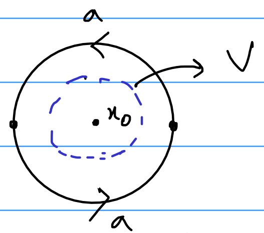
by induction hypothesis $\pi_1(X_{n-1}, x_0) = \overbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}^{n-1 \text{ times}}$

and by Van-Kampen theorem we have

$$\pi_1(X_n, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0) = \overbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}^{n \text{ times}}$$

2. Fundamental group of $\mathbb{P}^2_{\mathbb{R}}$.

$\Rightarrow$ $\mathbb{P}^2_{\mathbb{R}}$ can be visualized as



we take $U = \mathbb{P}^2_{\mathbb{R}} \setminus \{x_0\}$ and V be the disk centered at x_0.

U deformation retracts to S^1

V deformation retracts to a point. let $y_0 \in U \cap V$

$U \cap V$ deformation retracts to S^1.

by Van-Kampen theorem $\pi_1(\mathbb{P}^2_{\mathbb{R}}, \ ) = \dfrac{\pi_1(U, y_0) * \pi_1(V, y_0)}{N}$

$\pi_1(U, y_0) = \mathbb{Z}$, $\pi_1(V, y_0) = \{e\}$

and $N = \langle a^2 \rangle$

Hence $\pi_1(\mathbb{P}^2_{\mathbb{R}}, y_0) = \dfrac{\mathbb{Z} * \{e\}}{\langle a^2 \rangle} = \mathbb{Z}/\langle a^2 \rangle \cong \mathbb{Z}_2$

3. Fundamental group of $\mathbb{P}_{\mathbb{C}}^{n}$.

$\Rightarrow$ We first compute fundamental group of $\mathbb{P}_{\mathbb{C}}^{2}$.

Let $U = \mathbb{P}_{\mathbb{C}}^{2} \setminus [1,0]$, $V = \mathbb{C}^{2}$ centered at $[1,0]$

U deformation retracts to $\mathbb{P}_{\mathbb{C}}^{1} \cong S^{2}$

V deformation retracts to a point

$U \cap V$ deformation retracts to S^{3}.

Now we know $\pi_1(S^k) = \{e\}$ $\forall$ $k \geq 2$

So $\pi_1(U, x_0) = \{e\}$, $\pi_1(V, x_0) = \{e\}$ where $x_0 \in U \cap V$

Hence by Van-Kampen theorem $\pi_1(\mathbb{P}_{\mathbb{C}}^{2}, x_0) = \{e\}$.

Now we use induction and find $\pi_1(\mathbb{P}_{\mathbb{C}}^{n})$. $n > 2$

we take $U = \mathbb{P}_{\mathbb{C}}^{n} \setminus [1, 0, \ldots, 0]$

$V = \mathbb{C}^{n}$ centered at $[1, 0, \ldots, 0]$

U deformation retracts to $\mathbb{P}_{\mathbb{C}}^{n-1}$ (See 2.1)

V deformation retracts to a point let $x_0 \in U \cap V$

$U \cap V$ deformation retracts to S^{2n-1}

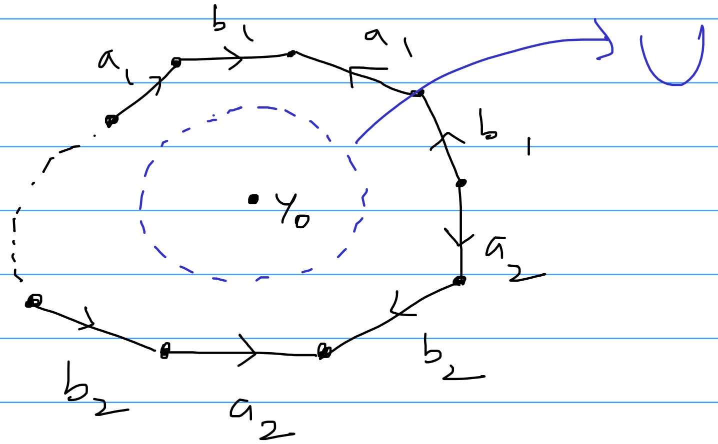
by induction we have $\pi_1(U, x_0) = \{e\}$

and $\pi_1(V, x_0) = \{e\}$

so $\pi_1(\mathbb{P}_{\mathbb{C}}^{n}, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0) = \{e\}$.

4. Fundamental group of compact orientable surface of genus k (M_k).

$\Rightarrow$ M_k can be visualized in the following way



i.e a 4k gon with edges and vertices identified as given in the picture.

Now we take U as given in the picture and V be the open set $M_k \setminus \{y_0\}$ let $x_0 \in U \cap V$

Now U deformation retracts to a point

V deformation retracts to a wedge of 2k circles

$U \cap V$ deformation retracts to S^1.

So by Van-Kampen theorem

$$\pi_1(M_k, x_0) = \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{N}$$

$$\pi_1(U, x_0) = \{e\}, \quad \pi_1(V, x_0) = \overbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}^{2k \text{ times}}$$

and N is given by the inclusion of $V \hookrightarrow M_k$ and as V deformation retracts to the wedge of $2k$ circles hence N is generated by the relation $\prod_{i=1}^{k} a_i b_i a_i^{-1} b_i^{-1}$

Hence $\pi_1(M_k, x_0) \simeq \dfrac{\mathbb{Z} * \cdots * \mathbb{Z}}{\langle \prod_{i=1}^{k} a_i b_i a_i^{-1} b_i^{-1} \rangle}$

Homology

1. Compute homology groups of $\mathbb{P}_{\mathbb{C}}^n$.

$\Rightarrow$ let $p = [1, 0, 0, --, 0]$ and take two open sets U, V of $\mathbb{P}_{\mathbb{C}}^n$ such that $\mathbb{P}_{\mathbb{C}}^n = U \cup V$ and we will use these two sets to compute the homology groups of $\mathbb{P}_{\mathbb{C}}^n$.

let $U = \mathbb{P}_{\mathbb{C}}^n \setminus \{p\}$,

$$V = \{[z_0, z_1, --, z_n] \mid z_0 \neq 0\} \text{ and } p \in V.$$

<u>Claim 1.</u> (i) U deformation retracts to $\mathbb{P}_{\mathbb{C}}^{n-1}$.
(ii) V is homeomorphic to $\mathbb{C}^n$.

(i) Define $F : U \times I \to U$ by

$$F([z_0, \dots, z_n], t) = [(1-t) z_0, z_1, --, z_n]$$

also observe that

$$\mathbb{P}_{\mathbb{C}}^{n-1} = \{[z_0, z_1, --, z_n] \mid z_0 = 0\} \subseteq U$$

Now F is induced by the continuous map

$$(\mathbb{C}^{n+1} \setminus \{0\}) \times I \longrightarrow \mathbb{C}^{n+1} \setminus \{0\} \text{ by}$$

$$((z_0, z_1, --, z_n), t) \longrightarrow ((1-t) z_0, z_1, ---, z_n)$$

So F is continuous. Now

$$F([z_0, \dots, z_n], 0) = [z_0, z_1, --, z_n]$$

$$F([z_0, ---, z_n], 1) = [0, z_1, --, z_n] \in \mathbb{P}_{\mathbb{C}}^{n-1}$$

$$F([0, z_1, \ldots, z_n], t) = [0, z_1, \ldots, z_n] \qquad \forall t \in [0, 1]$$

So U deformation retracts to $\mathbb{P}_{\mathbb{C}}^{n-1}$!

(ii) Now we define

$$\phi_1 : V \longrightarrow \mathbb{C}^n \text{ by } \phi_1([1, z_1, \ldots, z_n]) = (z_1, \ldots, z_n)$$

Clearly ϕ_1 is a homeomorphism.

<u>Claim-2</u> $\mathbb{P}_{\mathbb{C}}^1 \simeq S^2$

Now we know If X is a locally compact, Hausdorff space and let $\tilde{X}$ be its one point compactification. If $X \subset T$ is an open set and T is compact then there exists a unique map $\phi : T \to \tilde{X}$ for which we have

$$\begin{tikzcd}
T \arrow[rr, "\phi"] & & \tilde{X} \\
& X \arrow[ul] \arrow[ur] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

and ϕ maps $T \setminus X$ to ∞.

Now we take $T = \mathbb{P}_{\mathbb{C}}^1$ and $X = \mathbb{C}$ and $\tilde{X} = S^2$

So we have $\phi : \mathbb{P}_{\mathbb{C}}^1 \to S^2$ is a continuous bijection.
Now $\mathbb{P}_{\mathbb{C}}^1$ is compact and S^2 is Hausdorff.

Hence ϕ is a homeomorphism.

$\Rightarrow \mathbb{P}_{\mathbb{C}}^1 \simeq S^2$.

Now we use M-V sequence to compute homology groups of $\mathbb{P}_{\mathbb{C}}^n$.

First we compute for $\mathbb{P}_{\mathbb{C}}^2$

$U = \mathbb{P}_{\mathbb{C}}^2 \setminus \{p\}$, $V = \mathbb{C}^2$, $U \cap V = \mathbb{C}^2 \setminus \{p\}$.

By M-V sequence we have

$$\to H_n(U\cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(\mathbb{P}_{\mathbb{C}}^2)$$
$$\longrightarrow H_{n-1}(U \cap V) \longrightarrow \cdots$$

Now If $n > 4$ then

$$\to \cancelto{0}{H_n(U\cap V)} \longrightarrow \cancelto{0}{H_n(U) \oplus H_n(V)} \longrightarrow H_n(\mathbb{P}_{\mathbb{C}}^2) \to \cancelto{0}{H_{n-1}(U\cap V)} \to \cdots$$

as $U \cap V = \mathbb{C}^2 \setminus \{p\}$ deformation retracts to S^3 and U deformation retracts to $\mathbb{P}_{\mathbb{C}}^1 = S^2$

we have $H_n(U\cap V) = H_n(U) = H_n(V) = H_{n-1}(U\cap V) = 0$ $\forall n > 4$

$$\Rightarrow \boxed{H_n(\mathbb{P}_{\mathbb{C}}^2) = 0 \quad \forall n > 4}$$

Now if $n = 4$ then

$$\to \cancelto{0}{H_4(U) \oplus H_4(V)} \longrightarrow H_4(\mathbb{P}_{\mathbb{C}}^2) \longrightarrow H_3(U \cap V)$$
$$\longrightarrow \cancelto{0}{H_3(U) \oplus H_3(V)}$$

$$\Rightarrow \boxed{H_4(\mathbb{P}_{\mathbb{C}}^2) \simeq H_3(U\cap V) = H_3(S^3) = \mathbb{Z}}$$

Now if $1 \leq n < 4$ we have the reduced M-V sequence

$$\begin{aligned}
&\to \cancelto{0}{\tilde{H}_3(S^2) \oplus \tilde{H}_3(\mathbb{C}^2)} \longrightarrow \tilde{H}_3(\mathbb{P}_{\mathbb{C}}^2) \longrightarrow \cancelto{0}{\tilde{H}_2(S^3)} \longrightarrow \\
&\tilde{H}_2(S^2) \oplus \tilde{H}_3(\mathbb{C}^2) \longrightarrow \tilde{H}_2(\mathbb{P}_{\mathbb{C}}^2) \longrightarrow \cancelto{0}{\tilde{H}_1(S^3)} \longrightarrow \\
&\cancelto{0}{\tilde{H}_1(S^2) \oplus \tilde{H}_1(\mathbb{C}^2)} \longrightarrow \tilde{H}_1(\mathbb{P}_{\mathbb{C}}^2) \longrightarrow \cancelto{0}{\tilde{H}_0(S^3)} \longrightarrow
\end{aligned}$$

$$\Rightarrow \quad H_1(\mathbb{P}_{\mathbb{C}}^2) \simeq \tilde{H}_1(\mathbb{P}_{\mathbb{C}}^2) = 0$$

$$H_2(\mathbb{P}_{\mathbb{C}}^2) \simeq \tilde{H}_2(\mathbb{P}_{\mathbb{C}}^2) = \tilde{H}_2(S^2) \oplus \tilde{H}_3(\mathbb{C}^2) = \mathbb{Z}$$

$$H_3(\mathbb{P}_{\mathbb{C}}^2) \simeq \tilde{H}_3(\mathbb{P}_{\mathbb{C}}^2) = 0 \ , \quad H_0(\mathbb{P}_{\mathbb{C}}^2) = \mathbb{Z}$$ as $\mathbb{P}_{\mathbb{C}}^2$ is path connected.

Hence

$$\boxed{H_k(\mathbb{P}_{\mathbb{C}}^2) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 4 \text{ and } k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}}$$

Now we use induction hypothesis and assume that

$$\begin{aligned} H_k(\mathbb{P}_{\mathbb{C}}^n) &= \mathbb{Z} \quad \text{if } 0 \leq k \leq 2n \text{ and } k \text{ is even} \\ &= 0 \quad \text{if otherwise.} \end{aligned}$$

We use this and M-V sequence to find homology groups of $H_k(\mathbb{P}_{\mathbb{C}}^{n+1})$

In this case $U = \mathbb{P}_{\mathbb{C}}^{n+1} \setminus \{p\} \xrightarrow[\text{retract}]{\text{deform.}} \mathbb{P}_{\mathbb{C}}^n$

$$V = \mathbb{C}^{n+1} \ , \quad U \cap V = \mathbb{C}^{n+1} \setminus \{p\} \xrightarrow[\text{retract}]{\text{deform}} S^{2n+1}$$

Now if $k > 2n+2$ then by M-V sequence we have

$$\to \cancelto{0}{H_k(U) \oplus H_k(V)} \to H_k(\mathbb{P}_{\mathbb{C}}^{n+1}) \to \cancelto{0}{H_{k-1}(U \cap V)} \to \cdots$$

$$\Rightarrow \boxed{H_k(\mathbb{P}_{\mathbb{C}}^{n+1}) = 0 \quad \forall k > 2n+2}$$

Now if $k = 2n+2$

$$\to \cancelto{0}{H_{2n+2}(U) \oplus H_{2n+2}(V)} \to H_{2n+2}(\mathbb{P}_{\mathbb{C}}^{n+1}) \to H_{2n+1}(U \cap V)$$

$$\to \cancelto{0}{H_{2n+1}(U) \oplus H_{2n+1}(V)} \to \cdots$$

$$\Rightarrow \boxed{H_{2n+2}(\mathbb{P}_{\mathbb{C}}^{n+1}) \simeq H_{2n+1}(U \cap V) \simeq H_{2n+1}(S^{2n+1}) = \mathbb{Z}}$$

Now if $k = 2n+1$ we have

$$\to \cancelto{0}{H_{2n+1}(U) \oplus H_{2n+1}(V)} \to H_{2n+1}(\mathbb{P}_{\mathbb{C}}^{n+1}) \to \cancelto{0}{H_{2n}(U \cap V)}$$

$$\Rightarrow \boxed{H_{2n+1}(\mathbb{P}_{\mathbb{C}}^{n+1}) = 0}$$

Now if $1 < k < 2n+1$ then

$$\to \cancelto{0}{H_k(U \cap V)} \to H_k(U) \oplus H_k(V) \to H_k(\mathbb{P}_{\mathbb{C}}^{n+1}) \to \cancelto{0}{H_{k-1}(U \cap V)} \to \cdots$$

$$\Rightarrow \boxed{H_k(\mathbb{P}_{\mathbb{C}}^{n+1}) \simeq H_k(U) \oplus H_k(V) = H_k(\mathbb{P}_{\mathbb{C}}^{n})}$$

Now if $k=1$ then we have

$$\to \cancelto{0}{H_1(U)\oplus H_1(V)} \to H_1(\mathbb{P}_{\mathbb{C}}^{n+1}) \to H_0(U\cap V) \to H_0(U)\oplus H_0(V)$$
$$\to H_0(\mathbb{P}_{\mathbb{C}}^{n+1}) \to 0$$

and $U\cap V$, U, V, $\mathbb{P}_{\mathbb{C}}^{n+1}$ are all path connected, hence

$0 \to H_0(U\cap V) \to H_0(U)\oplus H_0(V) \to H_0(\mathbb{P}_{\mathbb{C}}^{n+1}) \to 0$ is exact.

$\Rightarrow 0 \to H_1(\mathbb{P}_{\mathbb{C}}^{n+1}) \to 0$ is exact.

$$\Rightarrow \boxed{H_1(\mathbb{P}_{\mathbb{C}}^{n+1}) = 0}$$

Hence using all the calculations we have

$$H_k(\mathbb{P}_{\mathbb{C}}^{n+1}) = \begin{cases} 0 & \text{when } k > 2n+2 \\ \mathbb{Z} & \text{when } k = 2n+2 \\ 0 & \text{when } k = 2n+1 \\ H_k(\mathbb{P}_{\mathbb{C}}^{n}) & \text{when } 1 < k < 2n+1 \\ 0 & \text{when } k = 1 \\ \mathbb{Z} & \text{when } k = 0 \text{ (as } \mathbb{P}_{\mathbb{C}}^{n+1} \text{ is path connected)} \end{cases}$$

So summing it up we have

$$H_k(\mathbb{P}_{\mathbb{C}}^{n+1}) = \begin{cases} \mathbb{Z} & \text{if } 0 \le k \le 2n+2 \text{ and } k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

2: Compute the homology groups of $\mathbb{R}^m - \{P_1 \cdots P_r\}$ for $m>2$

Ans $X = \mathbb{R}^n$, $A = \mathbb{R}^m - \{P_1, \cdots P_r\{$

Let B_i's are disjoint closed balls centered at P_i's respectively.

Let $Z = (\bigsqcup_{i=1}^{r} B_i)^c$. Then $Z \subset A^o$, and $X \setminus Z = \bigsqcup_{i=1}^{r} B_i$, $A \setminus Z = \bigsqcup_{i=1}^{r} (B_i - \{P_i\})$

Therefore we can apply exicision.

$$H_n(X,A) \cong H_n(X\setminus Z, A \setminus Z). \quad \forall\, n \geq 0$$

Step1 In This step we will compute the homology groups $H_n(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} (B_i - \{P_i\}))$ for $n \geq 1$ using The long enact sequence for pair $(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} (B_i - \{P_i\}))$

For $n>1$

$$\cdots \to H_n(\bigsqcup_{i=1}^{r} B_i) \longrightarrow H_n(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} (B_i - \{P_i\})) \longrightarrow H_{n-1}(\bigsqcup_{i=1}^{r} (B_i - \{P_i\})) \longrightarrow H_{n-1}(\bigsqcup_{i=1}^{r} B_i) \to$$

Since $\quad H_n(\bigsqcup_{i=1}^{r} B_i) = H_{n-1}(\bigsqcup_{i=1}^{r} B_i) = 0 \quad$ for $n>1$

we get, $\quad H_n(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} B_i - \{P_i\}) \cong H_{n-1}(\bigsqcup_{i=1}^{r} (B_i - \{P_i\})) \quad n>1$

For $n=1$,

$$H_1(\bigsqcup_{i=1}^{r} B_i) \to H_1(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} (B_i - \{P_i\})) \to H_0(\bigsqcup_{i=1}^{r} (B_i - P_i)) \to H_0(\bigsqcup_{i=1}^{r} B_i) \to \cdots$$

$$0 \longrightarrow H_1(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} (B_i - P_i)) \xrightarrow{\phi} \bigoplus_{i=1}^{r} \mathbb{Z} \xrightarrow{i_*} \bigoplus_{i=1}^{r} \mathbb{Z} \longrightarrow \cdots$$

Claim: $i_* : H_0(\bigsqcup_{i=1}^{r} (B_i - \{P_i\})) \longrightarrow H_0(\bigsqcup_{i=1}^{r} B_i)$ is an isomorphism.

we know if $f: X \to Y$ be continuous function between two path-connected spaces then $f_* : H_0(X) \to H_0(Y)$ is an isomorphism.

Therefore, $i_* : H_0(B_i - \{P_i\}) \longrightarrow H_0(B_i)$ is isomorphism for each $i = 1, \dots, r$

Hence $i_* : \bigoplus_{i=1} H_0(B_i - \{P_i\}) \longrightarrow \bigoplus_{i=1}^{r} H_0(B_i)$ is an isomorphism.

Then $\quad \operatorname{Im} \phi = \ker i_* = 0$. Also ϕ is injective.

$\Rightarrow \quad H_1(\bigsqcup_{i=1}^{r} B_i, \bigsqcup_{i=1}^{r} (B_i - \{P_i\})) = 0$

So we have $\quad H_n(X \setminus Z, A \setminus Z) = \begin{cases} H_{n-1}(\bigsqcup_{i=1}^{r} (B_i - \{P_i\})) & n>1 \\ 0 & n=1 \end{cases}$

Since $B_i - \{P_i\}$ deformation retracts to it's boundary which is homeomorphic to S^{m-1} for all $i = 1 - - - r$, we have

$$H_n(X \setminus Z, A - Z) = \begin{cases} \bigoplus_{i=1}^{r} \mathbb{Z} & \text{if } n = m \\ 0 & n \geq 1, n \neq m \end{cases}$$

Therefore by excision, $H_n(\mathbb{R}^m, \mathbb{R}^m - \{P_1 - - P_r\}) = \begin{cases} \bigoplus_{i=1}^{r} \mathbb{Z} & \text{if } n = m \\ 0 & \text{if } n > 1,\ n \neq m \end{cases}$ —— ①

<u>step 2</u>: In this step we will compute homology of $\mathbb{R}^m - \{P_1, - - - P_r\}$ for $n \geq 1$ using long exact sequence of pairs $(\mathbb{R}^m, \mathbb{R}^m - \{P_1, - - -, P_r\})$ and ①

<u>For $n > 1$</u>

$$\to H_n(\mathbb{R}^m) \to H_n(\mathbb{R}^m, \mathbb{R}^m - \{P_1 - - - P_r\}) \to H_{n-1}(\mathbb{R}^m - \{P_1, - - - P_r\}) \to H_{n-1}(\mathbb{R}^m) \to$$

Since $H_n(\mathbb{R}^m) = H_{n-1}(\mathbb{R}^m) = 0$ for $n > 1$

we have, $H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{P_1 - - P_r\}) \cong H_{n-1}(\mathbb{R}^m - \{P_1 - - P_r\})$ $n > 1$

Therefore from (1)

$$H_{n-1}(\mathbb{R}^m - \{P_1 - - - P_r\}) = \begin{cases} \bigoplus_{i=1}^{r} \mathbb{Z} & \text{when } n = m \\ 0 & \text{when } n \neq m,\ n > 1 \end{cases}$$

or $$H_n(\mathbb{R}^m - \{P_1 - - - P_r\}) = \begin{cases} \bigoplus_{i=1}^{r} \mathbb{Z} & \text{when } n = m - 1 \\ 0 & \text{when } n \geq 1\ n \neq m - 1 \end{cases}$$

Since $\mathbb{R}^m - \{P_1, - - - P_r\}$ is path connected, $H_0(\mathbb{R}^m - \{P_1 - - P_r\}) \cong \mathbb{Z}$

Therefore, $$H_n(\mathbb{R}^m - \{P_1 - - P_r\} = \begin{cases} \mathbb{Z} & n = 0 \\ \bigoplus_{i=1}^{r} \mathbb{Z} & n = m - 1 \\ 0 & \text{else} \end{cases}$$ ☒

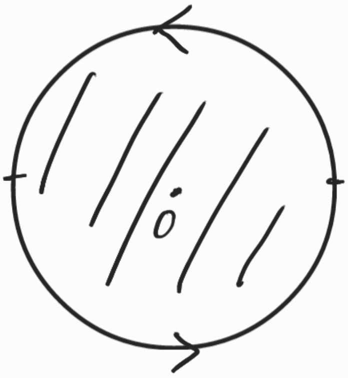
Alternative solution: - Also it is easy to see that $\mathbb{R}^m \setminus \{P_1, P_2, \dots, P_r\}$ is homotopy equivalent to $\bigvee_{i=1}^{r} S^{m-1}$. So we have

$$H_k(\mathbb{R}^m \setminus \{P_1, P_2, \dots, P_r\}) \cong H_k\left(\bigvee_{i=1}^{r} S^{m-1}\right) \quad \forall k \geq 0$$

For homologies of $\bigvee_{i=1}^{r} S^{m-1}$ See 2.5.

3. Compute the homology groups of $\mathbb{P}_{\mathbb{R}}^2$.

Ans: We get $\mathbb{P}_{\mathbb{R}}^2$ by identifying the boundary of the disc as follows:



(i.e. we identify the antipodal points on the boundary of D^2)

Let, $\mathcal{U}$ be the open subset $\mathbb{P}_{\mathbb{R}}^2 \setminus \{0\}$ and V be the open subset

$$\{ z \in D^2 : |z| < 1 \}.$$

Now, $\mathcal{U}$ deformation retracts onto the boundary which is homeomorphic to S^1 and V is contractible.

So, $H_n(U) \cong H_n(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Z} & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$

$$H_n(V) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

Now, $U \cap V = \{ z \in D^2 \setminus \{0\} : |z| < 1 \}$

$$\simeq S^1$$

So, $H_n(U \cap V) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Z} & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$

By Mayer-Vietoris sequence, we get

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(\mathbb{P}_{\mathbb{R}}^2)$$

$$\longrightarrow H_{n-1}(U \cap V) \longrightarrow \cdots$$

Now for $n > 2$, it is clear that

$$H_n(\mathbb{P}_{\mathbb{R}}^2) = 0$$

Now, we have,

$$\cdots \to \cancelto{0}{H_2(U)} \oplus \cancelto{0}{H_2(V)} \to H_2(\mathbb{P}_{\mathbb{R}}^2) \to H_1(U\cap V)$$

$$\xrightarrow{j_*} H_1(U) \oplus \cancelto{0}{H_1(V)} \to H_1(\mathbb{P}_{\mathbb{R}}^2) \to \cdots$$

Now, $j : U\cap V \to U$ is the inclusion

it induces a commutative diagram

$$\begin{tikzcd}
\pi_1(U\cap V) \arrow[r, "j_*"] \arrow[d] & \pi_1(U) \arrow[d] \\
(\pi_1(U\cap V))_{ab} \arrow[r, "\bar{j}_*"] \arrow[d, "\cong"'] & (\pi_1(U))_{ab} \arrow[d, "\cong"] \\
H_1(U\cap V) \arrow[r, "j_*"] & H_1(U)
\end{tikzcd}$$

Now, we know that

$$\begin{tikzcd}
j_* : \pi_1(U\cap V) \arrow[r] \arrow[d, phantom, sloped, "="] & \pi_1(U) \arrow[d, phantom, sloped, "="] \\
\mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

is given by $1 \longmapsto 2$

So,

$$\to 0 \to H_2(\mathbb{P}^2_{\mathbb{R}}) \to H_1(U\cap V) \xrightarrow{j_*} H_1(U) \to$$

$$\to H_1(\mathbb{P}^2_{\mathbb{R}}) \to H_0(U\cap V) \to \cdots$$

$$\to 0 \to H_2(\mathbb{P}^2_{\mathbb{R}}) \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to H_1(\mathbb{P}^2_{\mathbb{R}})$$

$$\xrightarrow{\delta} H_0(U\cap V) \to H_0(U)\oplus H_0(V) \to H_0(\mathbb{P}^2_{\mathbb{R}}) \to 0$$

Since $U\cap V$, U, V and $\mathbb{P}^2_{\mathbb{R}}$ are path connected, so get,

$$0 \to H_0(U\cap V) \to H_0(U)\oplus H_0(V) \to H_0(\mathbb{P}^2_{\mathbb{R}}) \to 0$$

is exact.

This implies that,

$$0 \to H_2(\mathbb{P}^2_{\mathbb{R}}) \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to H_1(\mathbb{P}^1_{\mathbb{R}}) \to 0$$

is exact.

This implies that

$$H_2(\mathbb{P}_{\mathbb{R}}^2) = 0$$

and $H_1(\mathbb{P}_{\mathbb{R}}^2) = \mathbb{Z}/2\mathbb{Z}$

---

4. C is the closed oriented surface of genus k. Compute the homogy groups of C.

Ans: Let $p \in C$.

<u>step I</u>: We will compute homology groups of the pair $(C, C-p)$

<u>step II</u>: We will compute Homology groups of $C-p$.

<u>step III</u>: We will compute Homology groups of C.

<u>Step I</u>: Let B be an open disc around p in C.

Let $Z = C \setminus B$

Then $Z \subseteq C - p$

By excision theorem,

$$H_n(C, C-\{p\}) \cong H_n(C \setminus Z, (C \setminus \{p\}) \setminus Z)$$

$$\forall n \geqslant 0$$

We observe that,

$$(C \setminus Z, (C \setminus \{p\}) \setminus Z) = (B, B \setminus p)$$

Let us compute $H_n(B, B \setminus p)$:

Using long exact sequence of the pair $(B, B \setminus p)$, we get,

$$\cdots \rightarrow H_n(B) \rightarrow H_n(B, B \setminus p) \rightarrow H_{n-1}(B \setminus p)$$

$$\rightarrow H_{n-1}(B) \rightarrow \cdots$$

Now, for $n > 1$, the ends in the above long exact sequence are 0.

So, $H_n(B, B \setminus p) = 0$ for $n > 2$

Now we have,

$$\to H_2(B) \to H_2(B, B\setminus p) \to H_1(B\setminus p)$$
$$\to H_1(B) \to H_1(B, B\setminus p) \to \cdots$$

Now, $H_2(B) = 0 = H_1(B)$

So, $H_2(B, B\setminus p) \cong H_1(B\setminus p)$
$\cong H_1(S^1) \cong \mathbb{Z}$

Again,

$$0 \to H_1(B, B\setminus p) \to H_0(B\setminus p) \to H_0(B)$$
$$\to H_0(B, B\setminus p) \to 0$$

Now since $B\setminus p$ & B are path connected & the inclusion $B\setminus p \hookrightarrow B$ induces isomorphism on H_0.

So, $H_1(B, B\setminus p) = 0$

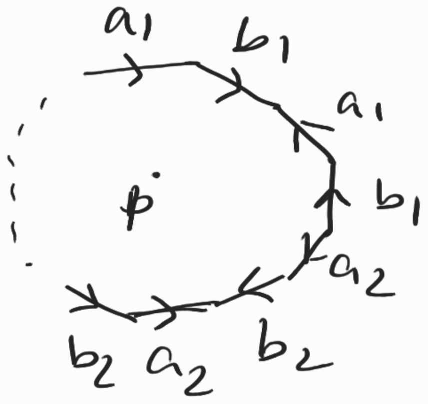
and $H_0(B, B\setminus p) = 0$

So, we have,

$$H_n(B, B\setminus p) = \begin{cases} \mathbb{Z} & \text{if } n=2 \\ 0 & \text{if } n\neq 2 \end{cases}$$

This finishes step I.

<u>step II</u>: We can present C as a $4k$-gon with identification on boundary as follows:



From diagram, it is clear that $C\setminus p$ deformation retracts onto the boundary which is homeomorphic to wedge of $2k$ circles.

So, $H_n(C\setminus p) \cong H_n(\vee_{2k} S^1) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Z}^{2k} & \text{if } n=1 \\ 0 & \text{otherwis} \end{cases}$

This finishes step II.

---

step III :

We have long exact sequence of the pair $(C, C\setminus p)$:

$$\cdots \to H_n(C) \longrightarrow H_n(C, C\setminus p) \longrightarrow H_{n-1}(C\setminus p) \longrightarrow H_{n-1}(C) \longrightarrow \cdots$$

For $n>2$, $H_n(C\setminus p)=0$

& $H_n(C, C\setminus p)=0$

So, $H_n(C)=0$ for $n>2$

Now we have,

$$\cdots \longrightarrow \overset{0}{\cancel{H_2(C\setminus p)}} \longrightarrow H_2(C) \longrightarrow H_2(C, C\setminus p) \longrightarrow H_1(C\setminus p) \longrightarrow H_1(C) \longrightarrow \overset{0}{\cancel{H_1(C, C\setminus p)}} \longrightarrow \cdots$$

So, we have

$$0 \to H_2(C) \to \mathbb{Z} \to H_1(C\setminus p) \xrightarrow{i_*} H_1(C) \to 0$$

is exact.

We have commutative diagram:

$$\begin{tikzcd}
\pi_1(C\setminus p) \arrow[r, "i_*"] \arrow[d] & \pi_1(C) \arrow[d] \\
(\pi_1(C\setminus p))_{ab} \arrow[r, "\bar{i}_*"] \arrow[d, "\cong"'] & (\pi_1(C))_{ab} \arrow[d, "\cong"] \\
H_1(C\setminus p) \arrow[r, "i_*"] & H_1(C)
\end{tikzcd}$$

Now, $\pi_1(C\setminus p)$ is the free group on 2K generators and

$$\pi_1(C) = \pi_1(C\setminus p) \Big/ \left\langle \prod_{1}^{K} a_i b_i a_i^{-1} b_i^{-1} \right\rangle$$

Since, $\prod_{1}^{K} (a_i b_i a_i^{-1} b_i^{-1}) \in [\pi_1(C\setminus p), \pi_1(C\setminus p)]$

It follows that.

$$\overline{i_*} : (\pi_1(C - p))_{ab} \longrightarrow (\pi_1(C))_{ab}$$

is an isomorphism.

From the commutative diagram, we get

$$i_* : H_1(C - p) \longrightarrow H_1(C)$$

is an isomorphism.

So from (#) we have,

$$0 \to H_2(C) \to \mathbb{Z} \to H_1(C-p) \xrightarrow{\cong} H_1(C) \to 0$$

So, $H_2(C) \cong \mathbb{Z}$

and $H_1(C) \cong H_1(C \setminus p) \cong \mathbb{Z}^{2K}$

Since C is connected, so $H_0(C) = \mathbb{Z}$.

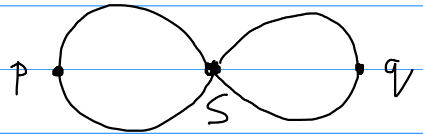
$$\text{So, } H_n(C) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2K} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n > 2 \end{cases}$$

∎

5. Homology group of $\bigvee_{i=1}^{r} S^m$ $m \geq 1$

$\Rightarrow$ First we compute homology groups of $X = S^m \vee S^m$.

let $U = X \setminus \{q\}$
$V = X \setminus \{p\}$
$U \cap V = X \setminus \{p, q\}$



U, V deformation retracts to S^m. and $U \cap V$ deformation retracts to the point s.

Now by M-V sequence we have
If $k > 1$

$$\to H_k(\overset{0}{\cancel{U\cap V}}) \longrightarrow H_k(U) \oplus H_k(V) \longrightarrow H_k(X)$$
$$\to H_{k-1}(\overset{0}{\cancel{U\cap V}}) \longrightarrow \cdots$$

$\Rightarrow$ $H_k(X) \simeq H_k(S^m) \oplus H_k(S^m)$ $\forall k > 1$

If $k = 1$ then we have

$$\to H_1(\overset{0}{\cancel{U\cap V}}) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(X)$$
$$\to H_0(U\cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(X) \longrightarrow 0$$

Now $U \cap V$, U, V, X are path connected. So
$0 \to H_0(U\cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0$ is exact.

$\Rightarrow$ $H_1(X) \simeq H_1(S^m) \oplus H_1(S^m)$

and $H_0(X) = \mathbb{Z}$ as X is path connected.

So we have $H_k(\bigvee_{i=1}^{2} S^m) = H_k(S^m) \oplus H_k(S^m)$ $k \geq 1$

$= \mathbb{Z}$ $k = 0$

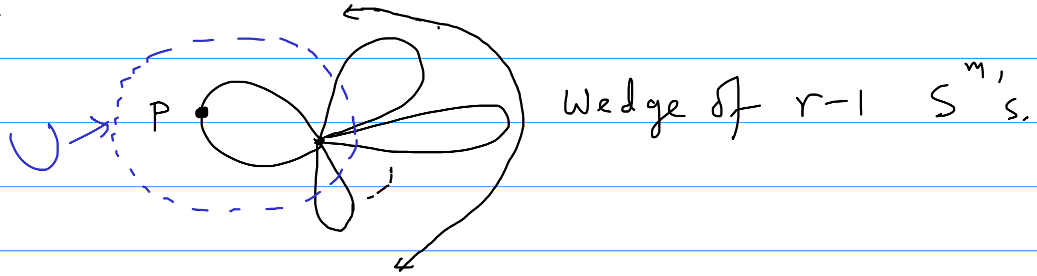
So let us use induction on r to compute homology groups of $\bigvee_{i=1}^{r} S^m$.

let us assume

$$H_k(\bigvee_{i=1}^{r-1} S^m) = \bigoplus_{i=1}^{r-1} H_i(S^m) \quad k \geq 1$$

$$= \mathbb{Z} \quad k = 0$$

let $X = \bigvee_{i=1}^{r} S^m$



let U is given in the picture above.

$V = X \setminus \{p\}$

U deformation retracts to S^m

V deformation retracts to $\bigvee_{i=1}^{r-1} S^m$

$U \cap V$ deformation retracts to a point.

Now by M-V sequence we have

$\forall k > 1$

$$\longrightarrow \overset{0}{\cancel{H_k(U \cap V)}} \longrightarrow H_k(U) \oplus H_k(V) \longrightarrow H_k(X)$$

$$\longrightarrow \overset{0}{\cancel{H_{k-1}(U \cap V)}} \longrightarrow \cdots$$

$$\Rightarrow \quad H_k(X) \simeq H_k(S^m) \oplus H_k(\bigvee_{i=1}^{r-1} S^m) = \bigoplus_{i=1}^{r} H_k(S^m)$$

If $k = 1$ then

$$\to \cancelto{0}{H_1(U \cap V)} \to H_1(U) \oplus H_1(V) \to H_1(X)$$

$$\to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0$$

Now $U \cap V$, U, V, X are pathconnected so

$$0 \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0$$

$$\Rightarrow \quad H_1(X) \simeq H_1(S^m) \oplus H_1\left(\bigvee_{i=1}^{r-1} S^m\right) = \bigoplus_{i=1}^{r} H_1(S^m)$$

and $H_0(X) = \mathbb{Z}$ as X is pathconnected.

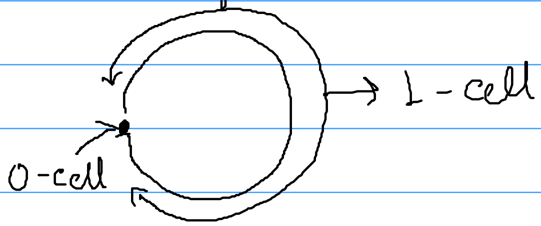
So we have

$$H_k\left(\bigvee_{i=1}^{r} S^m\right) = \bigoplus_{i=1}^{r} H_k(S^m) \qquad k \geq 1$$

$$= \mathbb{Z} \qquad k = 0$$

CW- Complex

1. CW structure of S^1



$k \geq 2$

$$CW_k(X) = H_k\left(X^k / X^{k-1}\right) = 0$$

$$CW_1(X) = H_1\left(X^1 / X^0\right) = H_1(S^1) = \mathbb{Z}$$

$$CW_0(X) = H_0(X^0) = \mathbb{Z}$$

Homology sequence of the pair (X^1, X^0) we have

$$\begin{tikzcd}
{} \arrow[r] & H_1(X^0) \arrow[r] & H_1(X^1) \arrow[r] & H_1(X^1, X^0) \arrow[r] \arrow[d, phantom, "\cong" sloped] & H_0(X^0) \arrow[r, "\sim"] & \cdots \\
 & & & H_1\left(X^1 / X^0\right) & H_0(X^1) &
\end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd}
{} \arrow[r] & H_1(X^0) \arrow[r] & H_1(X^1) \arrow[r] & H_1(X^1, X^0) \arrow[r] \arrow[d, phantom, "\cong" sloped] & 0 \\
 & & & H_1\left(X^1 / X_0\right) &
\end{tikzcd}$$

is exact

$\Rightarrow$ the map from $CW_1(X) \to CW_0(X)$ is 0.

$\Rightarrow$ $d_1 : \underset{\underset{\mathbb{Z}}{\parallel}}{CW_1(X)} \longrightarrow \underset{\underset{\mathbb{Z}}{\parallel}}{CW_0(X)}$ is 0.

$$\begin{tikzcd}
CW_2(X) \arrow[r, "d_2"] \arrow[d, phantom, "\parallel"] & CW_1(X) \arrow[r, "d_1 = 0"] \arrow[d, phantom, "\parallel"] & CW_0(X) \\
0 & \mathbb{Z} &
\end{tikzcd}$$

$$\Rightarrow \quad 0 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1 = 0} \mathbb{Z} \longrightarrow 0$$

$k \geq 2$

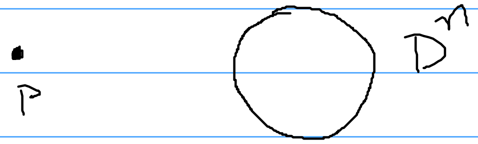
$$H_k^{CW}(X) = 0$$

If $k = 1$

$$H_1^{CW}(X) = \mathbb{Z}$$

$$H_0^{CW}(X) = \mathbb{Z}$$

2. CW structure of S^n

$\Rightarrow$ 

$f: \partial D^n \longrightarrow \{p\}$.

No 1, 2, .., n-1 - cells are present.

$$
\begin{aligned}
CW_k(X) &= 0 \quad \text{if } k \geq n+1 \\
&= \mathbb{Z} \quad \text{if } k = n \\
&= 0 \quad \text{if } 0 < k < n \\
&= \mathbb{Z} \quad \text{if } k = 0
\end{aligned}
$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "d_n"] \arrow[d, phantom, "\parallel"] & 0 \arrow[r] & \cdots \arrow[r, "d_2"] & 0 \arrow[r, "d_1"] \arrow[d, phantom, "\parallel"] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
 & CW_n(X) & & & CW_1(X) & CW_0(X) &
\end{tikzcd}$$

$$
\begin{aligned}
H_k^{CW}(X) &= 0 \quad \text{if } k \geq n+1 \\
&= \mathbb{Z} \quad \text{if } k = n \\
&= 0 \quad \text{if } 1 < k < n \\
&= \mathbb{Z} \quad \text{if } k = 0
\end{aligned}
$$

3. $\mathbb{P}_{\mathbb{C}}^1 \cong S^2$, Hence the CW-homology of $\mathbb{P}_{\mathbb{C}}^1$'s are computed.

4. We first compute CW-homology of $\mathbb{P}_{\mathbb{C}}^2$.

There is no n-cells for $n > 4$ and $n = 1, 3$.

$X^4 = \mathbb{P}_{\mathbb{C}}^2$, $X^2 = \mathbb{P}_{\mathbb{C}}^1$, $X^0 = \{p\}$ p is the 0-cell.

If $k > 4$

$$CW_k(X) = H_k(X^k/X^{k-1}) = 0$$

If $k = 4$

$$CW_4(X) = H_4(X^4/X^3) = H_4(\mathbb{P}_{\mathbb{C}}^2) = \mathbb{Z}$$

If $k = 3, 1$

$$CW_3(X) = 0 = CW_1(X)$$

If $k = 2$

$$CW_2(X) = H_2(X^2/X^1) = H_2(\mathbb{P}_{\mathbb{C}}^1) = \mathbb{Z}$$

If $k = 0$

$$CW_0(X) = H_0(X^0) = H_0(\{p\}) = \mathbb{Z}$$

So we have the chain complex.

$$\begin{tikzcd}
 & & & & & \mathbb{Z} \arrow[d, equal] & \\
0 \arrow[r, "d_5"] & CW_4(X) \arrow[r, "d_4"] \arrow[d, equal] & CW_3(X) \arrow[r, "d_3"] \arrow[d, equal] & CW_2(X) \arrow[r, "d_2"] \arrow[d, equal] & CW_1(X) \arrow[r, "d_1"] \arrow[d, equal] & CW_0(X) \arrow[r, "d_0"'] & 0 \\
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & &
\end{tikzcd}$$

Now we have

$$H_0^{cw}(X) = \ker d_0 / \operatorname{Im} d_1 = \mathbb{Z}/\{0\} = \mathbb{Z}$$

$$H_1^{cw}(X) = \ker d_1 / \operatorname{Im} d_2 = \{0\}$$

$$H_2^{cw}(X) = \ker d_2 / \operatorname{Im} d_3 = \mathbb{Z}/\{0\} = \mathbb{Z}$$

$$H_3^{cw}(X) = \ker d_3 / \operatorname{Im} d_4 = \{0\}$$

$$H_4^{cw}(X) = \ker d_4 / \operatorname{Im} d_5 = \mathbb{Z}/\{0\} = \mathbb{Z}$$

and clearly $H_k^{cw}(X) = 0 \quad \forall k > 4$

So we have

$$\begin{aligned} H_k^{cw}(X) &= \mathbb{Z} \quad \text{if } k = 0, 2, 4 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Now we compute CW-homology of $X = \mathbb{P}_{\mathbb{C}}^{n+1}$ using induction.

Let $H_k^{cw}(\mathbb{P}_{\mathbb{C}}^t) = \mathbb{Z}$ if $0 \leq k \leq 2t$ and k is even

$= 0$ if otherwise $\quad \forall\, t \leq n$

In $\mathbb{P}_{\mathbb{C}}^{n+1}$ we don't have m-cells for $m > 2n+2$ and $m = 2n+1, 2n-1, \cdots, 3, 1$

Now we have only one k-cell for each $k = 0, 2, 4, \cdots, 2n+2$

Now we have

$$CW_0(X) = H_0(X^0) = \mathbb{Z}$$

$$CW_1(X) = H_1(X^1/X_0) = 0$$

$$CW_2(X) = H_2(X^2/X^1) = H_2(\mathbb{P}_{\mathbb{C}}^1) = \mathbb{Z}$$

$$\vdots$$

$$CW_{2n+1}(X) = H_{2n+1}(X^{2n+1}/X^{2n}) = 0$$

$$CW_{2n+2}(X) = H_{2n+2}(X^{2n+2}/X^{2n-1}) = H_{2n+2}(\mathbb{P}_{\mathbb{C}}^{n+1}) = \mathbb{Z}$$

and $CW_k(X) = 0 \quad \forall k > 2n+2$

So we have the chain complex

$$0 \xrightarrow{d_{2n+3}} \overset{\mathbb{Z}}{\overset{\shortparallel}{CW_{2n+2}(X)}} \xrightarrow{d_{2n+2}} \cancelto{0}{CW_{2n+1}(X)} \xrightarrow{d_{2n+1}} \overset{\mathbb{Z}}{\overset{\shortparallel}{CW_{2n}(X)}} \xrightarrow{d_{2n}} --$$

$$--- \cdot \xrightarrow{d_4} \cancelto{0}{CW_3(X)} \xrightarrow{d_3} \overset{\mathbb{Z}}{\overset{\shortparallel}{CW_2(X)}} \xrightarrow{d_2} \cancelto{0}{CW_1(X)} \xrightarrow{d_1} \overset{\mathbb{Z}}{\overset{\shortparallel}{CW_0(X)}}$$

$$\xrightarrow{d_0} 0$$

So we have

$$H_0^{cw}(X) = \mathbb{Z}$$

$$H_1^{cw}(X) = 0$$

$$\vdots$$

$$H_{2n+1}^{cw}(X) = 0$$

$$H_{2n+2}^{cw}(X) = \mathbb{Z}$$

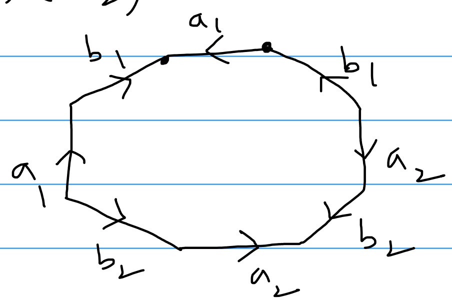
and $H_k^{cw}(X) = 0 \quad \forall k > 2n+2$

So we have

$$H_k^{cw}(X) = \mathbb{Z} \quad \forall 0 \leq k \leq 2n+2 \text{ and } k \text{ is even}$$

$$= 0 \quad \text{otherwise}$$

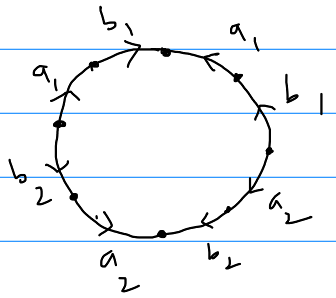
5. Compute the CW-homology of compact oriented surface of genus 2, (M_2) given below



Now X^0 is the point (0-cell),

X^1 is obtained by attaching 4 1-cells to X^0.

X^2 is obtained by attaching 1 2-cell by the attaching map. $S^1 \longrightarrow X^1$ by



Now $\forall k > 2$ we have $CW_k(X) = 0$

If $k=2$ then $CW_2(X) = H_2(X^2/X^1) = H_2(S^2) = \mathbb{Z}$

If $k=1$ then $CW_1(X) = H_1(X^1/X^0) = H_1(\bigvee_{i=1}^{4} S^1)$.

$= \bigoplus_{i=1}^{4} \mathbb{Z}$

If $k=0$ then $CW_0(X) = H_0(X^0) = \mathbb{Z}$.

Hence we have the complex

$$\begin{tikzcd}
0 \arrow[r, "d_3"] & CW_2(X) \arrow[r, "d_2"] \arrow[d, equal] & CW_1(X) \arrow[r, "d_1"] \arrow[d, equal] & CW_0(X) \arrow[r, "d_0"] \arrow[d, equal] & 0 \\
 & \mathbb{Z} & \bigoplus_{i=1}^{q} \mathbb{Z} & \mathbb{Z} &
\end{tikzcd}$$

Now we have the long exact sequence of the pair (X^1, X^0) which is

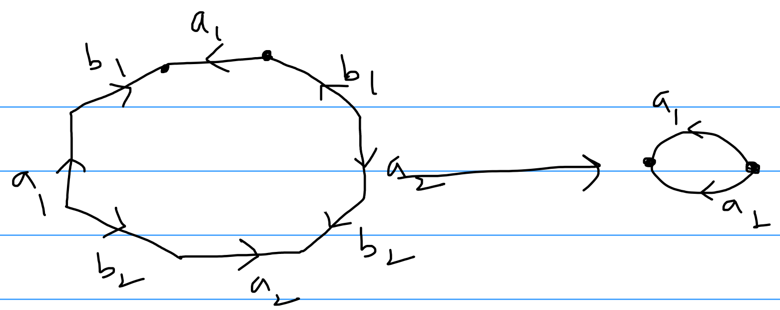
$$\begin{tikzcd}
{} \arrow[r] & H_1(X^0) \arrow[r] & H_1(X^1) \arrow[r] & H_1(X^1, X^0) \arrow[r] \arrow[d, equal] & H_0(X_0) \arrow[d] \\
 & & & H_1(X^1/X^0) & H_0(X^1) \arrow[d] \\
 & & & & \vdots
\end{tikzcd}$$

Now X^0, X^1 are pathconnected so $H_0(X^0) \simeq H_0(X^1)$

$\Rightarrow \quad \longrightarrow H_1(X^1) \longrightarrow H_1(X^1/X^0) \longrightarrow 0$ is exact.

$\Rightarrow \quad d_1 : CW_1(X) \to CW_0(X)$ is the zero map.

Now we see the behavior of d_2.
we have $S^1 \longrightarrow X^1$ the attaching map, then we have the collapsing maps which collapses all but one circle which gives the component of $d_2(1)$.
let $X^1 \longrightarrow S^1$ be the map which is given by



So as the orientation of the circle given by attaching map

$S^1 \longrightarrow X^1 \longrightarrow S^1$ maps S^1 to $a_1 - a_1 = 0$

$\Rightarrow$ 1st component of $d_2(1)$ is 0.

and by the description of the attaching map we see each component of $d_2(1)$ is 0.

Hence $d_2 = 0$

Another explanation using Fundamental group.

We have the attaching map takes the generator in $\pi_1(S^1)$ to the relation $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$ in $\pi_1(\bigvee_{i=1}^{4} S^1)$ which clearly belongs to $[\pi_1(\bigvee_{i=1}^{4} S^1), \pi_1(\bigvee_{i=1}^{4} S^1)]$

So we have the diagram

$$\begin{tikzcd}
\mathbb{Z} = \pi_1(S^1) \arrow[rrr] \arrow[ddd, "ab"'] & & & \pi_1\left(\bigvee_{i=1}^{4} S^1\right) = \mathop{*}_{i=1}^{4} \mathbb{Z} \arrow[ddd, "ab"] \\
& a \arrow[r] \arrow[d] & \prod_{i=1}^{4} a_i b_i a_i^{-1} b_i^{-1} \arrow[d] & \\
& a \arrow[r] & 0 & \\
\mathbb{Z} = H_1(S^1) \arrow[rrr] & & & H_1\left(\bigvee_{i=1}^{4} S^1\right) = \bigoplus_{i=1}^{4} \mathbb{Z}
\end{tikzcd}$$

$\Rightarrow$ d_2 takes a to 0

$\Rightarrow$ $d_2 = 0$

So we have

$$H_0^{cw}(X) = \ker d_0 / \operatorname{Im} d_1 = \mathbb{Z}/\{0\} \cong \mathbb{Z}$$

$$H_1^{cw}(X) = \ker d_1 / \operatorname{Im} d_2 = \bigoplus_{i=1}^{4} \mathbb{Z} / \{0\} = \bigoplus_{i=1}^{4} \mathbb{Z}$$

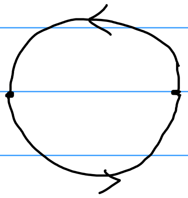
$$H_2^{cw}(X) = \ker d_2 / \operatorname{Im} d_3 = \mathbb{Z}/\{0\} \cong \mathbb{Z}$$

and $H_k^{cw}(X) = 0 \quad \forall k > 2$

[In case of genus > 2 the method is exactly same and in that case we have to use the relation defined as 1.4].

6. CW - Homology of $\mathbb{P}^2_{\mathbb{R}}$.

Here X^0 is a point, $X^1 = S^1$, X^2 is obtained by the attaching maps $f : \partial \mathbb{D}^2 \to X^1$ by $f(z) = z^2$ and no k-cells $\forall k > 2$.



$$CW_0(X) = H_0(X^0) = \mathbb{Z}$$
$$CW_1(X) = H_1(X^1/X^0) = H_1(S^1) = \mathbb{Z}$$
$$CW_2(X) = H_2(X^2/X^1) = H_2(S^2) = \mathbb{Z}$$

and $CW_k(X) = 0 \quad \forall k > 2$

So we have the complex

$$\begin{tikzcd} 0 \arrow[r] & CW_2(X) \arrow[r, "d_2"] \arrow[d, equal] & CW_1(X) \arrow[r, "d_1"] \arrow[d, equal] & CW_0(X) \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

like similar arguement of the previous problem $d_1 = 0$

Now we see how d_2 behaves.

We have the attaching map

$$\begin{tikzcd} S^1 \arrow[r] & X^1 \arrow[d, equal] & \text{by} & z \arrow[r] & z^2 \\ & S^1 & & & \end{tikzcd}$$

so $d_2(1) = 2$ as $\deg(z \longrightarrow z^2) = 2$

Now we show $\deg(z \longrightarrow z^2) = 2$

let $\gamma(t) = e^{2\pi i t}$ be the generator of $\pi_1(S^1, 1) \cong \mathbb{Z}$, let $f: S^1 \to S^1$ by $f(z) = z^2$

and $p: \mathbb{R} \to S^1$ is the covering map

$$p(x) = e^{2\pi i x}$$

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d] \\
{[0,1]} \arrow[r, "\gamma"] \arrow[urr, dashed, "\widetilde{f\circ\gamma}"] & S^1 \arrow[r, "f"] & S^1
\end{tikzcd}$$

where $\widetilde{f\circ\gamma}$ is the unique lift of the path $f\circ\gamma$.

Now if we define $g: [0,1] \to \mathbb{R}$ by $g(t) = 2t$ then we have

$$g(0) = 0 \text{ and } p\circ g = f\circ\gamma = p\circ\widetilde{f\circ\gamma}$$

$$\Rightarrow g = \widetilde{f\circ\gamma} \text{ (by uniqueness).}$$

So $\deg f = g(1) = 2$

Hence

$$H_0^{cw}(X) = \ker d_0 / \operatorname{Im} d_1 = \mathbb{Z}/\{0\} = \mathbb{Z}$$

$$H_1^{cw}(X) = \ker d_1 / \operatorname{Im} d_2 = \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2$$

$$H_2^{cw}(X) = \ker d_2 / \operatorname{Im} d_3 = \{0\}/\{0\} = 0$$

$$H_k^{cw}(X) = 0 \quad \forall k > 2.$$

7. CW-homology of $\mathbb{P}^n_{\mathbb{R}}$.

$\Rightarrow$ We have only one k-cell $\forall k = 0, 1, 2, \dots, n$ and we have $X^0 = \{p\}, X^1 = \mathbb{P}^1_{\mathbb{R}}, \dots, X^{n-1} = \mathbb{P}^{n-1}_{\mathbb{R}}$

Now we take n-cell and define $f: S^{n-1} \to \mathbb{P}^{n-1}_{\mathbb{R}}$ to be usual quotient map with antipodal points identified. Then we get $X^n = \mathbb{P}^n_{\mathbb{R}}$.

$$CW_0(X) = H_0(X^0) = \mathbb{Z}$$

$$CW_1(X) = H_1(X^1/X_0) = H_1(S^1) = \mathbb{Z}$$

$$\vdots$$

$$CW_n(X) = H_n(X^n/X^{n-1}) = H_n(S^n) = \mathbb{Z}$$

and $CW_k(X) = 0 \quad \forall k \geq n+1$

So we have the complex

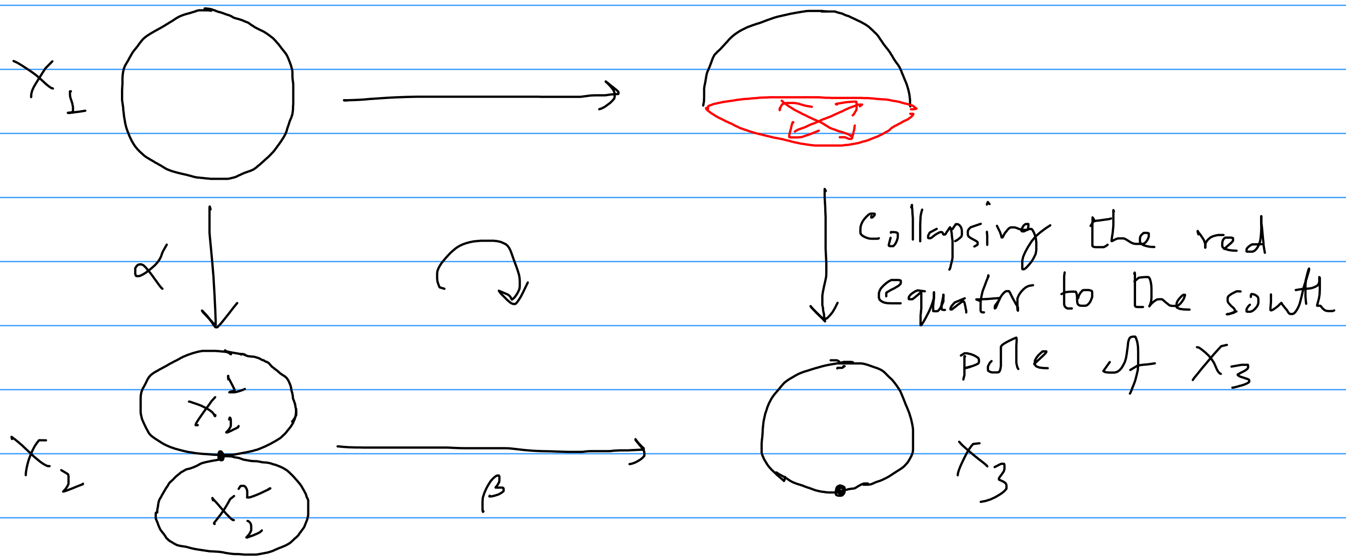
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "d_n"] \arrow[d] & \mathbb{Z} \arrow[r, "d_{n-1}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \cdots \arrow[r, "d_2"] & \mathbb{Z} \arrow[r, "d_1"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & n & n-1 & n-2 & \cdots & 1 & 0 &
\end{tikzcd}$$

So we compute $d_{k+1} \quad \forall \ k = 0, 1, \dots, n-1$

To compute this we see the map

$$\begin{tikzcd}
S^k \arrow[r] & X^k \arrow[r] & X^k/X^{k-1} \arrow[d, phantom, "\parallel"] \\
 & & S^k
\end{tikzcd}$$

and find its degree.



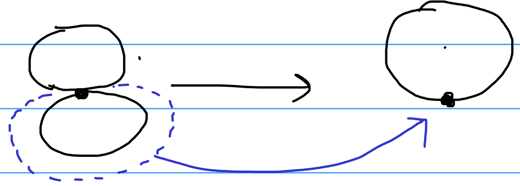
Collapsing the red equator to the south pole of X_3

If we take the map α which collapses the equator of X_1 to a point, then the map $X_1 = S^k \to X^k \to X^k/X^{k-1} = S^k = X_3$ factors through α and we get a map from β from $X_2 \to X_3$.

So we compute $(\beta \circ \alpha)_*(1)$ and we will find d_{k+1}

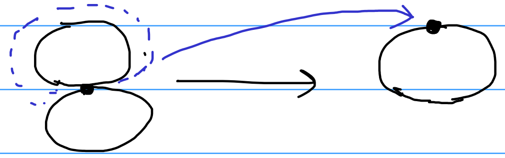
First let us take two quotient maps

$r_1 : X_2 \to X_2/X_2^2$ given by



So we have $r_1|_{X_2^1} = Id_{X_2^1}$ and $r|_{X_2^2}$ = constant map to the point

$r_2 : X_2 \to X_2/X_2^1$ given by



So we have $r_2|_{X_2^1}$ = constant map on the point and $r_2|_{X_2^2} = Id_{X_2^2}$

Hence by the description of r_1, r_2 we have the map

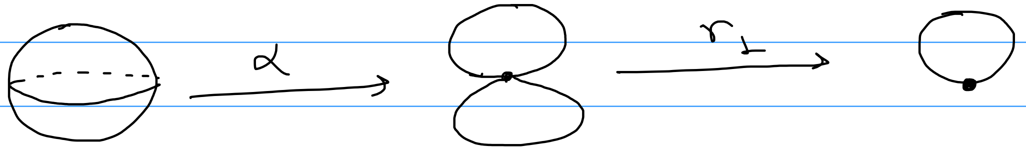
$$\begin{array}{ccc} H_k(X_2) & \longrightarrow & H_k(X_2/X_2^2) \oplus H_k(X_2/X_2^1) \text{ by} \\ \| & & \| \\ \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \\ (a,b) & \longrightarrow & ((r_1)_*(a), (r_2)_*(b)) = (a,b) \end{array}$$

is an isomorphism.

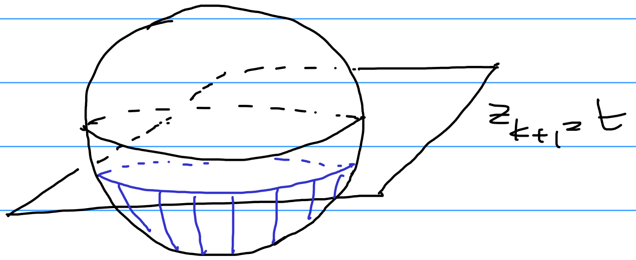
So to find $\alpha_*(1)$ we find $(r_1 \circ \alpha)_*(1)$ and $(r_2 \circ \alpha)_*(1)$

claim 1:— . $r_1 \circ \alpha$ and $r_2 \circ \alpha$ are homotopic to identity map. which implies $(r_1 \circ \alpha)_*(1) = (r_2 \circ \alpha)_*(1) = 1$

Now we see the geometric description of $r_1 \circ \alpha$



Now $r_1 \circ \alpha$ is homotopic to Id_{S^k} because



at time $z_{k+1} = t$ we can collapse the lower blue-colored part of S^k.

$\Rightarrow r_1 \circ \alpha \simeq Id_{S^k} \Rightarrow (r_1 \circ \alpha)_*(1) = 1$

Similarly $r_2 \circ \alpha \simeq Id_{S^k} \Rightarrow (r_2 \circ \alpha)_*(1) = 1$, Hence we have the map

$$H_k(X_1) \xrightarrow{\alpha_*} H_k(X_2/X_2^2) \oplus H_k(X_2/X_2^1)$$

by $1 \longrightarrow (1,1)$

Now observe we have inclusions $X_2^1 \xhookrightarrow{i_1} X_2$, $X_2^2 \xhookrightarrow{i_2} X_2$ and an isomorphism

$$H_k(X_2^1) \oplus H_k(X_2^2) \longrightarrow H_k(X_2)$$

defined by $(a,b) \longrightarrow ((i_1)_*(a), (i_2)_*(b))$
$= (a,b)$

So we have

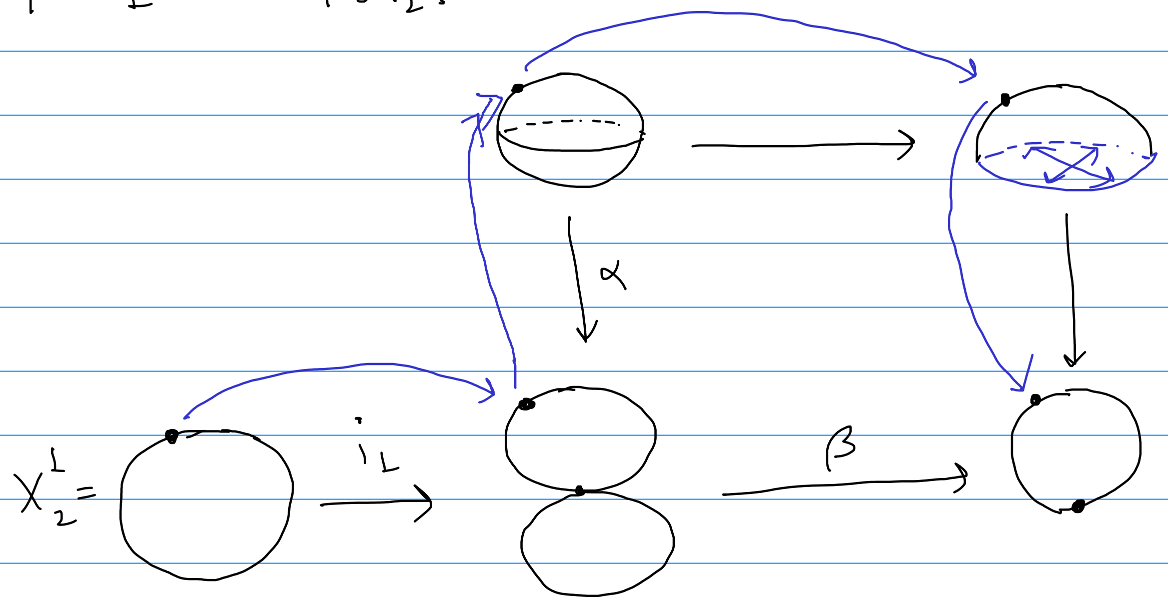
$$\begin{tikzcd}
H_k(X_2^1) \oplus H_k(X_2^2) \arrow[d, "\wr"] & (a,b) \arrow[dd] \\
H_k(X_2) \arrow[d, "\wr"] & \\
H_k(X_2/X_2^2) \oplus H_k(X_2/X_2^1) & (a,b)
\end{tikzcd}$$

because $r_1|_{X_2^1} = Id_{S^k}$ and $r_1|_{X_2^2}$ = constant map

and $r_2|_{X_2^1}$ = constant map and $r_2|_{X_2^2} = Id_{S^k}$

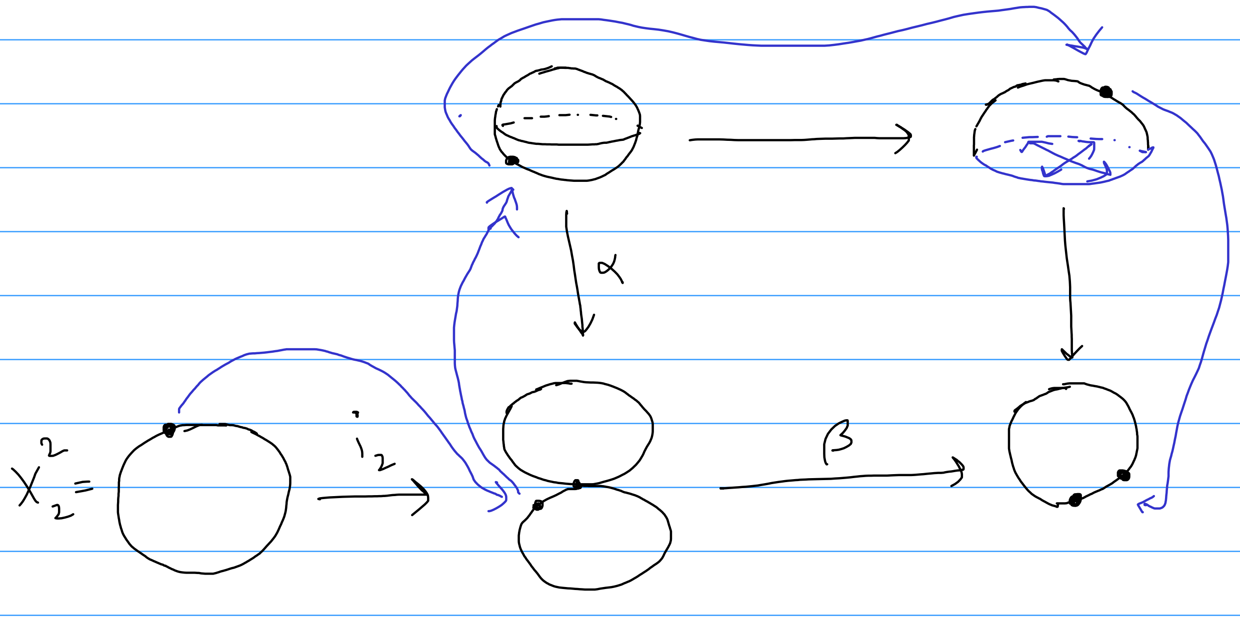
Claim 2:- $\beta \circ i_1 = Id$, $\beta \circ i_2 =$ Antipodal map on S^k.

Now to compute $\beta_*(1,1)$ it is enough to compute $(\beta\circ i_1)_*(1)$ and $(\beta \circ i_2)_*(1)$. Now we see the description of $\beta\circ i_1$ and $\beta\circ i_2$.



Hence by the description $\beta \circ i_1 = Id_{S^k}$

Now we see $\beta\circ i_2 =$ Antipodal map on S^k as below.



So we have $\beta \circ i_2 =$ Antipodal map on S^k.

Claim 3 :- $d_{k+1}(1) = 1 + (-1)^{k+1}$

Now we have the diagram

$$\begin{tikzcd}
& H_k(X_2^1) \oplus H_k(X_2^2) \arrow[d, phantom, sloped, "\cong"] & \\
H_k(X_1) \arrow[r, "\alpha"] & H_k(X_2) \arrow[r, "\beta"] \arrow[d, phantom, sloped, "\cong"] & H_k(X_3) \\
& H_k(X_2/X_2^2) \oplus H_k(X_2/X_2^1) &
\end{tikzcd}$$

Now $\beta_*(a,b) = (\beta \circ i_1)_*(a) + (\beta \circ i_2)_*(b)$ as we have the isomorphism $H_k(X_2^1) \oplus H_k(X_2^2) \simeq H_k(X_2)$

Hence $(\beta \circ \alpha)_*(1) = \beta_*(1,1) = (\beta \circ i_1)_*(1) + (\beta \circ i_2)_*(1)$

$$= 1 + (-1)^{k+1}$$

(By Question-11 on chapter 16 we have degree of Antipodal map on S^k is $(-1)^{k+1}$).

So we have proven our claim and so we found d_{k+1} $\forall k$.

So our chain complex becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "d_n"] \arrow[d, dash] & \mathbb{Z} \arrow[r, "d_{n-1}"] \arrow[d, dash] & \cdots \arrow[r, "d_2"] & \mathbb{Z} \arrow[r, "d_1"] \arrow[d, dash] & \mathbb{Z} \arrow[r] \arrow[d, dash] & 0 \\
& n & n-1 & & 1 & 0 &
\end{tikzcd}$$

where $d_{k+1}(1) = 1 + (-1)^{k+1}$ $\forall$ $k = 0, 1, \ldots, n-1$

i.e $d_{k+1}(1) = 0$ if k is even

$= 2$ if k is odd

So we have if $X = P_{\mathbb{R}}^{n}$

$$H_0^{cw}(X) = \frac{\ker d_0}{\operatorname{Im} d_1} = \mathbb{Z}/\{0\} \cong \mathbb{Z}$$

$$H_1^{cw}(X) = \frac{\ker d_1}{\operatorname{Im} d_2} = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$$

$$H_2^{cw}(X) = \frac{\ker d_2}{\operatorname{Im} d_3} = 0$$

$$H_3^{cw}(X) = \frac{\ker d_3}{\operatorname{Im} d_4} = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$$

$$\vdots$$

So we have different answer depending on n is even or odd.

Hence if n is even then

$$H_k^{cw}(X) = \mathbb{Z}_2 \text{ if } k \text{ is odd}$$
$$= 0 \text{ if } k \text{ is even } k \neq 0$$
$$= \mathbb{Z} \text{ if } k = 0$$

and when n is odd we have

$$d_n(1) = (-1)^n + 1 = 0$$

So $\ker d_n = \mathbb{Z}$ and $\operatorname{Im} d_{n+1} = 0$

$$\Rightarrow H_n^{cw}(X) = \mathbb{Z}/0 = \mathbb{Z}$$

Hence for n odd we have

$$H_k^{cw}(X) = \mathbb{Z}_2 \text{ if } k \text{ is odd, } k \neq n$$
$$= 0 \text{ if } k \text{ is even } k \neq 0$$
$$= \mathbb{Z} \text{ if } k = 0, n$$

Homology with Coefficient.

1. $X = \{p\}$ with Coefficient in $\mathbb{Z}_m$.

$\Rightarrow$ We have the Universal coefficient theorem we have the split exact sequence

$$0 \longrightarrow H_n(X)\otimes \mathbb{Z}_m \longrightarrow H_n(C_\bullet(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(H_{n-1}(X), \mathbb{Z}_m) \to 0$$

Now we have

$$\begin{aligned} H_i(X) &= \mathbb{Z} \text{ if } i=0 \\ &= 0 \text{ if otherwise} \end{aligned}$$

If $n=0$

$$0 \longrightarrow \overset{\mathbb{Z}}{\overset{\parallel}{H_0(X)}} \otimes \mathbb{Z}_m \longrightarrow H_0(C_\bullet(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(\cancelto{0}{\overset{0}{\overset{\parallel}{H_{-1}(X)}}}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H_0(C_\bullet(X)\otimes \mathbb{Z}_m) = \mathbb{Z}\otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n>0$

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\parallel}{H_n(X)}} \otimes \mathbb{Z}_m} \to H_n(C_\bullet(X)\otimes \mathbb{Z}_m) \longrightarrow \mathrm{Tor}^1(\cancelto{0}{H_{n-1}(X), \mathbb{Z}_m}) \to 0$$

$$\Rightarrow H_n(C_\bullet(X)\otimes \mathbb{Z}_m) = 0$$

So we have the homology of coefficients

$$\begin{aligned} H_n(C_\bullet(X)\otimes \mathbb{Z}_m) &= \mathbb{Z}_m \quad \text{if } n=0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

2. $X = S^1$ with coefficients in $\mathbb{Z}_m$.

⇒ We have the Universal Coefficient theorem we have the split exact sequence

$$0 \longrightarrow H_n(X)\otimes \mathbb{Z}_m \longrightarrow H_n(C_\cdot(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(H_{n-1}(X), \mathbb{Z}_m) \to 0$$

We have $H_i(X) = \mathbb{Z}$ $i=0,1$

$= 0$ otherwise

If $n=0$ then

$$0 \longrightarrow \overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(X)}} \otimes \mathbb{Z}_m \to H_0(C_\cdot(X)\otimes \mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{0}{\overset{\shortparallel}{H_{-1}(X)}}, \mathbb{Z})} \to 0$$

$$\Rightarrow H_0(C_\cdot(X)\otimes \mathbb{Z}_m) = \mathbb{Z}\otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n=1$ then

$$0 \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H_1(X)}} \otimes \mathbb{Z}_m \to H_1(C_\cdot(X)\otimes \mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(X)}}, \mathbb{Z})} \to 0$$

$$\Rightarrow H_1(C_\cdot(X)\otimes \mathbb{Z}_m) = \mathbb{Z}\otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n>1$ then

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\shortparallel}{H_n(X)}} \otimes \mathbb{Z}_m} \longrightarrow H_n(C_\cdot(X)\otimes \mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{0}{\overset{\shortparallel}{H_{n-1}(X)}}, \mathbb{Z}_m)} \to 0$$

$$\Rightarrow H_n(C_\cdot(X)\otimes \mathbb{Z}_m) = 0$$

So we have the homology with coefficient in $\mathbb{Z}_m$

$H_n(C_\cdot(X)\otimes \mathbb{Z}_m) = \mathbb{Z}_m$ $n=0,1$

$= 0$ otherwise

3. $X = S^k$ $(k>1)$ with coefficients in $\mathbb{Z}_m$.

$\Rightarrow$ We have the Universal Coefficient theorem we have the split exact sequence

$$0 \longrightarrow H_n(X) \otimes \mathbb{Z}_m \longrightarrow H_n(C_\cdot(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(H_{n-1}(X), \mathbb{Z}_m) \rightarrow 0$$

We have $H_i(X) = \mathbb{Z}$ $i = 0, k$

$= 0$ otherwise

If $n=0$

$$0 \rightarrow \overset{\mathbb{Z}}{\overset{\parallel}{H_0(X)}} \otimes \mathbb{Z}_m \rightarrow H_0(C_\cdot(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{\overset{0}{\overset{\parallel}{H_{-1}(X)}}}, \mathbb{Z}_m) \rightarrow 0$$

$\Rightarrow H_0(C_\cdot(X) \otimes \mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$

If $0 < n < k$

$$0 \rightarrow \cancelto{0}{\overset{0}{\overset{\parallel}{H_n(X)}} \otimes} \mathbb{Z}_m \rightarrow H_n(C_\cdot(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{H_{n-1}(X)}, \mathbb{Z}_m) \rightarrow 0$$

$\Rightarrow H_n(C_\cdot(X) \otimes \mathbb{Z}_m) = 0$

If $n=k$

$$0 \rightarrow \overset{\mathbb{Z}}{\overset{\parallel}{H_k(X)}} \otimes \mathbb{Z}_m \rightarrow H_k(C_\cdot(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{\overset{0}{\overset{\parallel}{H_{k-1}(X)}}}, \mathbb{Z}) \rightarrow 0$$

$\Rightarrow H_k(C_\cdot(X) \otimes \mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$

If $n>k$

$$0 \rightarrow \cancelto{0}{\overset{0}{\overset{\parallel}{H_n(X)}} \otimes} \mathbb{Z}_m \rightarrow H_n(C_\cdot(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{H_{n-1}(X)}, \mathbb{Z}) \rightarrow 0$$

$\Rightarrow H_n(C_\cdot(X) \otimes \mathbb{Z}_m) = 0$

So the homology with coefficient in $\mathbb{Z}_m$ is given by

$$H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathbb{Z}_m \quad n = 0, k$$
$$= 0 \quad \text{otherwise}$$

4. $X = M_k$ with coefficients in $\mathbb{Z}_m$.

⇒ We have the Universal Coefficient theorem we have the split exact sequence

$$0 \longrightarrow H_n(X)\otimes \mathbb{Z}_m \longrightarrow H_n(C_\bullet(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(H_{n-1}(X),\mathbb{Z}_m) \to 0$$

We have $H_i(X) = \mathbb{Z}$ $\quad i=0$

$$= \bigoplus_{i=1}^{2k} \mathbb{Z} \quad i=1$$

$= 0$ otherwise

If $n=0$

$$0 \longrightarrow \overset{\mathbb{Z}}{\overset{\parallel}{H_0(X)}} \otimes \mathbb{Z}_m \to H_0(C_\bullet(X)\otimes \mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{0}{\overset{\parallel}{H_{-1}}}(X),\mathbb{Z}_m)} \to 0$$

$$\Rightarrow H_0(C_\bullet(X)\otimes \mathbb{Z}_m) = \mathbb{Z}\otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n=1$

$$0 \longrightarrow \overset{\bigoplus_{i=1}^{2k}\mathbb{Z}}{\overset{\parallel}{H_1(X)}} \otimes \mathbb{Z}_m \to H_1(C_\bullet(X)\otimes \mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{\mathbb{Z}}{\overset{\parallel}{H_0}}(X),\mathbb{Z}_m)} \to 0$$

$$\Rightarrow H_1(C_\bullet(X)\otimes \mathbb{Z}_m) = \left(\bigoplus_{i=1}^{2k}\mathbb{Z}\right)\otimes \mathbb{Z}_m = \bigoplus_{i=1}^{2k}\mathbb{Z}\otimes \mathbb{Z}_m = \bigoplus_{i=1}^{2k}\mathbb{Z}_m$$

If $n>1$

$$0 \to \cancelto{0}{\overset{0}{\overset{\parallel}{H_n}}(X)\otimes \mathbb{Z}_m} \to H_n(C_\bullet(X)\otimes \mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(H_{n-1}(X),\mathbb{Z}_m)} \to 0$$

$$\Rightarrow H_n(C_\bullet(X)\otimes \mathbb{Z}_m) = 0$$

So the homology of coefficients in $\mathbb{Z}_m$ is given by

$$H_n(C_\circ(X) \otimes \mathbb{Z}_m) = \mathbb{Z}_m \quad n=0$$

$$= \bigoplus_{i=1}^{2K} \mathbb{Z}_m \quad n=1$$

$$= 0 \quad \text{otherwise}$$

5. $X = \mathbb{P}^k_{\mathbb{C}}$ with coefficients in $\mathbb{Z}_m$.

$\Rightarrow$ We have the Universal coefficient theorem we have the split exact sequence

$$0 \longrightarrow H_n(X) \otimes \mathbb{Z}_m \longrightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(H_{n-1}(X), \mathbb{Z}_m) \rightarrow 0$$

We have $H_i(X) = \mathbb{Z}$ $\quad 0 \leq i \leq 2k$ $\quad i$ is even

$= 0$ otherwise

If $n = 0$

$$0 \rightarrow \overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(X)}} \otimes \mathbb{Z}_m \longrightarrow H_0(C_\bullet(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{\overset{0}{\overset{\shortparallel}{H_{-1}}}(X), \mathbb{Z}_m}) \rightarrow 0$$

$$\Rightarrow H_0(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n = 1$

$$0 \rightarrow \cancelto{0}{\overset{0}{\overset{\shortparallel}{H_1(X)}} \otimes} \mathbb{Z}_m \rightarrow H_1(C_\bullet(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{\overset{\mathbb{Z}}{\overset{\shortparallel}{H_0}}(X), \mathbb{Z}_m}) \rightarrow 0$$

$$\Rightarrow H_1(C_\bullet(X) \otimes \mathbb{Z}_m) = 0$$

If $1 < n \leq 2k$ and n is even

$$0 \rightarrow \overset{\mathbb{Z}}{\overset{\shortparallel}{H_n(X)}} \otimes \mathbb{Z}_m \rightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{\overset{0}{\overset{\shortparallel}{H_{n-1}}}(X), \mathbb{Z}_m}) \rightarrow 0$$

$$\Rightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $1 < n < 2k$ and n is odd

$$0 \rightarrow \cancelto{0}{\overset{0}{\overset{\shortparallel}{H_n(X)}} \otimes} \mathbb{Z}_m \rightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) \rightarrow \mathrm{Tor}^1(\cancelto{0}{\overset{\mathbb{Z}}{\overset{\shortparallel}{H_{n-1}}}(X), \mathbb{Z}_m}) \rightarrow 0$$

$\Rightarrow H_n(C_\bullet(X)\otimes\mathbb{Z}_m) = 0$

If $n > 2k$ then

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\shortparallel}{H_n(X)}}\otimes\mathbb{Z}_m} \longrightarrow H_n(C_\bullet(X)\otimes\mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(H_{n-1}(X),\mathbb{Z}_m)} \to 0$$

$\Rightarrow H_n(C_\bullet(X)\otimes\mathbb{Z}_m) = 0$

So we have the homology with coefficients in $\mathbb{Z}_m$

$$\begin{aligned} H_n(C_\bullet(X)\otimes\mathbb{Z}_m) &= \mathbb{Z}_m \qquad 0\le n\le 2k \quad n \text{ is even} \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

6. $X = \mathbb{P}^k_{\mathbb{R}}$ with coefficient in $\mathbb{Z}_m$.

$\Rightarrow$ We have the Universal Coefficient theorem we have the split exact sequence

$$0 \longrightarrow H_n(X)\otimes \mathbb{Z}_m \longrightarrow H_n(C_\bullet(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(H_{n-1}(X),\mathbb{Z}_m) \to 0$$

Now if k is even then

$$H_i(X) = \mathbb{Z}_2 \quad i \text{ is odd}$$
$$= 0 \quad i \text{ is even } i \neq 0$$
$$= \mathbb{Z} \quad i = 0$$

if k is odd then

$$H_i(X) = \mathbb{Z}_2 \quad i \text{ is odd } i \neq n$$
$$= 0 \quad i \text{ is even } i \neq 0$$
$$= \mathbb{Z} \quad i = 0, k$$

let k be even

If $n=0$

$$0 \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(X)}}\otimes \mathbb{Z}_m \longrightarrow H_0(C_\bullet(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1\cancelto{0}{(\overset{0}{\overset{|}{H_{-1}(X)}}, \mathbb{Z}_m)} \to 0$$

$\Rightarrow H_0(C_\bullet(X)\otimes \mathbb{Z}_m) = \mathbb{Z}\otimes\mathbb{Z}_m = \mathbb{Z}_m$

If $n=1$

$$0 \to \overset{\mathbb{Z}_2}{\overset{\shortparallel}{H_1(X)}}\otimes \mathbb{Z}_m \longrightarrow H_1(C_\bullet(X)\otimes \mathbb{Z}_m) \to \mathrm{Tor}^1\cancelto{0}{(\overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(X)}}, \mathbb{Z}_m)} \to 0$$

$\Rightarrow H_1(C_\bullet(X)\otimes \mathbb{Z}_m) = \mathbb{Z}_2\otimes\mathbb{Z}_m = \mathbb{Z}_{\gcd(2,m)}$

If $n = 2$

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\|}{H_2(X)}} \otimes \mathbb{Z}_m} \longrightarrow H_2(C_\bullet(X) \otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(\overset{\mathbb{Z}_2}{\overset{\|}{H_1(X)}}, \mathbb{Z}_m) \longrightarrow 0$$

$$\Rightarrow H_2(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathrm{Tor}^1(\mathbb{Z}_2, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(2,m)}$$

If $1 < n \leq k$ is even

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\|}{H_n(X)}} \otimes \mathbb{Z}_m} \longrightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) \longrightarrow \mathrm{Tor}^1(\overset{\mathbb{Z}_2}{\overset{\|}{H_{n-1}(X)}}, \mathbb{Z}_m) \longrightarrow 0$$

$$\Rightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathrm{Tor}^1(\mathbb{Z}_2, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(2,m)}$$

If $1 < n < k$ is odd

$$0 \longrightarrow \overset{\mathbb{Z}_2}{\overset{\|}{H_n(X)}} \otimes \mathbb{Z}_m \longrightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) \longrightarrow \mathrm{Tor}^1\cancelto{0}{(\overset{0}{\overset{\|}{H_{n-1}(X)}}, \mathbb{Z}_m)} \longrightarrow 0$$

$$\Rightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathbb{Z}_2 \otimes \mathbb{Z}_m = \mathbb{Z}_{\gcd(2,m)}$$

If $n > k$

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\|}{H_n(X)}} \otimes \mathbb{Z}_m} \longrightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) \to \mathrm{Tor}^1(\cancelto{0}{H_{n-1}(X)}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = 0$$

So the homology in coefficient $\mathbb{Z}_m$ is given by (where k is even)

$$H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathbb{Z}_m \qquad n=0$$
$$= \mathbb{Z}_{gcd(2,m)} \qquad 1 \leq n \leq k$$
$$= 0 \qquad \text{otherwise}$$

Similarly if k is odd then using the same results we have

$$H_n(C_\bullet(X) \otimes \mathbb{Z}_m) = \mathbb{Z} \qquad n=0,k$$
$$= \mathbb{Z}_{gcd(2,m)} \qquad 1 \leq n < k$$
$$= 0 \qquad \text{otherwise}.$$

Cohomology.

1. Cohomology of $X = \{p\}$ with
(a) Coefficients in $\mathbb{Z}$.
(b) Coefficients in $\mathbb{Z}_m$.

$\Rightarrow$ (a) We have the universal coefficient theorem which says $\forall n \geq 0$ we have the split exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(X),\mathbb{Z}) \longrightarrow 0$$

Now we have $H_i(X) = \mathbb{Z}$ if $i=0$
$= 0$ if $i>0$

So if $n=0$ we have

$$0 \longrightarrow \cancel{\mathrm{Ext}^1(\overset{0}{\overset{\|}{H_{-1}}}(X), \mathbb{Z})} \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow \underset{\mathbb{Z}}{\underset{\|}{\mathrm{Hom}(\overset{\mathbb{Z}}{\overset{\|}{H_0}}(X), \mathbb{Z})}} \longrightarrow 0$$

$\Rightarrow H^0(X,\mathbb{Z}) = \mathbb{Z}$

If $n>0$ then we have $\mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) = 0$
$\mathrm{Hom}(H_n(X), \mathbb{Z}) = 0$

Hence $H^n(X,\mathbb{Z}) = 0$

So the Cohomology of X is

$$H^n(X,\mathbb{Z}) = \mathbb{Z} \quad n=0$$
$$= 0 \quad n>0$$

(b) We have the split exact sequence $\forall n \geq 0$

$$0 \to H^n(X;\mathbb{Z}) \otimes \mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(H^{n+1}(X,\mathbb{Z}),\mathbb{Z}_m) \to 0$$

If $n=0$ we have

$$0 \to \overset{\mathbb{Z}}{\overset{\|}{H^0(X,\mathbb{Z})}} \otimes \mathbb{Z}_m \to H^0(X,\mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{0}{\overset{\|}{H^1(X,\mathbb{Z})}},\mathbb{Z}_m)} \to 0$$

$$\Rightarrow \quad H^0(X,\mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m \simeq \mathbb{Z}_m$$

If $n>0$ we have

$$0 \to \cancelto{0}{\overset{0}{\overset{\|}{H^n(X,\mathbb{Z})}} \otimes \mathbb{Z}_m} \to H^n(X,\mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}(\overset{0}{\overset{\|}{H^{n+1}(X,\mathbb{Z})}},\mathbb{Z}_m)} \to 0$$

$$\Rightarrow \quad H^n(X,\mathbb{Z}_m) = 0$$

Hence we have the cohomology with coefficients in $\mathbb{Z}_m$

$$\begin{aligned} H^n(X,\mathbb{Z}_m) &= \mathbb{Z}_m \quad n=0 \\ &= 0 \quad n>0 \end{aligned}$$

2. Cohomology of S^1 with

(a) Coefficients in $\mathbb{Z}$

(b) Coefficients in $\mathbb{Z}_m$

$\Rightarrow$ (a) We have the universal coefficient theorem which says $\forall n \geq 0$ we have the split exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(X),\mathbb{Z}) \longrightarrow 0$$

Now we have

$$H_i(X) = \mathbb{Z} \quad \text{if } i=0,1$$
$$= 0 \quad \text{if otherwise}$$

If $n=0$ then

$$0 \to \cancelto{0}{\mathrm{Ext}^1(\overset{0}{\overset{\shortparallel}{H_{-1}(X)}}, \mathbb{Z})} \to H^0(X,\mathbb{Z}) \to \underset{\mathbb{Z}}{\underset{\shortparallel}{\overset{\mathbb{Z}}{\overset{\shortparallel}{\mathrm{Hom}(H_0(X), \mathbb{Z})}}}} \to 0$$

$\Rightarrow \quad H^0(X,\mathbb{Z}) = \mathbb{Z}$

If $n=1$ then

$$0 \to \cancelto{0}{\mathrm{Ext}^1(\overset{\mathbb{Z}}{\overset{\shortparallel}{H_0(X)}}, \mathbb{Z})} \to H^1(X,\mathbb{Z}) \longrightarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{\overset{\mathbb{Z}}{\overset{\shortparallel}{\mathrm{Hom}(H_1(X), \mathbb{Z})}}}} \longrightarrow 0$$

$\Rightarrow \quad H^1(X,\mathbb{Z}) = \mathbb{Z}$

If $n > 1$ then $\mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) = 0$

$\mathrm{Hom}(H_n(X), \mathbb{Z}) = 0$

$\Rightarrow \quad H^n(X,\mathbb{Z}) = 0$

So the cohomology of X is

$$H^n(X,\mathbb{Z}) = \mathbb{Z} \quad \text{if } n=0,1$$
$$= 0 \quad \text{otherwise}$$

(b) We have the split exact sequence $\forall n \geq 0$

$$0 \to H^n(X;\mathbb{Z})\otimes\mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(H^{n+1}(X,\mathbb{Z}),\mathbb{Z}_m) \to 0$$

If $n=0$

$$0 \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H^0(X,\mathbb{Z})}}\otimes\mathbb{Z}_m \to H^0(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\overset{\mathbb{Z}}{\overset{\shortparallel}{\cancelto{0}{H^1(X,\mathbb{Z}),\mathbb{Z}_m}}}) \to 0$$

$$\Rightarrow H^0(X,\mathbb{Z}_m) = \mathbb{Z}\otimes\mathbb{Z}_m = \mathbb{Z}_m$$

If $n=1$

$$0 \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H^1(X,\mathbb{Z})}}\otimes\mathbb{Z}_m \to H^1(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\overset{0}{\overset{\shortparallel}{\cancelto{0}{H^2(X,\mathbb{Z}),\mathbb{Z}_m}}}) \to 0$$

$$\Rightarrow H^1(X,\mathbb{Z}_m) = \mathbb{Z}\otimes\mathbb{Z}_m = \mathbb{Z}_m$$

If $n>1$ then

$$0 \to \cancelto{0}{H^n(X,\mathbb{Z})}\otimes\mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\cancelto{0}{H^{n+1}(X,\mathbb{Z})},\mathbb{Z}_m) \to 0$$

$$\Rightarrow H^n(X,\mathbb{Z}_m) = 0$$

So we have Cohomology with coefficients in $\mathbb{Z}_m$

$$H^n(X, \mathbb{Z}_m) = \mathbb{Z}_m \quad n = 0, 1$$
$$= 0 \quad \text{otherwise}$$

3. Cohomology of $X = S^k$ $(k > 1)$ with

(a) Coefficients in $\mathbb{Z}$

(b) Coefficients in $\mathbb{Z}_m$

$\Rightarrow$ (a) We have the universal coefficient theorem which says $\forall n \geq 0$ we have the split exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \mathrm{Hom}(H_n(X), \mathbb{Z}) \longrightarrow 0$$

Now we have

$$H_i(X) = \mathbb{Z} \quad i = 0, k$$
$$= 0 \quad \text{otherwise}$$

If $n = 0$

$$0 \to \cancelto{0}{\mathrm{Ext}^1(H_{-1}(\overset{\overset{0}{\shortparallel}}{X}), \mathbb{Z})} \to H^0(X, \mathbb{Z}) \to \underset{\underset{\mathbb{Z}}{\shortparallel}}{\mathrm{Hom}}(\overset{\overset{\mathbb{Z}}{\shortparallel}}{H_0(X)}, \mathbb{Z}) \to 0$$

$$\Rightarrow H^0(X, \mathbb{Z}) = \mathbb{Z}.$$

if $0 < n < k$

$$0 \longrightarrow \cancelto{0}{\mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z})} \to H^n(X, \mathbb{Z}) \to \cancelto{0}{\mathrm{Hom}(\overset{\overset{0}{\shortparallel}}{H_n(X)}, \mathbb{Z})} \to 0$$

$$\Rightarrow H^n(X, \mathbb{Z}) = 0$$

if $n = k$

$$0 \to \cancelto{0}{\mathrm{Ext}^1(\overset{\overset{0}{\shortparallel}}{H_{k-1}(X)}, \mathbb{Z})} \to H^k(X, \mathbb{Z}) \to \underset{\underset{\mathbb{Z}}{\shortparallel}}{\mathrm{Hom}}(\overset{\overset{\mathbb{Z}}{\shortparallel}}{H_k(X)}, \mathbb{Z}) \to 0$$

$\Rightarrow \quad H^k(X,\mathbb{Z}) = \mathbb{Z}$

If $n>k$

$$0 \longrightarrow \operatorname{Ext}^1(\cancelto{0}{H_{n-1}(X)},\mathbb{Z}) \to H^n(X,\mathbb{Z}) \to \operatorname{Hom}(\cancelto{0}{\overset{\overset{0}{\shortparallel}}{H_n(X)}},\mathbb{Z}) \to 0$$

$\Rightarrow \quad H^n(X,\mathbb{Z}) = 0$

So the Cohomology of X is

$$H^n(X,\mathbb{Z}) = \mathbb{Z} \qquad n=0,k$$
$$= 0 \qquad \text{otherwise}$$

(b) We have the split exact sequence $\forall n \geq 0$

$$0 \to H^n(X,\mathbb{Z}) \otimes \mathbb{Z}_m \longrightarrow H^n(X,\mathbb{Z}_m) \to \operatorname{Tor}^1(H^{n+1}(X,\mathbb{Z}),\mathbb{Z}_m) \hookrightarrow 0$$

If $n=0$

$$0 \longrightarrow \overset{\overset{\mathbb{Z}}{\shortparallel}}{H^0(X,\mathbb{Z})} \otimes \mathbb{Z}_m \to H^0(X,\mathbb{Z}_m) \to \operatorname{Tor}^1(\cancelto{0}{\overset{\overset{0}{\shortparallel}}{H^1(X,\mathbb{Z})}},\mathbb{Z}_m) \to 0$$

$\Rightarrow \quad H^0(X,\mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$

If $0<n<k$

$$0 \to \cancelto{0}{\overset{\overset{0}{\shortparallel}}{H^n(X,\mathbb{Z})}} \otimes \mathbb{Z}_m \longrightarrow H^n(X,\mathbb{Z}_m) \to \operatorname{Tor}^1(\cancelto{0}{H^{n+1}(X,\mathbb{Z})},\mathbb{Z}_m) \to 0$$

$\Rightarrow \quad H^n(X,\mathbb{Z}_m) = 0$

If $n=k$

$$0 \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H^k(X,\mathbb{Z})}} \otimes \mathbb{Z}_m \to H^k(X,\mathbb{Z}_m) \to \mathrm{Tor}^1\big(\cancelto{0}{\overset{0}{\overset{\shortparallel}{H^{k+1}(X,\mathbb{Z})}}}, \mathbb{Z}_m\big) \to 0$$

$$\Rightarrow \quad H^k(X,\mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n>k$

$$0 \to \cancelto{0}{\overset{0}{\overset{\shortparallel}{H^n(X,\mathbb{Z})}}} \otimes \mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1\big(\cancelto{0}{\overset{0}{\overset{\shortparallel}{H^{n+1}(X,\mathbb{Z})}}}, \mathbb{Z}_m\big) \to 0$$

$$\Rightarrow \quad H^n(X,\mathbb{Z}_m) = 0$$

So we have the Cohomology of X with coefficients in $\mathbb{Z}_m$.

$$H^n(X,\mathbb{Z}_m) = \mathbb{Z}_m \qquad n=0,k$$

$$= 0 \qquad \text{otherwise}$$

4. $X = M_k$

(a) Coefficient in $\mathbb{Z}$

(b) Coefficient in $\mathbb{Z}_m$

$\Rightarrow$ (a) We have the universal coefficient theorem which says $\forall n \geq 0$ we have the split exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \longrightarrow H^n(X, \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(X), \mathbb{Z}) \longrightarrow 0$$

Now we have

$$\begin{aligned} H_i(X) &= \mathbb{Z} & i=0 \\ &= \bigoplus_{i=1}^{2k} \mathbb{Z} & i=1 \\ &= 0 & \text{otherwise} \end{aligned}$$

If $n = 0$

$$0 \to \cancelto{0}{\mathrm{Ext}^1(\overset{0}{\overset{\|}{H_{-1}}}(X), \mathbb{Z})} \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow \underset{\overset{\|}{\mathbb{Z}}}{\mathrm{Hom}(\overset{\mathbb{Z}}{\overset{\|}{H_0}}(X), \mathbb{Z})} \to 0$$

$\Rightarrow H^0(X, \mathbb{Z}) = \mathbb{Z}$

If $n = 1$

$$0 \to \cancelto{0}{\mathrm{Ext}^1(\overset{\mathbb{Z}}{\overset{\|}{H_0}}(X), \mathbb{Z})} \longrightarrow H^1(X, \mathbb{Z}) \to \mathrm{Hom}(\overset{\bigoplus_{i=1}^{2k} \mathbb{Z}}{\overset{\|}{H_1}}(X), \mathbb{Z}) \longrightarrow 0$$

$$\Rightarrow H^1(X, \mathbb{Z}) = \mathrm{Hom}\left(\bigoplus_{i=1}^{2k} \mathbb{Z}, \mathbb{Z}\right) = \bigoplus_{i=1}^{2k} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \bigoplus_{i=1}^{2k} \mathbb{Z}$$

If $n > 1$

$$0 \to \cancelto{0}{\mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z})} \to H^n(X, \mathbb{Z}) \to \cancelto{0}{\mathrm{Hom}(\overset{0}{\overset{\|}{H_n}}(X), \mathbb{Z})} \longrightarrow 0$$

$$\Rightarrow H^n(X,\mathbb{Z}) = 0$$

So the Cohomology is given by

$$\begin{aligned} H^n(X,\mathbb{Z}) &= \mathbb{Z} & n=0 \\ &= \bigoplus_{i=1}^{2k} \mathbb{Z} & n=1 \\ &= 0 & \text{otherwise} \end{aligned}$$

(b) We have the split exact sequence $\forall n \geq 0$

$$0 \to H^n(X,\mathbb{Z}) \otimes \mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(H^{n+1}(X,\mathbb{Z}),\mathbb{Z}_m) \to 0$$

If $n=0$

$$0 \to \overset{\overset{\mathbb{Z}}{\|}}{H^0(X,\mathbb{Z})} \otimes \mathbb{Z}_m \to H^0(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\overset{\overset{\bigoplus_{i=1}^{2k}\mathbb{Z}}{\|}}{\cancelto{0}{H^1(X,\mathbb{Z})}}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H^0(X,\mathbb{Z}) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $n=1$

$$0 \to \overset{\overset{\bigoplus_{i=1}^{2k}\mathbb{Z}}{\|}}{H^1(X,\mathbb{Z})} \otimes \mathbb{Z}_m \to H^1(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\overset{\overset{0}{\|}}{\cancelto{0}{H^2(X,\mathbb{Z})}}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H^1(X,\mathbb{Z}_m) = \left(\bigoplus_{i=1}^{2k} \mathbb{Z}\right) \otimes \mathbb{Z}_m = \bigoplus_{i=1}^{2k} (\mathbb{Z} \otimes \mathbb{Z}_m) = \bigoplus_{i=1}^{2k} \mathbb{Z}_m$$

If $n>1$

$$0 \to \cancelto{0}{H^n(X,\mathbb{Z})} \otimes \mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\cancelto{0}{H^{n+1}(X,\mathbb{Z})}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H^n(X,\mathbb{Z}_m) = 0$$

So the Cohomology with coefficients in $\mathbb{Z}_m$ is

$$H^n(X, \mathbb{Z}_m) = \mathbb{Z}_m \qquad n=0$$

$$= \bigoplus_{i=1}^{2k} \mathbb{Z}_m \qquad n=1$$

$$= 0 \qquad \text{otherwise}$$

5. $X = \mathbb{P}^k_{\mathbb{C}}$ with

(a) Coefficient in $\mathbb{Z}$

(b) Coefficient in $\mathbb{Z}_m$

$\Rightarrow$ (a) We have the universal coefficient theorem which says $\forall n \geq 0$ we have the split exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(H_{n-1}(X),\mathbb{Z}) \to H^n(X,\mathbb{Z}) \to \operatorname{Hom}(H_n(X),\mathbb{Z}) \longrightarrow 0$$

Now $H_i(X) = \mathbb{Z}$ $\quad 0 \leq i \leq 2k \quad i$ is even

$\qquad\qquad = 0$ otherwise

If $n = 0$

$$0 \longrightarrow \cancelto{0}{\operatorname{Ext}^1(\overset{\overset{0}{\shortparallel}}{H_{-1}}(X),\mathbb{Z})} \to H^0(X,\mathbb{Z}) \to \underset{\underset{\mathbb{Z}}{\shortparallel}}{\operatorname{Hom}(\overset{\overset{\mathbb{Z}}{\shortparallel}}{H_0}(X),\mathbb{Z})} \longrightarrow 0$$

$\Rightarrow \quad H^0(X,\mathbb{Z}) = \mathbb{Z}$

If $n = 1$

$$0 \longrightarrow \cancelto{0}{\operatorname{Ext}^1(\overset{\overset{\mathbb{Z}}{\shortparallel}}{H_0}(X),\mathbb{Z})} \to H^1(X,\mathbb{Z}) \to \cancelto{0}{\operatorname{Hom}(\overset{\overset{0}{\shortparallel}}{H_1}(X),\mathbb{Z})} \longrightarrow 0$$

$\Rightarrow \quad H^1(X,\mathbb{Z}) = 0$

If $1 < n \leq 2k$ and n is even

$$0 \longrightarrow \cancelto{0}{\operatorname{Ext}^1(\overset{\overset{0}{\shortparallel}}{H_{n-1}}(X),\mathbb{Z})} \to H^n(X,\mathbb{Z}) \to \underset{\underset{\mathbb{Z}}{\shortparallel}}{\operatorname{Hom}(\overset{\overset{\mathbb{Z}}{\shortparallel}}{H_n}(X),\mathbb{Z})} \longrightarrow 0$$

$\Rightarrow \quad H^n(X,\mathbb{Z}) = \mathbb{Z}$

If $1 < n < 2k$ and n is odd then

$$0 \longrightarrow \cancelto{0}{\mathrm{Ext}^1(\overset{\mathbb{Z}}{\overset{\|}{H_{n-1}(X)}}, \mathbb{Z})} \to H^n(X,\mathbb{Z}) \longrightarrow \cancelto{0}{\mathrm{Hom}(\overset{0}{\overset{\|}{H_n(X)}}, \mathbb{Z})} \longrightarrow 0$$

$\Rightarrow H^n(X,\mathbb{Z}) = 0$

So the Cohomology of X is given by

$$\begin{aligned} H^n(X,\mathbb{Z}) &= \mathbb{Z} \qquad 0 \leq n \leq 2k \quad n \text{ is even} \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

(b) We have the split exact sequence $\forall\, n \geq 0$

$$0 \to H^n(X;\mathbb{Z}) \otimes \mathbb{Z}_m \longrightarrow H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(H^{n+1}(X,\mathbb{Z}), \mathbb{Z}_m) \to 0$$

If $n=0$

$$0 \longrightarrow \overset{\mathbb{Z}}{\overset{\|}{H^0(X,\mathbb{Z})}} \otimes \mathbb{Z}_m \to H^0(X,\mathbb{Z}_m) \to \cancelto{0}{\mathrm{Tor}^1(\overset{0}{\overset{\|}{H^1(X,\mathbb{Z})}}, \mathbb{Z}_m)} \to 0$$

$\Rightarrow H^0(X,\mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$

If $n=1$

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\|}{H^1(X,\mathbb{Z})}} \otimes \mathbb{Z}_m} \to H^1(X,\mathbb{Z}_m) \longrightarrow \cancelto{0}{\mathrm{Tor}^1(\overset{\mathbb{Z}}{\overset{\|}{H^2(X,\mathbb{Z})}}, \mathbb{Z}_m)} \to 0$$

$\Rightarrow H^1(X,\mathbb{Z}_m) = 0$

If $1 < n \leq 2k$ is even

$$0 \longrightarrow \overset{\mathbb{Z}}{\overset{\parallel}{H^n(X,\mathbb{Z})}} \otimes \mathbb{Z}_m \longrightarrow H^n(X,\mathbb{Z}_m) \to \operatorname{Tor}^1\big(\cancelto{0}{\overset{0}{\overset{\parallel}{H^{n+1}(X,\mathbb{Z})}}},\mathbb{Z}_m\big) \to 0$$

$$\Rightarrow \quad H^n(X,\mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$$

If $1 < n < 2k$ and n is odd.

$$0 \longrightarrow \cancelto{0}{\overset{0}{\overset{\parallel}{H^n(X,\mathbb{Z})}} \otimes \mathbb{Z}_m} \longrightarrow H^n(X,\mathbb{Z}_m) \to \operatorname{Tor}^1\big(\cancelto{0}{H^{n+1}(X,\mathbb{Z}),\mathbb{Z}_m}\big) \to 0$$

$$\Rightarrow \quad H^n(X,\mathbb{Z}_m) = 0$$

So the cohomology with coefficient in $\mathbb{Z}_m$ is

$$H^n(X,\mathbb{Z}_m) = \mathbb{Z}_m \qquad 0 \leq n \leq 2k,\ n \text{ is even}$$

$$= 0 \qquad \text{otherwise}$$

6. Cohomology of $P_{\mathbb{R}}^{k}$ with

(a) Coefficient in $\mathbb{Z}$

(b) Coefficient in $\mathbb{Z}_m$.

$\Rightarrow$ (a) We have the universal coefficient theorem which says $\forall n \geq 0$ we have the split exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z}) \rightarrow H^n(X, \mathbb{Z}) \rightarrow \mathrm{Hom}(H_n(X), \mathbb{Z}) \longrightarrow 0$$

Now if k is even then

$$\begin{aligned} H_i(X) &= \mathbb{Z}_2 && i \text{ is odd} \\ &= 0 && i \text{ is even } i \neq 0 \\ &= \mathbb{Z} && i = 0 \end{aligned}$$

if k is odd then

$$\begin{aligned} H_i(X) &= \mathbb{Z}_2 && i \text{ is odd } i \neq n \\ &= 0 && i \text{ is even } i \neq 0 \\ &= \mathbb{Z} && i = 0, k \end{aligned}$$

let k be even.

If $n = 0$

$$0 \rightarrow \cancelto{0}{\mathrm{Ext}^1(\overset{\overset{0}{\parallel}}{H_{-1}(X)}, \mathbb{Z})} \rightarrow H^0(X, \mathbb{Z}) \rightarrow \underset{\underset{\mathbb{Z}}{\parallel}}{\mathrm{Hom}(\overset{\overset{\mathbb{Z}}{\parallel}}{H_0(X)}, \mathbb{Z})} \longrightarrow 0$$

$\Rightarrow$ $H^0(X, \mathbb{Z}) = \mathbb{Z}$

If $n = 1$

$$0 \longrightarrow \cancelto{0}{\mathrm{Ext}^1(\overset{\overset{\mathbb{Z}}{\parallel}}{H_0(X)}, \mathbb{Z})} \rightarrow H^1(X, \mathbb{Z}) \rightarrow \cancelto{0}{\mathrm{Hom}(\overset{\overset{\mathbb{Z}_2}{\parallel}}{H_1(X)}, \mathbb{Z})} \longrightarrow 0$$

$\Rightarrow$ $H^1(X, \mathbb{Z}) = 0$

If $n=2$

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ext}^1(\overset{\mathbb{Z}_2}{\overset{\|}{H_1}}(X), \mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(X,\mathbb{Z}) \arrow[r] & \cancelto{0}{\operatorname{Hom}(\overset{0}{\overset{\|}{H_2}}(X),\mathbb{Z})} \arrow[r] & 0 \\ & \mathbb{Z}_2 & & & \end{tikzcd}$$

$\Rightarrow \quad H^r(X,\mathbb{Z}) = \mathbb{Z}_2$

[We have $\operatorname{Ext}^1(\mathbb{Z}/_{m\mathbb{Z}}, \mathbb{Z}) = \mathbb{Z}/_{m\mathbb{Z}}$

$\Rightarrow$ we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/_{m\mathbb{Z}} \arrow[r] & 0 \\ & x \arrow[r, mapsto] & mx & & \\ & & a \arrow[r, mapsto] & a+m\mathbb{Z} & \end{tikzcd}$$

$$\begin{tikzcd} \Rightarrow \; 0 \arrow[r] & \cancelto{0}{\operatorname{Hom}(\mathbb{Z}/_{m\mathbb{Z}}, \mathbb{Z})} \arrow[r] & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[d] \arrow[d, equal, shift left=8] \\ & & \mathbb{Z} \; x \arrow[r, "f"] & mx \; \mathbb{Z} \\ & & & \operatorname{Ext}^1(\mathbb{Z}/_{m\mathbb{Z}}, \mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow \operatorname{Ext}^1(\mathbb{Z}/_{m\mathbb{Z}}, \mathbb{Z}) = \mathbb{Z}/_{\operatorname{Im} f} = \mathbb{Z}/_{m\mathbb{Z}}$]

Now if n is odd $n<k$

$$0 \to \cancelto{0}{\operatorname{Ext}^1(\overset{0}{\overset{\|}{H_{n-1}}}(X), \mathbb{Z})} \to H^n(X,\mathbb{Z}) \to \cancelto{0}{\operatorname{Hom}(\overset{\mathbb{Z}_2}{\overset{\|}{H_n}}(X), \mathbb{Z})} \to 0$$

$\Rightarrow H^n(X,\mathbb{Z}) = 0$

If n is even $n \leq k$

$$0 \to \mathrm{Ext}^1(\overset{\mathbb{Z}_2}{\overset{\shortparallel}{H_{n-1}(X)}}, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \cancelto{0}{\mathrm{Hom}(\overset{0}{\overset{\shortparallel}{H_n(X)}}, \mathbb{Z})} \to 0$$

$$\Rightarrow H^n(X, \mathbb{Z}) = \mathbb{Z}_2$$

If $n > k$ then

$$0 \to \cancelto{0}{\mathrm{Ext}^1(H_{n-1}(X), \mathbb{Z})} \to H^n(X, \mathbb{Z}) \to \cancelto{0}{\mathrm{Hom}(\overset{0}{\overset{\shortparallel}{H_n(X)}}, \mathbb{Z})} \to 0$$

$$\Rightarrow H^n(X, \mathbb{Z}) = 0$$

So if k is even Cohomology is given by

$$\begin{aligned} H^n(X, \mathbb{Z}) &= \mathbb{Z}_2 && n \text{ is even } 0 < n \leq k \\ &= \mathbb{Z} && n = 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

Similarly if k is odd using the same results we have

$$\begin{aligned} H^n(X, \mathbb{Z}) &= \mathbb{Z}_2 && 0 < n < k \quad n \text{ is even} \\ &= \mathbb{Z} && n = 0, k \\ &= 0 && \text{otherwise} \end{aligned}$$

(b) We have the split exact sequence $\forall n \geq 0$

$$0 \to H^n(X;\mathbb{Z}) \otimes \mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to Tor^1(H^{n+1}(X,\mathbb{Z}),\mathbb{Z}_m) \to 0$$

let k be even

If $n=0$

$$0 \to \overset{\mathbb{Z}}{\overset{\shortparallel}{H^0(X,\mathbb{Z})}} \otimes \mathbb{Z}_m \to H^0(X,\mathbb{Z}_m) \to Tor^1(\cancelto{0}{\overset{0}{\overset{\shortparallel}{H^1}}(X,\mathbb{Z})},\mathbb{Z}_m) \to 0$$

$\Rightarrow \quad H^0(X,\mathbb{Z}_m) = \mathbb{Z} \otimes \mathbb{Z}_m = \mathbb{Z}_m$

If $n=1$

$$0 \to \cancelto{0}{\overset{0}{\overset{\shortparallel}{H^1(X,\mathbb{Z})}}} \otimes \mathbb{Z}_m \to H^1(X,\mathbb{Z}_m) \to Tor^1(\overset{\mathbb{Z}_2}{\overset{\shortparallel}{H^2(X,\mathbb{Z})}},\mathbb{Z}_m) \to 0$$

$\Rightarrow \quad H^1(X,\mathbb{Z}_m) = Tor^1(\mathbb{Z}_2,\mathbb{Z}_m) = \mathbb{Z}_{gcd(2,m)}$

[$Tor^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$ where $d = gcd(m,n)$

$\Rightarrow$ We have the exact seq.

$$\begin{tikzcd}
 & x \arrow[r] & mx & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/m\mathbb{Z} \arrow[r] & 0 \\
 & & a \arrow[r] & a+m\mathbb{Z} &
\end{tikzcd}$$

$$\Rightarrow \begin{tikzcd}
0 \arrow[r] & Tor^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \arrow[d, equal] \\
 & & \mathbb{Z}/n\mathbb{Z} & \mathbb{Z}_{n\mathbb{Z}} \\
 & & a+n\mathbb{Z} \arrow[r, "f"] & ma+n\mathbb{Z}
\end{tikzcd}$$

$\Rightarrow \mathrm{Tor}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \ker f$

Now we have $\ker f = n'\mathbb{Z}/n\mathbb{Z}$ where $n' = \frac{n}{d}$

$$= n'\mathbb{Z}/n'd\,\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}$$

Similarly using the same exact sequence we have

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & 0 \\ \mathbb{Z}/n\mathbb{Z} \arrow[r, "f", "x \mapsto mx"'] & \mathbb{Z}/n\mathbb{Z} & & \end{tikzcd}$$

$$\Rightarrow \mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \frac{\mathbb{Z}/n\mathbb{Z}}{\mathrm{Im} f}$$

$$= \frac{\mathbb{Z}/n\mathbb{Z}}{m(\mathbb{Z}/n\mathbb{Z})} = \frac{\mathbb{Z}/n\mathbb{Z}}{d\mathbb{Z}/n\mathbb{Z}}$$

$$= \mathbb{Z}/d\mathbb{Z} \quad]$$

If $1 < n \leq k$ is even then

$$\begin{tikzcd} & \mathbb{Z}_2 \arrow[d, equal] & & 0 \arrow[d, equal] & \\ 0 \arrow[r] & H^n(X, \mathbb{Z}) \otimes \mathbb{Z}_m \arrow[r] & H^n(X, \mathbb{Z}_m) \arrow[r] & \cancelto{0}{\mathrm{Tor}^1(H^{n+1}(X, \mathbb{Z}), \mathbb{Z}_m)} \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow H^n(X, \mathbb{Z}) = \mathbb{Z}_2 \otimes \mathbb{Z}_m = \mathbb{Z}_{\gcd(2,m)}$

If $1 < n < k$ is odd then

$$\begin{tikzcd} & 0 \arrow[d, equal] & & \mathbb{Z}_2 \arrow[d, equal] & \\ 0 \arrow[r] & \cancelto{0}{H^n(X, \mathbb{Z}) \otimes \mathbb{Z}_m} \arrow[r] & H^n(X, \mathbb{Z}_m) \arrow[r] & \mathrm{Tor}^1(H^{n+1}(X, \mathbb{Z}), \mathbb{Z}_m) \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow H^n(X, \mathbb{Z}_m) = \mathrm{Tor}^1(\mathbb{Z}_2, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(2,m)}$

If $n=k$ then

$$0 \to \overset{\overset{\mathbb{Z}_2}{\shortparallel}}{H^k(X,\mathbb{Z})} \otimes \mathbb{Z}_m \longrightarrow H^k(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\cancelto{0}{\overset{\overset{0}{\shortparallel}}{H^{k+1}(X,\mathbb{Z})}}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H^k(X,\mathbb{Z}_m) = \mathbb{Z}_2 \otimes \mathbb{Z}_m = \mathbb{Z}_{\gcd(2,m)}$$

If $n>k$ then

$$0 \to \cancelto{0}{\overset{\overset{0}{\shortparallel}}{H^n(X,\mathbb{Z})} \otimes} \mathbb{Z}_m \to H^n(X,\mathbb{Z}_m) \to \mathrm{Tor}^1(\cancelto{0}{\overset{\overset{0}{\shortparallel}}{H^{n+1}(X,\mathbb{Z})}}, \mathbb{Z}_m) \to 0$$

$$\Rightarrow H^n(X,\mathbb{Z}_m) = 0$$

So the Cohomology with coefficients in $\mathbb{Z}_m$ is given by

$$\begin{aligned} H^n(X,\mathbb{Z}_m) &= \mathbb{Z}_{\gcd(2,m)} && 0<n\leq k \\ &= \mathbb{Z}_m && n=0 \\ &= 0 && \text{otherwise} \end{aligned}$$

Similarly if k is odd using the same results we have

$$\begin{aligned} H^n(X,\mathbb{Z}_m) &= \mathbb{Z}_m && n=0,k \\ &= \mathbb{Z}_{\gcd(2,m)} && 0<n<k \\ &= 0 && \text{otherwise} \end{aligned}$$