Topology

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In Spring 2019 I taught Topology at IIT Bombay. The material up to Section 13.4 in these notes was covered in that course. The main reference was [Mun00]. In Fall 2019 I taught Basic Algebraic Topology at IIT Bombay. The material covered in that course comprises Chapters 12 to 16 of these notes. The main references were [Hat02] and [tD08]. (As mentioned in the mathscinet review, [tD08] is a very thorough and carefully-written text. The approach is more formal than Hatcher's in the sense that it is more of a theorem-proof exposition). Some more topics can be found at the course web page of a graduate level course I taught in Spring 2021: http://www.math.iitb.ac.in/ ronnie/Spring2021/MA816.html. The student interested in learning more advanced topics should consult [Hat02] or [tD08].

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Contents

Chapter 1

Definition of topological space and examples

1.1 Topological spaces

Definition 1.1.1 (Power set)**.** *Let X be a set. The power set of X is the set whose elements are all possible subsets of X. It is denoted $\mathscr{P}(X)$.*

In particular, the entire set X and the empty set $\emptyset$ are elements of $\mathscr{P}(X)$.

Definition 1.1.2 (Topology)**.** *Let X be a set and let $\mathcal{T} \subset \mathscr{P}(X)$ be a subset of the power set which satisfies the following three conditions:*

(1) The empty set $\emptyset$ and X are in $\mathcal{T}$,

(2) If $U_1, U_2, \ldots, U_r$ are finitely many subsets of X which are in $\mathcal{T}$, then the intersection $\bigcap_{i=1}^{r} U_i$ is in $\mathcal{T}$,

(3) Let I be any set and suppose for each $i \in I$ we are given a subset $U_i \in \mathcal{T}$. Then $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$.

Then we say that $\mathcal{T}$ defines a topology on X. The elements of $\mathcal{T}$ will be called open subsets of X for the topology $\mathcal{T}$.

Remark 1.1.3. Often we will write "Let $(X, \mathcal{T})$ be a topological space". By this we shall mean that X is a set and $\mathcal{T}$ is a topology on X. □

Let us see some examples of topological spaces.

1.2 Trivial topology

Let X be any set. Let $\mathcal{T} := \{\emptyset, X\}$. In this example the topology consists of only two open subsets. It is easy to check that the three defining conditions for $\mathcal{T}$ to be a topology are satisfied. This topology is called the trivial topology on X.

1.3 Discrete topology

Let X be any set. Let $\mathcal{T} = \mathscr{P}(X)$. In this example, every subset of X is open. It is easy to check that the three defining conditions for $\mathcal{T}$ to be a topology are satisfied. This topology is called the discrete topology on X.

1.4 Finite complement topology

Let X be any set. Let S be the collection of all subsets $U \subset X$ such that $X \setminus U$ is a finite set (possibly empty). Let $\mathcal{T} = S \bigcup \{\emptyset\}$. Let us check that the three conditions for being a topology are satisfied.

1. The empty set is in $\mathcal{T}$ since $\mathcal{T} = S \bigcup \{\emptyset\}$. The set X is in S since $X \setminus X = \emptyset$. Thus, X is in $\mathcal{T}$.

2. Let $U_1, U_2, \ldots, U_r$ be in $\mathcal{T}$. We need to show that $\bigcap_{i=1}^r U_i$ is in $\mathcal{T}$. This is true if any one of the U_i is empty, since then $\bigcap_{i=1}^r U_i = \emptyset$. So let us assume that none of the U_i are empty. We will show that $X \setminus \bigcap_{i=1}^r U_i$ is a finite set. But
$$X \setminus \bigcap_{i=1}^{r} U_i = \bigcup_{i=1}^{r} (X \setminus U_i)\,.$$
The right hand side is a finite union of finite sets and so is finite. This shows that the second condition is satisfied.

3. For the third condition, let I be any set. Suppose we are given a collection of subsets U_i, for every $i \in I$, such that U_i is in $\mathcal{T}$. Then we need to show that $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$. If all the U_i are empty, then the union is also empty and so it is in $\mathcal{T}$. So now assume that one of the U_i is not empty. We will be done if we can show that $X \setminus \bigcup_{i=1}^r U_i$ is a finite set. But
$$X \setminus \bigcup_{i=1}^{r} U_i = \bigcap_{i=1}^{r} (X \setminus U_i)$$
and the right hand side is a subset of $X \setminus U_i$. Since one of the $X \setminus U_i$ is a finite set, it follows that $\bigcap_{i=1}^r (X \setminus U_i)$ is a finite set. This verifies the third condition.

1.5 Standard topology on $\mathbb{R}$

Let $\mathbb{R}$ denote the set of real numbers. Let $\mathcal{T}$ be the collection of all subsets $U \subset \mathbb{R}$ which satisfy the following condition $(*_1)$.

$(*_1)$ For every $x \in U$ there is an $\epsilon > 0$ (which depends on x) such that the interval $(x-\epsilon, x+\epsilon) \subset U$.

Let us check that $\mathcal{T}$ satisfies the three conditions for defining a topology.

1. The empty set is in $\mathcal{T}$, this is vacuously true. The entire set $\mathbb{R}$ is in $\mathcal{T}$, since if $x \in \mathbb{R}$, then we may take $\epsilon = 1$ and we see that $(x-1, x+1) \subset \mathbb{R}$.

2. Let $U_1, U_2, \ldots, U_r$ be in $\mathcal{T}$. We need to show that $\bigcap_{i=1}^r U_i$ is in $\mathcal{T}$. This is true if $\bigcap_{i=1}^r U_i = \emptyset$. So let us assume that $\bigcap_{i=1}^r U_i \neq \emptyset$. Choose $x \in \bigcap_{i=1}^r U_i$. For every i, there is an $\epsilon_i > 0$ such that $(x-\epsilon_i, x+\epsilon_i) \subset U_i$. Let $\epsilon = \min\{\epsilon_i\}$. Then $(x-\epsilon, x+\epsilon) \subset U_i$ for every i, and so $(x-\epsilon, x+\epsilon) \subset \bigcap_{i=1}^r U_i$. This verifies the second condition.

3. For the third condition, let I be any set. Suppose we are given subsets U_i, for every $i \in I$, such that U_i is in $\mathcal{T}$. Then we need to show that $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$. Let $x \in \bigcup_{i \in I} U_i$. Then $x \in U_i$ for some i. Since U_i is in $\mathcal{T}$, there is ϵ such that $(x-\epsilon, x+\epsilon) \subset U_i$. Since $(x-\epsilon, x+\epsilon) \subset U_i \subset \bigcup_{i \in I} U_i$, this shows that $\mathcal{T}$ satisfies the third condition.

The above topology is called the **standard topology** on $\mathbb{R}$.

1.6 Standard topology on $\mathbb{R}^2$

Let $\mathbb{R}^2$ denote the set of tuples (x, y) with $x, y \in \mathbb{R}$.

Definition 1.6.1. *Let $\epsilon > 0$ and $(x, y) \in \mathbb{R}^2$. Define*

$$B_\epsilon(x, y) := \{(a, b) \in \mathbb{R}^2 \bigm| |x-a| < \epsilon, |y-b| < \epsilon\}.$$

Let $\mathcal{T}$ be the collection of all subsets $U \subset \mathbb{R}$ which satisfy the following condition $(*_2)$.

$(*_2)$ For every $(x, y) \in U$ there is an $\epsilon > 0$ (which depends on (x, y)) such that $B_\epsilon(x, y) \subset U$.

Let us check that $\mathcal{T}$ satisfies the three conditions for defining a topology.

1. The empty set is in $\mathcal{T}$, this is vacuously true. The whole set $\mathbb{R}^2$ is in $\mathcal{T}$, since if $(x, y) \in \mathbb{R}^2$, then we may take $\epsilon = 1$ and we see that $B_\epsilon(x, y) \subset \mathbb{R}^2$.

2. Let $U_1, U_2, \ldots, U_r$ be in $\mathcal{T}$. We need to show that $\bigcap_{i=1}^r U_i$ is in $\mathcal{T}$. This is true if $\bigcap_{i=1}^r U_i = \emptyset$. So let us assume that $\bigcap_{i=1}^r U_i \neq \emptyset$. Choose $(x, y) \in \bigcap_{i=1}^r U_i$. For every i, there is an $\epsilon_i > 0$ such that $B_\epsilon(x, y) \subset U_i$. Let $\epsilon = \min\{\epsilon_i\}$. Then $B_\epsilon(x, y) \subset U_i$ for every i, and so $B_\epsilon(x, y) \subset \bigcap_{i=1}^r U_i$. This verifies the second condition.

3. For the third condition, let I be any set. Suppose we are given subsets U_i, for every $i \in I$, such that U_i is in $\mathcal{T}$. Then we need to show that $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$. Let $(x, y) \in \bigcup_{i \in I} U_i$. Then $(x, y) \in U_i$ for some i. Since U_i is in $\mathcal{T}$, there is ϵ such that $B_\epsilon(x, y) \subset U_i$. Since $B_\epsilon(x, y) \subset U_i \subset \bigcup_{i \in I} U_i$, this shows that $\mathcal{T}$ satisfies the third condition.

1.7 Standard topology on $\mathbb{R}^n$

Let $\mathbb{R}^n$ denote the vector space of n-tuples of real numbers.

Definition 1.7.1. *Let $\epsilon > 0$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Define*

$$B_\epsilon(x_1, x_2, \ldots, x_n) := \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \mid |x_i - a_i| < \epsilon\}.$$

Let $\mathcal{T}$ be the collection of all subsets $U \subset \mathbb{R}^n$ which satisfy the following condition $(*_n)$.

$(*_n)$ For every $(x_1, x_2, \ldots, x_n) \in U$ there is an $\epsilon > 0$ (which depends on $(x_1, x_2, \ldots, x_n)$) such that $B_\epsilon(x_1, x_2, \ldots, x_n) \subset U$.

Let us check that $\mathcal{T}$ satisfies the three conditions for defining a topology.

1. The empty set is in $\mathcal{T}$, this is vacuously true. The whole set $\mathbb{R}^n$ is in $\mathcal{T}$, since if $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then we may take $\epsilon = 1$ and we see that $B_\epsilon(x_1, x_2, \ldots, x_n) \subset \mathbb{R}^n$.

2. Let $U_1, U_2, \ldots, U_r$ be in $\mathcal{T}$. We need to show that $\bigcap_{i=1}^r U_i$ is in $\mathcal{T}$. This is true if $\bigcap_{i=1}^r U_i = \emptyset$. So let us assume that $\bigcap_{i=1}^r U_i \neq \emptyset$. Choose $(x_1, x_2, \ldots, x_n) \in \bigcap_{i=1}^r U_i$. For every i, there is an $\epsilon_i > 0$ such that $B_\epsilon(x_1, x_2, \ldots, x_n) \subset U_i$. Let $\epsilon = \min\{\epsilon_i\}$. Then $B_\epsilon(x_1, x_2, \ldots, x_n) \subset U_i$ for every i, and so $B_\epsilon(x_1, x_2, \ldots, x_n) \subset \bigcap_{i=1}^r U_i$. This verifies the second condition.

3. For the third condition, let I be any set. Suppose we are given subsets U_i, for every $i \in I$, such that U_i is in $\mathcal{T}$. Then we need to show that $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$. Let $(x_1, x_2, \ldots, x_n) \in \bigcup_{i \in I} U_i$. Then $(x_1, x_2, \ldots, x_n) \in U_i$ for some i. Since U_i is in $\mathcal{T}$, there is ϵ such that $B_\epsilon(x_1, x_2, \ldots, x_n) \subset U_i$. Since $B_\epsilon(x_1, x_2, \ldots, x_n) \subset U_i \subset \bigcup_{i \in I} U_i$, this shows that $\mathcal{T}$ satisfies the third condition.

1.8 Exercises

1.8.1. Let X be a set and suppose we are given a collection of topologies $\{\mathcal{T}_i\}_{i\in I}$ on X. Let

$$\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$$

Since each $\mathcal{T}_i \subset \mathscr{P}(X)$ their intersection $\mathcal{T}$ is also a subset of $\mathscr{P}(X)$. Show that $\mathcal{T}$ is a topology on X. Is $\bigcup\limits_{i\in I} \mathcal{T}_i$ is a topology on X ?

1.8.2. Let $\{\mathcal{T}_i\}_{i\in I}$ be a family of topologies on X. Show that there is a topology $\mathcal{T}$ contained in all $\mathcal{T}_i$ which has the following property: If $\mathcal{T}'$ is another topology which is contained in all $\mathcal{T}_i$ then $\mathcal{T}' \subset \mathcal{T}$. Thus, $\mathcal{T}$ is the unique largest topology contained in all $\mathcal{T}_i$.

1.8.3. Let $\{\mathcal{T}_i\}_{i\in I}$ be a family of topologies on X. Show that there is a topology $\mathcal{T}$ containing all $\mathcal{T}_i$ which has the following property: If $\mathcal{T}'$ is another topology containing all $\mathcal{T}_i$ then $\mathcal{T} \subset \mathcal{T}'$. Thus, $\mathcal{T}$ is the unique smallest topology containing all $\mathcal{T}_i$.

1.8.4. Show that

$$[0,1] := \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$$

is not an open subset of $\mathbb{R}$ in the standard topology.

1.8.5. Let $a, b \in \mathbb{R}$. Show that for $a < b$ the interval

$$(a,b) := \{x \in \mathbb{R} \,|\, a < x < b\}$$

is an open set in the standard topology.

1.8.6. Let X be a topological space. Let A be a subset of X such that for every $x \in A$ there is an open subset U such that $U \subset A$ and $x \in U$. Show that A is open.

1.8.7. Show that

$$\bigcap_{n=1}^{\infty} \Big(\frac{-1}{n}, \frac{1}{n} \Big)$$

is not open in standard topology.

1.8.8. Consider the sets $[a,b), (a,b]$ and $[a,b]$

1. $[a,b) = \{x \mid a \leqslant x < b\}$,
2. $(a,b] = \{x \mid a < x \leqslant b\}$,
3. $[a,b] = \{x \mid a \leqslant x \leqslant b\}$.

Show that for any $a < b$ the above sets are not open in the standard topology.

1.8.9. This problem uses some concepts from a first course in Real Analysis

1. Let $X = C[0,1]$ be the space of all continuous functions $f : [0,1] \to \mathbb{R}$. For any $f \in C[0,1]$, the supremum norm of f is defined as

$$\|f\|_\infty := \sup\{|f(x)| : x \in [0,1]\}\,.$$

Notice that $\|f\|_\infty$ is well defined in this case. Given any $\epsilon > 0$ and $f \in C[0,1]$, let

$$B_\epsilon(f) := \{g \in C[0,1] : \|f - g\|_\infty < \epsilon\}\,.$$

Let $\mathcal{T}$ be the collection of all subsets $U \subseteq C[0,1]$ which satisfy the following condition: for every $f \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(f) \subset U$. Show that $(C[0,1], \mathcal{T})$ is a topological space.

2. Let $p \in [0,1]$. Consider the subset $F_p \subset C[0,1]$ defined as

$$F_p = \{f \in C[0,1] \,|\, f(p) = 0\}\,.$$

Show that $C[0,1] \setminus F_p$ is an open set in $(C[0,1], \mathcal{T})$.

Chapter 2

Basis for a topology

Now that we know what a topology on a set X is, we will see methods of how to specify a topology on a set X. In this chapter we will see that, roughly speaking, to define a topology it is enough to specify "sufficiently small" open sets. The open sets will then be arbitrary unions of these "sufficiently small" sets.

2.1 Basis for a topology

Definition 2.1.1 (Basis)**.** *Let $\mathcal{T}$ be a topology on a set X. A subset $\mathscr{B} \subset \mathcal{T}$ is called a basis for $\mathcal{T}$ if it satisfies the following property. Let U be in $\mathcal{T}$ and let $x \in U$. Then there is a V in $\mathscr{B}$ such that $x \in V$ and $V \subset U$.*

Lemma 2.1.2. *Let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$. Then*

$$\bigcup_{U \in \mathscr{B}} U = X\,.$$

Proof. Let $x \in X$. Then there is a V in $\mathscr{B}$ such that $x \in V$. Thus,

$$x \in \bigcup_{U \in \mathscr{B}} U\,.$$

This shows that

$$X \subset \bigcup_{U \in \mathscr{B}} U$$

But every $U \subset X$, and so we also have

$$\bigcup_{U \in \mathscr{B}} U \subset X\,.$$

$\square$

Proposition 2.1.3. *Let $\mathscr{B}$ be a subset of $\mathscr{P}(X)$ which satisfies the following two conditions*

(1) $\bigcup_{U \in \mathscr{B}} U = X$,

(2) If $U_1, U_2, \ldots, U_r$ *are in* $\mathscr{B}$, *then for every* $x \in \bigcap_{i=1}^r U_i$, *there is a* $W \in \mathscr{B}$ *such that* $x \in W$ *and* $W \subset \bigcap_{i=1}^r U_i$.

Let $\mathcal{T}$ *be the collection of subsets* U *of* X *which satisfy the following condition:*

$(*)$ *If* $x \in U$, *then there is a* $W \in \mathscr{B}$ *such that* $x \in W$ *and* $W \subset U$.

Then $\mathcal{T}$ *is a topology on* X. *This topology is called the topology generated by* $\mathscr{B}$.

Proof. Let us check that $\mathcal{T}$ satisfies the three conditions defining a topology. The empty set is in $\mathcal{T}$ because $(*)$ is vacuously true for $\emptyset$. Since $\bigcup_{U \in \mathscr{B}} U = X$, for every $x \in X$ there is a $U \in \mathscr{B}$ such that $x \in U \subset X$. This shows that X is in $\mathcal{T}$. This verifies the first condition.

For the second condition we need to show that a finite intersection of members of $\mathcal{T}$ is in $\mathcal{T}$. Let $U_1, U_2, \ldots, U_r$ be members of $\mathcal{T}$. If the intersection is empty then it lies in $\mathcal{T}$. So assume that the intersection is not empty and let $x \in \bigcap_{i=1}^r U_i$. Now there is a subset W in $\mathscr{B}$ such that $x \in W$ and $W \subset \bigcap_{i=1}^r U_i$. This shows that $\bigcap_{i=1}^r U_i$ satisfies $(*)$ and so it is in $\mathcal{T}$.

The third condition is easily checked and is left as an exercise. □

Lemma 2.1.4. *Let* $\mathcal{T}$ *be the topology generated by* $\mathscr{B}$. *(In particular, this means that* $\mathscr{B}$ *satisfies the two conditions of the previous proposition.) Then* $\mathscr{B}$ *is a basis for* $\mathcal{T}$.

Proof. Obvious from the definition of $\mathcal{T}$. □

2.2 Exercises

2.2.1. Let $\mathscr{B}$ be the collection of subsets of $\mathbb{R}$ of the type (a,b) with $a,b \in \mathbb{Q}$. Show that $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}$.

2.2.2. Let $\mathscr{B}$ be the collection of subsets of $\mathbb{R}^2$ of the type $B_\epsilon(a,b)$ with $\epsilon, a, b \in \mathbb{Q}$. Show that $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}^2$.

2.2.3. Let $\mathscr{B}$ be the collection of subsets of $\mathbb{R}^n$ of the type $B_\epsilon(a_1, a_2, \ldots, a_n)$ with $\epsilon, a_i \in \mathbb{Q}$. Show that $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}^2$.

2.2.4. Let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$. Note that $\mathscr{B}$ satisfies the two conditions

1. $\bigcup_{U \in \mathscr{B}} U = X$,
2. For finitely many U_i in $\mathscr{B}$, for any $x \in \bigcap_{i=1}^r U_i$, there is a $W \in \mathscr{B}$ such that $x \in W$ and $W \subset \cap_{i=1}^r U_i$.

Therefore, by the Proposition 2.1.3, $\mathscr{B}$ generates a topology $\mathcal{T}_{\mathscr{B}}$. Show that $\mathcal{T}_{\mathscr{B}} = \mathcal{T}$.

Chapter 3

Producing new topological spaces

In this chapter we will see how to construct new topological spaces out of known ones.

3.1 Subspace topology

Let X be a topological space and let $Y \subset X$ be a subset. Then we can specify a topology on Y as follows. Let $\mathcal{T}$ denote the topology on X. Let $\mathcal{T}_Y$ be the following collection of subsets of Y.

$$\mathcal{T}_Y := \{U \bigcap Y \mid U \in \mathcal{T}\}\,.$$

Proposition 3.1.1. *$\mathcal{T}_Y$ is a topology on Y.*

Proof. Taking $U = \emptyset$ and $U = X$ we see that $\emptyset, Y \in \mathcal{T}_Y$. This verifies the first condition for a topology. If $U_i \cap Y \in \mathcal{T}_Y$ for $i = 1, 2, \ldots, r$, then $\cap_{i=1}^r U_i \cap Y$ is in $\mathcal{T}_Y$ since $\cap_{i=1}^r U_i \in \mathcal{T}$. This verifies the second condition for being a topology. Finally if $U_i \cap Y \in \mathcal{T}_Y$ for $i \in I$, then $\cup_{i \in I} U_i \cap Y$ is in $\mathcal{T}_Y$ since $\cup_{i \in I} U_i \in \mathcal{T}$. This proves that $\mathcal{T}_Y$ is a topology on Y. $\square$

The following statements are easy to check:

1. If X is any set with the trivial topology and $Y \subset X$, then the subspace topology on Y is the trivial topology.

2. If X is any set with the discrete topology and $Y \subset X$, then the subspace topology on Y is the discrete topology.

3. Let $X = \mathbb{R}^2$ with the standard topology and let $Y = \mathbb{Z}$ (the set of integers on the X-axis). Then the subspace topology on Y is the same as the discrete topology. To see this it is enough to check that every $\{n\} \subset \mathbb{Z}$ is in $\mathcal{T}_Y$. This is true since $\{n\} = B_{1/4}(n, 0) \cap \mathbb{Z}$ (see Definition 1.6.1).

Proposition 3.1.2. *Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Let $\mathscr{B}_1$ be a basis for $\mathcal{T}_1$. If $\mathscr{B}_1 \subset \mathcal{T}_2$ then $\mathcal{T}_1 \subset \mathcal{T}_2$.*

Proof. Let $U \in \mathcal{T}_1$ be an open subset in the first topology. We need to show that $U \in \mathcal{T}_2$. By the definition of a basis, the set U is a union of subsets $U = \bigcup_{i \in I} V_i$, where each $V_i \in \mathscr{B}_1$. As $\mathscr{B}_1 \subset \mathcal{T}_2$ we get $V_i \in \mathcal{T}_2$. By the third condition in the definition of a topology, it follows that $U \in \mathcal{T}_2$. □

Corollary 3.1.3. *Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Let $\mathscr{B}_1$ be a basis for $\mathcal{T}_1$ and $\mathscr{B}_2$ be a basis for $\mathcal{T}_2$. If $\mathscr{B}_1 \subset \mathcal{T}_2$ and $\mathscr{B}_2 \subset \mathcal{T}_1$ then $\mathcal{T}_1 = \mathcal{T}_2$.*

Proposition 3.1.4. *Let X be a topological space. Suppose $\mathscr{B}$ is a basis for the topology. If $Y \subset X$ then*

$$\mathscr{B}_Y := \{V \bigcap Y \mid V \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

Proof. Left as an exercise to the reader. □

The point of the next Proposition is the following. On the one hand we have the standard topology on $\mathbb{R}$, which we denote by S_Y. On the other hand we have the following topology on $\mathbb{R}$. We may embed $\mathbb{R}$ into $\mathbb{R}^2$ by the map $i : \mathbb{R} \to \mathbb{R}^2$ given by $x \mapsto (x, 0)$. This map identifies $\mathbb{R}$ with the horizontal axis in $\mathbb{R}^2$ and using this identification we can transfer the subspace topology from the horizontal axis to $\mathbb{R}$. More precisely we mean the following. Let $\mathcal{T}$ denote the standard topology on $\mathbb{R}^2$. Let $\mathcal{T}_Y$ denote the subspace topology on the horizontal axis. Then

$$\mathcal{T}_Y = \{i^{-1}(U) \mid U \in \mathcal{T}\}$$

defines a topology on $\mathbb{R}$. The claim of the next proposition is that the above two topologies are the same.

Proposition 3.1.5. *Let $X = \mathbb{R}^2$ and let $Y = \{(x, 0) \mid x \in \mathbb{R}\}$. Then the subspace topology on Y is the same as the standard topology on $\mathbb{R}$. Here we have identified $\mathbb{R}$ with the subset Y by the map $x \mapsto (x, 0)$.*

Proof. Let S_Y denote the standard topology on $\mathbb{R}$ and let $\mathcal{T}_Y$ denote the topology on $\mathbb{R}$ described above. A basis for $\mathcal{T}_Y$ is given by sets of the type

$$i^{-1}(B_\epsilon(x, y)).$$

The set $i^{-1}(B_\epsilon(x, y))$ is nonempty iff $0 \in (y - \epsilon, y + \epsilon)$. If it is nonempty then it is equal to

$$(x - \epsilon, x + \epsilon) = B_\epsilon(x).$$

This shows that the collection $i^{-1}(B_\epsilon(x,y))$ is equal to $\{B_\epsilon(x)\}\cup\{\emptyset\}$. Thus,

$$\mathcal{B} := \{B_\epsilon(x)\}\cup\{\emptyset\}$$

is a basis for $\mathcal{T}_Y$. But this is also basis for S_Y. Now using Corollary 3.1.3 it follows that $S_Y = \mathcal{T}$. □

3.2 Product topology(1)

Let X and Y be topological spaces. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ denote the topologies on X and Y. Consider the set $\mathscr{B}\subset\mathscr{P}(X\times Y)$ defined as

$$\mathscr{B} := \{U\times V \mid U\in\mathcal{T}_X, V\in\mathcal{T}_Y\}.$$

Let us check that $\mathscr{B}$ satisfies the two conditions in Proposition 2.1.3. The first is true since $X\times Y$ is in $\mathscr{B}$. For the second condition, let us take $(x,y)\in\bigcap_{i=1}^r U_i\times V_i$. In particular, this means that $x\in\bigcap_{i=1}^r U_i$ and $y\in\bigcap_{i=1}^r V_i$. Thus, $(x,y)\in(\bigcap_{i=1}^r U_i)\times(\bigcap_{i=1}^r V_i)$. Since

$$\left(\bigcap_{i=1}^r U_i\right)\times\left(\bigcap_{i=1}^r V_i\right)$$

is in $\mathscr{B}$ and

$$(x,y)\in\left(\bigcap_{i=1}^r U_i\right)\times\left(\bigcap_{i=1}^r V_i\right)\subset\bigcap_{i=1}^r U_i\times V_i$$

we see that the second condition is also satisfied. Let $\mathcal{T}$ denote the topology on $X\times Y$ generated by $\mathscr{B}$. This is called the product topology on $X\times Y$. As we saw in Lemma 2.1.4, $\mathscr{B}$ is a basis for the product topology.

We now have two topologies on $\mathbb{R}^2$, the first being the standard topology and the second the product topology, where each factor $\mathbb{R}$ is given the standard topology. The following proposition shows that both these topologies are the same.

Proposition 3.2.1. *The product topology on $\mathbb{R}\times\mathbb{R}$ is the same as the standard topology on $\mathbb{R}^2$.*

Proof. A basic open set in the standard topology for $\mathbb{R}^2$ is of the type $B_\epsilon(x,y)$. Since

$$B_\epsilon(x,y) = B_\epsilon(x)\times B_\epsilon(y)$$

it follows that $B_\epsilon(x,y)$ is open in the product topology.

It is easily seen that if $U,V\subset\mathbb{R}$ are open sets then $U\times V$ is open in the standard topology on $\mathbb{R}^2$. From this it follows that each basic open set in the product topology is open in the standard topology.

The Proposition now follows from Corollary 3.1.3. □

3.3 Product topology(2)

Let $X_1, X_2, \ldots, X_n$ be topological spaces. We can generalize the preceding discussion and make $X_1 \times X_2 \times \ldots \times X_n$ into a topological space. A basis for this topology is given by

$$\mathscr{B} := \{U_1 \times \ldots \times U_n \mid U_i \in \mathcal{T}_{X_i}\}.$$

It is easily checked that $\mathscr{B}$ satisfies the two conditions mentioned in Proposition 2.1.3. This check is left to the reader.

3.4 Product topology(3)

Now we consider an infinite collection of topological spaces X_i, for $i \in I$. In this case, we define a collection $\mathscr{B}$ as follows

$$\mathscr{B} := \{\prod_{i \in I} U_i \mid U_i = X_i \text{ for all but finitely many } i\}.$$

For each i, we have $U_i \subset X$. Let $S \subset I$ be the collection of those indices i such that $U_i \neq X$. The phrase "$U_i = X_i$ for all but finitely many i" is to say that the cardinality of S is finite.

It is easily checked that $\mathscr{B}$ satisfies the two conditions mentioned in Proposition 2.1.3. This check is left to the reader.

3.5 Exercises

3.5.1. Let X be a topological space. Suppose $\mathscr{B}$ is a basis for the topology. If $Y \subset X$ then show that

$$\mathscr{B}_Y := \{V \bigcap Y \mid V \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

3.5.2. Check that the product topology on $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ (n-fold product) is the same as the standard topology on $\mathbb{R}^n$.

3.5.3. Consider $\mathbb{R}^2$ with the standard topology (which we denote $\mathcal{T}$). Let $Y = \mathbb{R}$. Let $\Delta : Y \to \mathbb{R}^2$ denote the map $x \mapsto (x, x)$. Note that Δ embeds Y into $\mathbb{R}^2$ as the diagonal. Let S_Y be the standard topology on Y and let $\mathcal{T}_Y$ be the subspace topology from $\mathbb{R}^2$, that is,

$$\mathcal{T}_Y = \{\Delta^{-1}(U) \,|\, U \in \mathcal{T}\}.$$

Show that $S_\Delta = \mathcal{T}_\Delta$.

3.5.4. Consider the set $S^1 := \{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Let $\mathcal{T}$ be the subspace topology on S^1. Let $\mathscr{B}$ consist of subsets of the type $V_{a,b} := \{(\cos\theta, \sin\theta) \mid a < \theta < b\}$. Show that $\mathscr{B}$ is a basis for the topology $\mathcal{T}$.

3.5.5. Let X and Y be topological spaces. Let $A \subset X$ and let $B \subset Y$. We can give two topologies on $A \times B$. The first is the product topology, where A is given the subspace topology from X and B is given the subspace topology from Y. The second is the subspace topology from the topology on $X \times Y$. Show that these two topologies are the same.

3.5.6. Consider topological spaces Y_i, where $i \in I$ is an indexing set of infinite size. Let $\mathscr{B}$ be the collection of subsets of $\prod_{i\in I} Y_i$ of the type $\prod_{i\in I} U_i$, where each U_i is open in Y_i.

1. Show that $\mathscr{B}$ satisfies the two conditions required to generate a topology. The topology generated by this basis is called the **box topology**.

2. Give an example of topological spaces and continuous maps $f_i : X \to Y_i$ such that the map $f : X \to \prod_{i\in I} Y_i$, given by $x \mapsto (f_i(x))$, is not continuous when $\prod_{i\in I} Y_i$ is given the box topology.

Chapter 4

Continuous maps

4.1 Definition and basic properties

Definition 4.1.1. *Let X and Y be topological spaces and let $f : X \to Y$ be a map of sets. We say f is continuous if for every open subset $V \subset Y$, the set $f^{-1}(V)$ is open in X.*

Lemma 4.1.2. *Let $\mathscr{B}$ be a basis for the topology of Y. A map $f : X \to Y$ is continuous iff for every $V \in \mathscr{B}$ the set $f^{-1}(V)$ is open in X.*

Proof. Let f be continuous. If V is in $\mathscr{B}$, then it is open in Y. Thus, $f^{-1}(V)$ is open.

Conversely, suppose that $f^{-1}(V)$ is open for every V in $\mathscr{B}$. Let U be an open subset of Y and let $x \in U$. Then there is a V_x in $\mathscr{B}$ such that $x \in V_x$ and $V_x \subset U$. Thus, $U = \bigcup_{x \in U} V_x$. Then $f^{-1}(U) = \bigcup_{x \in U} f^{-1}(V_x)$. Each $f^{-1}(V_x)$ is open and since an arbitrary union of open sets is open, we see that $f^{-1}(U)$ is open. $\square$

Lemma 4.1.3. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.*

Proof. Easy exercise. $\square$

Theorem 4.1.4. *Let $\mathbb{R}^2$ and $\mathbb{R}$ have the standard topologies. Consider the addition map $A : \mathbb{R}^2 \to \mathbb{R}$ given by $A(x, y) := x + y$. This map is continuous.*

Proof. By Lemma 4.1.2 it is enough to check that the inverse image of a basic open set is open. Let $B_\epsilon(a)$ be a basic open set. Suppose $(x, y) \in A^{-1}(B_\epsilon(a))$. Then $A(x, y) = x + y =: b \in B_\epsilon(a)$. This means that $b \in (a - \epsilon, a + \epsilon)$. Therefore, we can find $\delta > 0$ such that $(b - \delta, b + \delta) \subset (a - \epsilon, a + \epsilon)$.

Let $(x', y') \in B_{\delta/2}(x, y)$. This means that $|x - x'| < \delta/2$ and $|y - y'| < \delta/2$. Then

$$|A(x, y) - A(x', y')| = |x + y - x' - y'| < \delta\,.$$

Thus, $|x + y - x' - y'| = |b - x' - y'| < \delta$. Thus,

$$A(x', y') \in (b - \delta, b + \delta) \subset B_\epsilon(a)\,.$$

That is, we have proved that $A(B_{\delta/2}(x, y)) \subset B_\epsilon(a)$. Equivalently,

$$B_{\delta/2}(x, y) \subset A^{-1}(B_\epsilon(a))\,.$$

This proves that $A^{-1}(B_\epsilon(a))$ is open. This completes the proof of the Theorem. □

Theorem 4.1.5. *The multiplication map $m : \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto xy$ is continuous.*

Proof. By Lemma 4.1.2, it suffice to show that the inverse image of a basic open set is open. For this it suffices to show that if $(x, y) \in m^{-1}(B_\epsilon(a))$, then there is a δ such that $B_\delta(x, y) \subset m^{-1}(B_\epsilon(a))$. Since $xy \in B_\epsilon(a)$, there is an $\epsilon' < 1$ such that $B_{\epsilon'}(xy) \subset B_\epsilon(a)$. Thus, it suffices to find δ such that

$$B_\delta(x, y) \subset m^{-1}(B_{\epsilon'}(xy)) \subset m^{-1}(B_\epsilon(a)).$$

Let $0 < \delta < 1$. If $|x - x'| < \delta$ and $|y - y'| < \delta$ then

$$\begin{aligned}
|xy - x'y'| &= |xy - x'y + x'y - x'y'| \\
&\leqslant |x - x'||y| + |x'||y - y'| \\
&< \delta(|y| + |x'|) \\
&\leqslant \delta(|y| + |x| + \delta) \\
&< \delta(|y| + |x| + 1)\,.
\end{aligned}$$

As $0 < \epsilon' < 1$ we have

$$\delta' := \frac{\epsilon'}{|y| + |x| + 1} < 1\,.$$

The above computation shows that if $(x', y') \in B_{\delta'}(x, y)$ then

$$|m(x', y') - m(x, y)| < \epsilon'\,.$$

This shows that $(x', y') \in m^{-1}(B_{\epsilon'}(xy))$ and so

$$B_{\delta'}(x, y) \subset m^{-1}(B_{\epsilon'}(xy)) \subset m^{-1}(B_\epsilon(a)).$$

This completes the proof of the Theorem. □

Theorem 4.1.6. *Let X be a topological space and let Y_i be a collection of topological spaces, for $i \in I$. Assume that we are given continuous maps $f_i : X \to Y_i$, for every $i \in I$. Then the map $f : X \to \prod_{i \in I} Y_i$ defined by $f(x) = (f_i(x))_{i \in I}$ is continuous.*

Proof. Let U be an open subset in $\prod_{i\in I} Y_i$. We need to check that $f^{-1}(U)$ is open in X. By Lemma 4.1.2 it suffices to check this when U is a basic open set. Recall that in the product topology, the basic open sets are of the form $\prod_{i\in I} U_i$, where $U_i = Y_i$ for all but finitely many i. Let S be a finite subset of I and let us assume that if $i \notin S$ then $U_i = Y_i$. We have

$$f^{-1}(\prod_{i\in I} U_i) = \bigcap_{i\in I} f_i^{-1}(U_i)\,.$$

If $U_i = Y_i$, then $f_i^{-1}(U_i) = X$. Thus, the above becomes

$$f^{-1}(\prod_{i\in I} U_i) = \bigcap_{i\in S} f_i^{-1}(U_i)\,.$$

Since each f_i is continuous, the above is a finite intersection of open subsets of X and so is open. □

Proposition 4.1.7. *Let $f, g : X \to \mathbb{R}$ be continuous functions. Then the function $h : X \to \mathbb{R}$ given by $x \mapsto f(x) + g(x)$ is continuous.*

Proof. We have seen that the map $(f, g) : X \to \mathbb{R}^2$ given by $x \mapsto (f(x), g(x))$ is continuous. Also the addition map from $\mathbb{R}^2 \to \mathbb{R}$ is continuous. Since h is the composite $A \circ (f, g)$, it follows that it is continuous. □

Proposition 4.1.8. *Let $f : X \to Z$ be a continuous map. Suppose that the image of f is contained in a subset $i : Y \subset Z$. Thus, there is a map $f_Y : X \to Y$ such that $f = i \circ f_Y$. If we give Y the subspace topology then the map $f_Y : X \to Y$ is continuous.*

Proof. Let V be an open subset of Y. Then there is an open subset U of Z such that $V = U \bigcap Z$, that is, $V = i^{-1}(U)$. Thus, $f^{-1}(U) = f_Y^{-1}(i^{-1}(U)) = f_Y^{-1}(V)$. As U is open and f is continuous, it follows that $f_Y^{-1}(V)$ is open. Thus, f_Y is continuous. □

Remark 4.1.9. We shall often abuse notation and denote the map f_Y by f. □

Proposition 4.1.10. *If $f : X \to Z$ is continuous and $Y \subset X$ is given the subspace topology, then the restriction of f to Y is continuous.*

Proof. This follows because the inclusion $i : Y \to X$ is continuous and the composite of continuous functions is continuous. □

Corollary 4.1.11. *Let $f : X \to \mathbb{R}$ be continuous such that $f(x) \neq 0$. Then this defines a continuous map $f : X \to \mathbb{R}^{\times}$.*

Proposition 4.1.12. *Let $f, g : X \to \mathbb{R}$ be continuous functions such that $f(x) \neq 0$. Then the function $x \mapsto g(x)/f(x)$ is continuous.*

Proof. Note that f defines a continuous map $X \to \mathbb{R}^\times$, which we continue to denote by f. Consider the functions

1. $(g, f) : X \to \mathbb{R} \times \mathbb{R}^\times$ given by $x \mapsto (g(x), f(x))$,
2. $\mathbb{R} \times \mathbb{R}^\times \to \mathbb{R} \times \mathbb{R}^\times$ given by $(x, y) \mapsto (x, y^{-1})$,
3. $\mathbb{R} \times \mathbb{R}^\times \to \mathbb{R}$ given by $(x, y) \mapsto xy$.

All these are continuous and their composite is the function $x \mapsto g(x)/f(x)$. $\square$

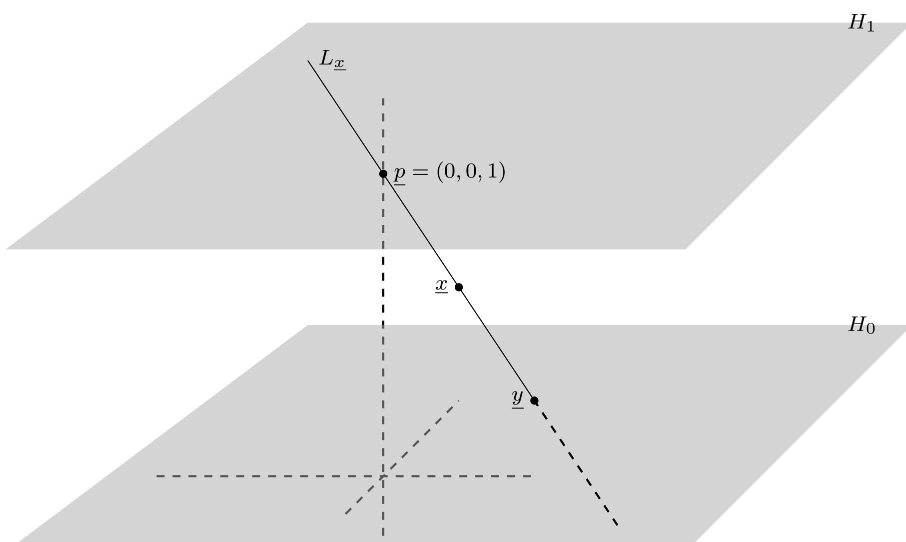
4.2 Projection from a point

Often in geometry we encounter certain natural constructions and maps. In this section we will give a geometric description of one such map and then use coordinates to check that this map is continuous.

Let $H_t \subset \mathbb{R}^{n+1}$ denote the hyperplane $\{x_{n+1} = t\}$. Let

$$X := \mathbb{R}^{n+1} \setminus H_1$$

Let $\underline{x}$ be a point in X. Consider the unique straight line in $\mathbb{R}^{n+1}$ which joins $\underline{x}$ and the point $\underline{p} := (0, 0, \ldots, 0, 1) \in H_1$. Denote this line by $L_{\underline{x}}$. Since $\underline{p} \notin H_0$, $L_{\underline{x}}$ is not contained in H_0. The line $L_{\underline{x}}$ meets H_0 in a point $\underline{y}$. Define $\pi(\underline{x}) = \underline{y}$. Notice that if $\underline{x}'$ is any other point of X which is on $L_{\underline{x}}$, then $\pi(\underline{x}') = \pi(\underline{x})$.



Let us check that this map is continuous. We will do the check using coordinates. Let $\underline{x} = (x_1, x_2, \ldots, x_n, x_{n+1})$. Points on the line joining $\underline{x}$ with $\underline{p}$ are of the form

$$t(x_1, x_2, \ldots, x_{n+1}) + (1-t)(0, 0, \ldots, 0, 1) = (tx_1, tx_2, \ldots, tx_n, tx_{n+1} + 1 - t)\,.$$

To solve for the point on this line which is on H_0, we set the last coordinate equal to 0. This gives

$$t = \frac{1}{1 - x_{n+1}}\,.$$

Notice that this is well defined since $x_{n+1} \neq 1$. Thus, we get

$$\pi(x_1, x_2, \ldots, x_{n+1}) = (\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}})\,. \tag{4.2.1}$$

This map is continuous since each of the coordinate functions is continuous.

4.3 Homeomorphism

Definition 4.3.1 (Homeomorphism)**.** *Let $f : X \to Y$ be a bijective continuous map. Let $g : Y \to X$ denote the set theoretic inverse map. If g is continuous then f is called a homeomorphism.*

Lemma 4.3.2. *Let $f : X \to Y$ be a homeomorphism. Let $g : Y \to X$ denote the inverse of f. Then g is a homeomorphism.*

Proof. Left as an exercise. □

We will need the following Lemma later.

Lemma 4.3.3. *Let X and Y be topological spaces. Let $x \in X$ be a point. Consider the subset $x \times Y \subset X \times Y$ with the subspace topology. This subspace is homeomorphic to Y.*

Proof. Consider the natural map $f : x \times Y \to Y$ given by $(x, y) \mapsto y$. We will show that this map is a homeomorphism. This map is a bijection with the inverse being given by $y \mapsto (x, y)$.

The composite map $x \times Y \to X \times Y \to Y$, where the first is the obvious inclusion and the second is the projection, is continuous. The inclusion is continuous since $x \times Y$ has the subspace topology. The projection is continuous is an exercise. This shows that f is continuous. To show that f is a homeomorphism, it suffices to show that the image of an open set is open. An open set of $x \times Y$ is the intersection of an open subset of $X \times Y$ with $x \times Y$. An open subset of $X \times Y$ is the union of basic open subsets, therefore looks like $\bigcup_{i \in I} U_i \times V_i$. We have

$$f\Big((x \times Y) \bigcap (\bigcup_{i \in I} U_i \times V_i)\Big) = \bigcup_{i \in I} f((x \times Y) \bigcap U_i \times V_i)\,.$$

Thus, it suffices to show that $f((x \times Y) \bigcap U \times V)$ is open in Y. If $x \notin U$, then $(x \times Y) \bigcap U \times V = \emptyset$ and so this is trivially true. If $x \in U$, then $x \times Y \bigcap U \times V = x \times V$, and under f it has image V, which is open. This proves the lemma. $\square$

4.4 Exercises

4.4.1. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Show that $g \circ f : X \to Z$ is continuous.

4.4.2. Let X be a topological space and let Y be a subset of X. Give Y the subspace topology. Show that the inclusion map $i : Y \to X$ is continuous.

4.4.3. Let X_t denote X with the trivial topology. Let X_d denote X with the discrete topology. Show that the identity map $X_d \to X_t$ is continuous. If the identity map $X_t \to X_d$ is continuous, show that X contains only one point.

4.4.4. Let $\mathbb{R}$ have the standard topology. Let $a \in \mathbb{R}$ and consider the translation map $T_a : \mathbb{R} \to \mathbb{R}$ given by $T_a(x) = a + x$. Show that T_a is continuous.

4.4.5. Let $\mathbb{R}$ have the standard topology. Consider the map $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto -x$. Show that this map is continuous.

4.4.6. Let $\mathbb{R}^\times := \mathbb{R}\backslash 0$. Give $\mathbb{R}^\times \subset \mathbb{R}$ the subspace topology. Show that the map $\mathbb{R}^\times \to \mathbb{R}^\times$ given by $x \mapsto x^{-1}$ is continuous.

4.4.7. Let Y_i be a collection of topological spaces. Consider $\prod_{i\in I} Y_i$ with the product topology. Show that the projection maps $p_i : \prod_{i\in I} Y_i \to Y_i$ are continuous.

4.4.8. Show that if $f, g : X \to \mathbb{R}$ are continuous, then the function $fg : X \to \mathbb{R}$ given by $x \mapsto f(x)g(x)$ is continuous.

4.4.9. Prove by induction on n that the addition and multiplication map from $\mathbb{R}^n \to \mathbb{R}$ are continuous. The base case for induction is $n = 2$, which we have already seen.

4.4.10. Use the composite map $\mathbb{R} \xrightarrow{\Delta} \mathbb{R}^n \to \mathbb{R}$, where the first denotes the diagonal and the second denotes multiplication, to show that the polynomial map $x \mapsto x^n$ is continuous.

4.4.11. Let P be a polynomial in n-variables. Then P defines a map $\mathbb{R}^n \to \mathbb{R}$, given by

$$(x_1, x_2, \ldots, x_n) \mapsto P(x_1, x_2, \ldots, x_n)$$

Show that this map is continuous.

4.4.12. Let x_i be a family of topological spaces indexed by $i \in I$. Show that the projection maps $\prod_{i\in I} X_i \to X_i$ are continuous.

4.4.13. Let $f : X \to Y$ be a bijective map which is continuous. Then f is a homeomorphism iff for every open set U, the set $f(U)$ is open.

4.4.14. Use a “linear” map to show that $(0, 1)$ is homeomorphic to (a, b).

4.4.15. Show that $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

4.4.16. Define the sphere as

$$S^n := \{(x_1, x_2, \ldots, x_{n+1}) \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$$

Give it the subspace topology from $\mathbb{R}^{n+1}$. Restricting the map π in equation (4.2.1) to $S^n \setminus (0, 0, \ldots, 0, 1)$ gives a continuous map

$$S^n \setminus (0, 0, \ldots, 0, 1) \to \mathbb{R}^n$$

Show that this restricted map is a bijection.

4.4.17. Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices with real entries. Then clearly $M_n(\mathbb{R})$ is an $\mathbb{R}$ vector space of dimension n^2. Considering $M_n(\mathbb{R})$ as a vector space, give it the standard topology.

(a) Show that the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$, which sends each matrix to its determinant, is continuous.

(b) Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be given by $A \mapsto AA^T$, where A^T is the transpose of the matrix A. Show that f is continuous.

(c) Show that the map $f : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $(A, B) \mapsto AB$ is continuous.

4.4.18. Let $GL_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid det(A) \neq 0\}$. Show that $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.

Chapter 5

Closed subsets and closures

5.1 Closed sets

Definition 5.1.1. *Let Y be a topological space. A subset $Z \subset Y$ is called closed if $Y \setminus Z$ is an open subset of Y.*

Proposition 5.1.2. *Let $f : X \to Y$ be a map. Then f is continuous iff for every closed subset $Z \subset Y$, $f^{-1}(Z)$ is a closed subset of X.*

Proof. First assume that f is continuous. Let $Z \subset Y$ be a closed subset. Then $V := Y \setminus Z$ is an open subset of Y. Since f is continuous it follows that $f^{-1}(V)$ is open. Since $f^{-1}(V) = X \setminus f^{-1}(Z)$ it follows by definition that $f^{-1}(Z)$ is a closed subset. This proves one direction of the Proposition.

Next let us assume that for every closed subset $Z \subset Y$ we have $f^{-1}(Z)$ is closed in X. If V is an open subset in Y, let $Z := Y \setminus V$. Then Z is closed in Y and so $f^{-1}(Z)$ is closed in X. This shows that $f^{-1}(V) = X \setminus f^{-1}(Z)$ is open in X. This proves that f is continuous. $\square$

5.2 Closure

Definition 5.2.1. *Let X be a topological space and let $A \subset X$ be a subspace. Define the* **closure of** A **in** X *to be the set $\bar{A}$ which contains all points $x \in X$ having the following property: If U is an open subset and $x \in U$, then $U \bigcap A \neq \emptyset$.*

As an example, let us check that the closure of $(0, 1)$ in $\mathbb{R}$ is $[0, 1]$. Let U be an open set which contains 0. Then there is an ϵ such that $B_\epsilon(0) \subset U$. Since $B_\epsilon(0) = (-\epsilon, \epsilon)$, it has a nonempty intersection with $(0, 1)$. Similarly, 1 is in the closure. Finally, we need to show that there is nothing else in the closure. Suppose $x \notin [0, 1]$, then we can find an ϵ such that $B_\epsilon(x) \bigcap [0, 1] = \emptyset$. This shows that x is not in the closure. In the same way we can see that the closure of (a, b) in $\mathbb{R}$ is $[a, b]$.

We emphasize that the closure depends on where the closure is being taken. For example, the closure of $(0,1)$ in $(0,3)$ is $(0,1]$. The same proof as above shows that 1 is in the closure. Note that $0 \notin (0,3)$ and so it is not in the closure of $(0,1)$ in $(0,3)$.

Proposition 5.2.2. *A set A is closed iff $A = \bar{A}$.*

Proof. For any set it is obvious $A \subset \bar{A}$. Let us assume that A is closed, which implies that $X \setminus A$ is open in X. Suppose that $x \notin A$. Then $x \in X \setminus A$ which is open. Further, $(X \setminus A) \bigcap A = \emptyset$ and this shows that $x \notin \bar{A}$. This implies that $(X \setminus A) \subset (X \setminus \bar{A})$. This shows that $\bar{A} \subset A$ which proves that $A = \bar{A}$.

Next assume that $A = \bar{A}$. Let $x \notin A$. Then $x \notin \bar{A}$. Thus, there is an open set which contains x and does not meet A. In other words, there is an open set which contains x and is completely contained in $X \setminus A$. Thus, for every $x \in X \setminus A$ there is an open subset U such that $x \in U \subset (X \setminus A)$. This shows that $X \setminus A$ is open. Thus, A is closed. □

Definition 5.2.3. *Let X be a topological space. A subset T is said to be dense in X if its intersection with every nonempty open subset is nonempty.*

Proposition 5.2.4. *A is dense in $\overline{A}$.*

Proof. If not, then there is an open subset $U \subset \overline{A}$ such that $U \bigcap A = \emptyset$. If $x \in U$, then on the one hand we have $x \in \bar{A}$. On the other hand we have, by the definition of closure, that $x \notin \overline{A}$. This gives a contradiction. □

Let X be a topological space and let A be a subspace. Let B be a closed subset of A. Apriori it is not clear if B is obtained by intersecting a closed subspace with A. We prove in the next Lemma that this is indeed the case.

Lemma 5.2.5. *Let A be a subset of X with the subspace topology. The closed subsets of A are precisely the subsets of the form $Z \cap A$, where Z is closed in X.*

Proof. Let T be a closed subspace of A. Then $A \setminus T$ is an open subset of A. By the definition of subspace topology it follows that there is an open subset $U \subset X$ such that $U \cap A = A \setminus T$.

A simple set theoretic check shows that

$$A \setminus (U \cap A) = (X \setminus U) \cap A\,.$$

From this it follows that

$$(X \setminus U) \cap A = A \setminus (U \cap A) = A \setminus (A \setminus T) = T\,.$$

Thus, we have written T as the intersection of a closed subset with A.

Conversely, if Z is a closed subspace of X then $(X \setminus Z) \cap A$ is open in A. One easily checks that the complement of this in A is $Z \cap A$, which shows that $Z \cap A$ is closed in A. □

Lemma 5.2.6. *Let X be a topological space and let A be a closed subspace. Let B be a closed subset of A. Then B is a closed subset of X.*

Proof. From the previous lemma it follows that there is a closed subspace $Z \subset X$ such that $B = Z \cap A$. Since the intersection of two closed subspaces is closed, it follows that B is closed in X. $\square$

5.3 Joining continuous maps

Theorem 5.3.1 (Joining continuous maps)**.** *Let X be a topological space and let A and B be closed subsets such that $X = A \bigcup B$. Suppose we have continuous functions $f : A \to Y$ and $g : B \to Y$ such that $f(x) = g(x)$ for all $x \in A \bigcap B$. Then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof. Let $Z \subset Y$ be closed. It is enough to check that $h^{-1}(Z)$ is closed. However,

$$\begin{aligned} h^{-1}(Z) &= (h^{-1}(Z) \bigcap A) \bigcup (h^{-1}(Z) \bigcap B) \\ &= f^{-1}(Z) \bigcup g^{-1}(Z) . \end{aligned}$$

Since f and g are continuous, it follows that $f^{-1}(Z)$ is closed in A and $g^{-1}(Z)$ is closed in B. Lemma 5.2.6 shows that $f^{-1}(Z)$ and $g^{-1}(Z)$ are closed in X, so their union is also closed. This proves that h is continuous. $\square$

5.4 Exercises

5.4.1. Show that the sphere S^n is a closed subspace of $\mathbb{R}^n$.

5.4.2. Let $\mathbb{I}_n$ denotes the $n \times n$ identity matrix. Show that $S := \{\mathbb{I}_n\}$ is a closed subset of $M_n(\mathbb{R})$. More generally, show that for any point $x \in \mathbb{R}^l$, the set $\{x\}$ is a closed subset in the standard topology.

5.4.3. Let $O_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid AA^T = \mathbb{I}_n\}$ and $SO_n(\mathbb{R}) := \{A \in O_n(\mathbb{R}) \mid det(A) = 1\}$, where $\mathbb{I}_n$ is the $n \times n$ identity matrix. Show that $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ are closed subsets of $M_n(\mathbb{R})$.

5.4.4. Let $f : X \to Y$ be a map. Then f is continuous iff for every closed subset $Z \subset Y$, $f^{-1}(Z)$ is a closed subset of X.

5.4.5. Let X and Y be topological spaces and let $f : X \to Y$. Show that f is continuous iff for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

5.4.6. Let A, B, and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$; give an example where equality fails.

5.4.7. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$, $\overline{A \times B} = \overline{A} \times \overline{B}$.

5.4.8. Show that S^n is a closed subset of $\mathbb{R}^{n+1}$.

5.4.9. Let $\mathcal{C} := \left\{ \left(\frac{1}{n}, y \right) \;\middle|\; n \in \mathbb{Z}_{\geqslant 1},\ 0 \leqslant y \leqslant 1 \right\} \subset \mathbb{R}^2$. Find the closure of C in $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{(0, 0)\}$.

5.4.10. Let $a < b$ be real numbers and let $A := \{x \in \mathbb{Q} \mid a < x < b\}$. Find the closure of A in $\mathbb{Q}$ and in $\mathbb{R}$.

Chapter 6

Metric spaces

In this chapter we will see an important class of topological spaces and explain some of the earlier ideas like closure, continuity in terms of the metric.

6.1 Topology on a metric space

Definition 6.1.1. *A metric on a set X is a function*

$$d : X \times X \to \mathbb{R}_{\geqslant 0}$$

which satisfies the following three conditions for all $x, y, z \in X$,

1. $d(x, y) = 0$ *iff* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leqslant d(x, y) + d(y, z)$.

The third condition is often called triangle inequality. The motivation for this inequality is the following: If $a, b, c \in \mathbb{R}^2$ are three points, then the side lengths of the triangle formed by these three points satisfy the above inequality.

If (X, d) is a metric space, then an open ball of radius r around x_0 is the set

$$B_r(x_0) := \{y \in X \mid d(y, x_0) < r\}.$$

The simplest examples of metric spaces are $\mathbb{R}^n$ equipped with the metric

$$d(x, y) = \|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Lemma 6.1.2. *The above function is a metric on $\mathbb{R}^n$.*

Proof. That d satisfies the first two conditions to be a metric is an exercise left to the reader. To prove triangle inequality it suffices to show that

$$\|x-y\|_2 \leqslant \|x\|_2 + \|y\|_2. \tag{6.1.3}$$

This is because $d(x,y) = \|x-y\|_2$ and using the above we get

$$d(x,y) = \|x-y\|_2 \leqslant \|x-z\|_2 + \|z-y\|_2 = d(x,z) + d(z,y)\,.$$

Consider the following inner product on $\mathbb{R}^n$

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i.$$

Note that $\langle x, x\rangle = \|x\|_2^2$. We will first prove the Cauchy-Schwarz inequality which says that

$$|\langle x, y\rangle| \leqslant \|x\|_2\|y\|_2. \tag{6.1.4}$$

For $t \in \mathbb{R}$ define $w := y - tx$. Then

$$0 \leqslant \langle w, w\rangle = \|y_2^2\| + t^2\|x\|_2^2 - 2t\langle x, y\rangle\,.$$

This shows that for fixed x, y, the above quadratic equation (which depends on x and y) in the variable t always takes non-negative values. Thus, it must have non-positive discriminant. This gives

$$|\langle x, y\rangle| \leqslant \|x\|_2\|y\|_2,$$

which proves (6.1.4). Now

$$\begin{aligned}
\|x-y\|_2^2 &= |\langle x-y, x-y\rangle| \\
&= |\langle x, x\rangle + \langle y, y\rangle - 2\langle x, y\rangle| \\
&\leqslant \langle x, x\rangle + \langle y, y\rangle + 2|\langle x, y\rangle| \\
&\leqslant \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 \\
&= (\|x\|_2 + \|y\|_2)^2\,.
\end{aligned}$$

This proves (6.1.3). $\square$

Given a metric on a space X, we can define a topology on X. This topology has as basis the open balls described above. More precisely, a set U is open in X if for every $x \in U$, there is an $r > 0$ (which depends on x) such that, the ball of radius r around x, $B_r(x)$ is contained in U.

6.2 Closures in metric spaces

Theorem 6.2.1. *Let (X, d) be a metric space. Let Y be a subset of X. Then $x \in X$ is in $\overline{Y}$ iff there is a sequence of points $y_n \in Y$ such that $\lim_{n\to\infty} d(x, y_n) = 0$.*

Proof. Let us assume that $x \in \overline{Y}$. Then $B_{1/n}(x)$ is an open subset containing x and by definition of x being in $\overline{Y}$ we see $B_{1/n}(x) \bigcap Y \neq \emptyset$. Let $y_n \in B_{1/n}(x) \bigcap Y$. Then y_n is the required sequence.

Conversely, assume that $x \notin \overline{Y}$. This means that there is an open subset U containing x such that $U \bigcap Y = \emptyset$. There is $r > 0$ such that $B_r(x) \subset U$. Thus, $B_r(x) \bigcap Y = \emptyset$. Thus, there is no sequence of points $y_n \in Y$ such that $\lim_{n\to\infty} d(x, y_n) = 0$. □

Definition 6.2.2. *Let (X, d) be a metric space. We say a sequence of points x_n converges to x if for every $\epsilon > 0$, there is an N such that for all $n \geqslant N$, $x_n \in B_\epsilon(x)$. We denote this by $x_n \to x$.*

6.3 Continuous maps between metric spaces

Proposition 6.3.1. *Let X and Y be metric spaces and consider the induced topologies on both. A map $f : X \to Y$ is continuous iff for every sequence $x_n \to x$, the sequence $f(x_n) \to f(x)$.*

Proof. Let us assume that f is continuous. Let x_n be a sequence of points converging to x. By definition, this means that given any $\epsilon > 0$, there is an N such that for all $n \geqslant N$, $x_n \in B_\epsilon(x)$. Let us consider the open set $B_\epsilon(f(x))$. The inverse image of this $f^{-1}(B_\epsilon(f(x)))$ is open and contains x. Thus, there is δ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Thus, there is N such that for all $n \geqslant N$, $x_n \in B_\delta(x)$. This shows that for all $n \geqslant N$, $f(x_n) \in B_\epsilon(f(x))$, which proves that $f(x_n) \to f(x)$.

Conversely, suppose that for every sequence $x_n \to x$, the sequence $f(x_n) \to f(x)$. To show that f is continuous, it suffice to show that the inverse image of a closed subset is closed. Let Z be a closed subset of Y, and let $T := f^{-1}(Z)$. We will show that $T = \overline{T}$. Suppose $x \in \overline{T}$, this means that there is a sequence of points $x_n \in T$ such that $x_n \to x$. It is given that $f(x_n) \to f(x)$. Since Z is closed and $f(x_n) \in Z$, this implies by Theorem 6.2.1 that $f(x) \in Z$, which implies that $x \in T$. Thus, $T = \overline{T}$ and this shows that T is closed. □

Theorem 6.3.2 (Uniform limit theorem)**.** *Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$, for $n \geqslant 1$, be continuous functions. Let $f : X \to Y$ be a map. Suppose that for every $\epsilon > 0$, there is N such that for all $x \in X$ and for all $n \geqslant N$ we have $d(f_n(x), f(x)) < \epsilon$. Then f is continuous.*

Proof. Let $U \subset Y$ be open. We need to show that $W = f^{-1}(U)$ is open. Let $x_0 \in W$. Choose $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset U$. Now choose N large enough so that

$$d(f_n(x), f(x)) < \epsilon/3 \qquad \forall\, x \in X, \quad \forall\, n \geqslant N\,. \tag{6.3.3}$$

Since f_N is continuous, there is an open subset V such that $x_0 \in V$ and

$$f_N(V) \subset B_{\epsilon/3}(f_N(x_0)). \tag{6.3.4}$$

Let $x \in V$, then

1. $d(f(x_0), f_N(x_0)) < \epsilon/3$ by (6.3.3)
2. $d(f_N(x_0), f_N(x)) < \epsilon/3$ by (6.3.4)
3. $d(f_N(x), f(x)) < \epsilon/3$ by (6.3.3)

Adding these three and using triangle inequality we get that

$$d(f(x), f(x_0)) < \epsilon \qquad \forall\, x \in V\,.$$

This shows that $x_0 \in V \subset f^{-1}(B_\epsilon(f(x_0))) \subset f^{-1}(U)$, which proves that $f^{-1}(U)$ is open. □

Lemma 6.3.5. *Let X be a metric space and let C be a subspace. Define $d_C : X \to \mathbb{R}$ by*

$$d_C(x) := \inf_{y \in C} d(x, y).$$

The function d_C is continuous.

Proof. Suppose $z \in B_\delta(x)$. Then by triangle inequality

$$d(z, y) \leqslant d(z, x) + d(x, y).$$

For $y \in C$ we have $d_C(z) \leqslant d(z, y)$ and so we get

$$d_C(z) \leqslant d(z, y) \leqslant d(z, x) + d(x, y).$$

Taking infimum over $y \in C$ we get

$$d_C(z) \leqslant d(z, x) + \inf_{y \in C} d(x, y) = d(z, x) + d_C(x).$$

If $z \in B_\delta(x)$ then $x \in B_\delta(z)$, so we can repeat the above to get

$$d_C(x) \leqslant d(z, x) + d_C(z).$$

This give that

$$|d_C(x) - d_C(z)| \leqslant d(z, x) < \delta.$$

Thus, for any $\epsilon > 0$ we can take $\delta = \epsilon$ and we get that if $z \in B_\delta(x)$ then

$$|d_C(x) - d_C(z)| < \epsilon$$

which proves continuity. □

6.4 Exercises

6.4.1. Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be given by

$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

Prove that ρ defines a metric on $\mathbb{R}^n$. Further, show that the topology induced by ρ is same as the standard topology of $\mathbb{R}^n$.

6.4.2. Let X be metric space and let C be a subspace of X. Show that the following are equivalent:

(a) C is a closed subspace of X.

(b) $x \in C$ if and only if $d_C(x) = 0$.

6.4.3. Give an example of a sequence of continuous functions $f_n : X \to Y$ from a metric space X to a metric space Y such that $\lim_{n\to\infty} f_n(x)$ exists for each $x \in X$ and the function $f : X \to Y$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous.

Chapter 7

Connected and Path connected spaces

In this chapter we will understand how to formulate the intuitive notion of being "connected". For example, intuitively speaking, the subspace $(0,1) \cup \{3\} \subset \mathbb{R}$ should surely not be described as being connected, whereas, the subspace $[0,1]$ should be described as being connected.

7.1 Connected topological spaces

Definition 7.1.1. *Let X be a topological space. We say that X is disconnected if there is a proper and nonempty subset U which is both open and closed in X. Equivalently, there are nonempty open sets U and V which are disjoint and such that $X = U \bigsqcup V$.*

Definition 7.1.2. *We say a topological space is connected if it is not disconnected.*

Proposition 7.1.3. *Let X be a topological space which is disconnected. Let $U \subset X$ be a dense subset. Then U is disconnected.*

Proof. In this proposition U has the subspace topology. We will prove the converse. Since X is disconnected, there is a proper and nonempty subset $W \subset X$ which is both open and closed. Since U is dense in X, it has nonempty intersection with every nonempty open subset. Thus,

$$U = (U \bigcap W) \sqcup (U \bigcap W^c).$$

This shows that U is disconnected since both W and W^c are open. □

Corollary 7.1.4. *Let X be a topological space and let U be a subspace. If U is connected then $\overline{U}$ is connected.*

Proof. U is dense in $\overline{U}$. Now apply the previous proposition. □

Theorem 7.1.5. *The interval* $[0,1]$ *is connected.*

Proof. Let us assume that $[0,1]$ is not connected. Then there are disjoint and nonempty sets U and V which are both open and closed in $[0,1]$ and $[0,1] = U \bigsqcup V$. One of these contains 0, say U. Let

$$S := \{x \in [0,1] \mid [0,x] \subset U\}.$$

Let $a = \sup_{x \in S} x$. By the definition of supremum, there is a sequence of points x_n such that $[0, x_n] \subset U$ and $x_n \to a$. Since U is closed and each x_n is in U, Theorem 6.2.1 shows that $a \in U$. If $a = 1$ then we get a contradiction, since this means that $[0,1] \subset U$, which would mean that V is empty. Thus, it follows that $0 < a < 1$. Since U is also open, there is an $\epsilon > 0$ such that $B_\epsilon(a) \subset U$. Thus, $[0, a + \epsilon/2] \subset U$, which shows that $a + \epsilon/2 \in S$. This is a contradiction since a was the supremum. □

Slightly modifying the above proof yields that the subspace $[a,b] \subset \mathbb{R}$ is connected. This is left as Exercise 7.4.1.

Theorem 7.1.6. $\mathbb{R}$ *is connected.*

Proof. Let us assume that $\mathbb{R}$ is not connected. Then there exist disjoint and nonempty open subsets U and V such that $\mathbb{R} = U \bigsqcup V$. Let $a \in U$ and $b \in V$. Without loss of generality we may assume that $a < b$. Thus,

$$[a,b] = ([a,b] \bigcap U) \bigsqcup ([a,b] \bigcap V).$$

Note that $[a,b] \bigcap U \neq \emptyset$ since it contains a and $[a,b] \bigcap V \neq \emptyset$ since it contains b. This shows that the interval $[a,b]$ is not connected, which is a contradiction. □

Note that by an interval we mean any one of the following:

1. $\mathbb{R} = (-\infty, \infty)$,
2. $(-\infty, b)$ or $(-\infty, b]$ where $b \in \mathbb{R}$,
3. (a, ∞) or $[a, \infty)$ where $a \in \mathbb{R}$,
4. (a,b) or $[a,b)$ or $(a,b]$ where $a < b$ are in $\mathbb{R}$,
5. $[a,b]$ where $a \leqslant b$ are in $\mathbb{R}$ (if $a = b$ then this has only one point).

Proposition 7.1.7. *Let* $Y \subset \mathbb{R}$ *be a nonempty connected subspace. Then* Y *is an interval.*

Proof. Let $a = \inf_{y \in Y} y$ and let $b = \sup_{y \in Y} y$. Then we claim that $(a,b) \subset Y$. If not, then there is $c \in (a,b)$ which is not in Y. But then

$$Y = \Big(Y \bigcap (-\infty, c)\Big) \bigsqcup \Big(Y \bigcap (c, \infty)\Big).$$

This shows that Y is disconnected, which is a contradiction. Thus, $(a, b) \subset Y$ as claimed.

Let us consider the case when $a, b \in \mathbb{R}$. Then as a and b are the infimum and supremum, for every $y \in Y$ we have $a \leqslant y \leqslant b$, that is, $Y \subset [a, b]$. Thus, Y is one of the following: (a, b) or $(a, b]$ or $[a, b)$ or $[a, b]$. The cases when $a = -\infty$ or $b = \infty$ are done similarly and we leave these to the reader. □

Theorem 7.1.8. *If X and Y are connected topological spaces then $X \times Y$ with the product topology is connected.*

Proof. Recall from Lemma 4.3.3 that $x \times Y$, with the subspace topology from $X \times Y$, is homeomorphic to Y. In view of this, $x \times Y$ with the subspace topology is connected.

Assume that $X \times Y$ is disconnected. Let $X \times Y = W \sqcup W'$ be a disconnection. For $x \in X$, the subspace $x \times Y$ is contained in W or W'. If not then we get a disconnection for $x \times Y$, which is not possible since it is connected. Define

$$U := \{x \in X \mid x \times Y \subset W\} .$$

Since W is nonempty, it contains a point (x_0, y_0) and it follows that $x_0 \times Y \subset W$, and so $x_0 \in U$. Now consider $X \times y$. It follows that the intersection of this set with W is nonempty, since both contain (x_0, y). Since X is connected, $X \times y$ is connected, and so $X \times y \subset W$. Since this happens for every $y \in Y$, we see that $X \times Y \subset W$. This contradicts the assumption that $W' \neq \emptyset$. □

Corollary 7.1.9. $\mathbb{R}^n$ *is connected.*

Proposition 7.1.10. *Let $f : X \to Y$ be a continuous map and assume that X is connected. Then $f(X)$ (with the subspace topology from Y) is connected.*

Proof. If this is not the case, then there is a disconnection for $f(X)$. This means that there are open sets U and V in Y such that

$$f(X) = (f(X) \bigcap U) \bigsqcup (f(X) \bigcap V) .$$

This gives a disconnection for X since

$$X = f^{-1}(U) \bigsqcup f^{-1}(V) .$$

□

Let X be a topological space. Consider the following relation on the points of X. We say $x \sim y$ if there is a connected subspace of X which contains both x and y. Let us check that this is an equivalence relation. We need to check 3 conditions.

1. $x \sim x$: This is clear since the subspace $\{x\}$ is connected.

2. If $x \sim y$ then $y \sim x$: This is also clear.

3. If $x \sim y$ and $y \sim z$ then $x \sim z$: Since $x \sim y$, there is a connected subspace $U \subset X$ such that $x, y \in U$. Similarly, there is a connected subspace $V \subset X$ such that $y, z \in V$. Since $U \bigcap V \neq \emptyset$, as $y \in U \bigcap V$, it follows from Exercise 7.4.4 that $U \bigcup V$ is connected. As $x, z \in U \bigcup V$, it follows that $x \sim z$.

This equivalence relation breaks X into equivalence classes.

Proposition 7.1.11. *The equivalence classes are connected and closed subsets.*

Proof. Let U be an equivalence class. Fix $x_0 \in U$. Let $y \in U$ be any point. Then there is a connected subset $V_y \subset X$ such that $x, y \in V_y$. Clearly, every point in V_y is equivalent to x and so $V_y \subset U$ (since U is the equivalence class of x). Since this happens for every $y \in U$, we see that $\bigcup_{y \in U} V_y \subset U$. However, we also have that $U \subset \bigcup_{y \in U} V_y$, since $y \in V_y$ for every $y \in U$. This shows that $\bigcup_{y \in U} V_y = U$. Further $x \in \bigcap_{y \in U} V_y$ and by the Exercise 7.4.4, since each V_y is connected, we see that there union is connected. Thus, U is connected.

It follows from Corollary 7.1.4 that $\overline{U}$ is connected. This shows that every point of $\overline{U}$ is in the equivalence class of x. Since the equivalence class of x is U, it follows that $\overline{U} \subset U$. Thus, $U = \overline{U}$ and so U is closed by Proposition 5.2.2. □

Definition 7.1.12 (Components)**.** *Each equivalence class is called a component (or more pedantically connected component) of X.*

7.2 Path connected topological spaces

Definition 7.2.1 (Path connected)**.** *A topological space X is called path connected if for any two points x and y, there is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Proposition 7.2.2. *A path connected space is connected.*

Proof. Let X be path connected. Suppose X is not connected, let $X = U \bigsqcup V$ be a disconnection. Let $x \in U$ and $y \in V$ and let $\gamma : [0, 1] \to X$ be a path joining x and y. Then $[0, 1] = \gamma^{-1}(U) \bigsqcup \gamma^{-1}(V)$ is a disconnection for $[0, 1]$, which is a contradiction. □

Proposition 7.2.3. *The sphere S^n is path connected.*

Proof. Recall the stereographic projection from 4.2. Let p denote the north pole of the sphere, that is, the point $(0, 0, \dots, 0, 1)$ and let q denote the south pole of the sphere, that is, the point $(0, 0, \dots, 0, -1)$. The stereographic projection from p shows that $S^n \setminus p$ is homeomorphic to $\mathbb{R}^n$. Let

$$\pi_p : S^n \setminus p \to \mathbb{R}^n$$

denote this homeomorphism. If $x, y \in S^n \setminus p$, then there is a path $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = \pi_p(x)$ and $\gamma(1) = \pi_p(y)$. Then $\pi_p^{-1} \circ \gamma$ is a path joining x and y in S^n.

Now we show that p can be joined to $x \in S^n \setminus \{p, q\}$. Use stereographic projection from q. Since $p, x \in S^n \setminus q$, it follows as above that p and x can be joined by a path.

Now let $x, y \in S^n$. We claim that x and y can be joined by a path. If $x \neq p$ and $y \neq p$ then both of them are in $S^n \setminus p$ and can be joined by a path as described above. If one of them is equal to p, say $x = p$ and $y \neq p$ then choose $z \in S^n \setminus \{p, q\}$. As described above, there is a path from x to z and there is a path from z to y. It follows from Theorem 5.3.1 that there is a path from x to y. Finally when $x = y = p$ we can simply take the constant path.

Alternatively, we can do the following. Let $x, y \in S^n$ Suppose $x \neq -y$. Then consider the straight line path in $\mathbb{R}^{n+1}$ which joins x and y, and divide it by the norm. This path is given by

$$\gamma(t) := \frac{tx + (1-t)y}{\|tx + (1-t)y\|}\,.$$

To join x and $-x$, choose a different point y and join x with y and y with $-x$. □

Similar to connected components, we can define path components. Let X be a topological space. Consider the following relation on the points of X. We say $x \sim y$ if there is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. It is easy to check that this is an equivalence relation. This equivalence relation breaks X into equivalence classes.

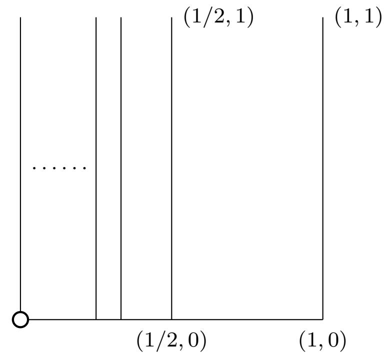
Definition 7.2.4 (Path components)**.** *Each equivalence class is called a path component.*

7.3 Topological space which is connected but not path connected

A connected topological space need not be path connected, as the following example will show. Let

$$\mathcal{C}_0 = \left\{\left(\frac{1}{n}, y\right) \,\middle|\, n \in \mathbb{Z}_{\geqslant 1},\ 0 \leqslant y \leqslant 1\right\} \bigcup \left\{(x, 0) \,\middle|\, 0 < x \leqslant 1\right\} \\ \bigcup \left\{(0, y) \,\middle|\, 0 < y \leqslant 1\right\}.$$

$\mathcal{C}_0 :=$



Let us first show that this topological space is connected. Note that $(0,0) \notin \mathcal{C}_0$. Consider $\mathcal{C}_0$ with the subspace topology from $\mathbb{R}^2 \setminus (0,0)$, which is the same as the topology coming from the metric on $\mathbb{R}^2$. Let us first consider the subset

$$\mathcal{C} := \left\{ \left(\frac{1}{n}, y\right) \,\middle|\, n \in \mathbb{Z}_{\geqslant 1},\ 0 \leqslant y \leqslant 1 \right\} \bigcup \left\{ (x,0) \,\middle|\, 0 < x \leqslant 1 \right\}.$$

It is easily seen that this subspace is path and hence connected. Thus, it's closure in $\mathbb{R}^2 \setminus (0,0)$ is connected, by Corollary 7.1.4. That $\mathcal{C}_0$ is contained in the closure of $\mathcal{C}$ can be checked in the following way. For any point $(0,y) \in \mathcal{C}_0$, the sequence $(\frac{1}{n}, y)$ is in $\mathcal{C}$ and converges to $(0,y)$. Thus, $\mathcal{C}_0$ is in the closure. One can easily check that the closure is exactly $\mathcal{C}_0$, by checking that if $(x,y) \in \mathbb{R}^2 \setminus (0,0)$ and is not in $\mathcal{C}_0$, then there is a small open ball around it which does not meet $\mathcal{C}$.

However, $\mathcal{C}_0$ is not path connected. The idea is that any path joining $(0,1)$ to $(1,1)$ will have to pass through $(0,0)$, which is not in $\mathcal{C}_0$. Let us write a formal proof.

We claim that for any path $\gamma : [0,1] \to \mathcal{C}_0$, such that $\gamma(0) = (0,1)$, the image is a subset of L_0, defined below.

$$(*) \qquad \gamma([0,1]) \subset \{(0,y) \mid 0 < y \leqslant 1\} =: L_0\,.$$

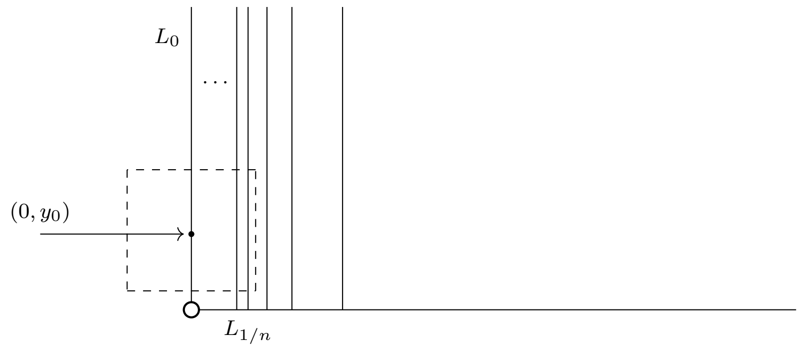
If we can show this, then this will mean that there is no continuous path which joins $(0,1)$ with $(1,1)$.

Let us first note that L_0 is a closed subset of $\mathcal{C}_0$. This is because it is equal to $\mathcal{C}_0 \bigcap \{x = 0\}$ and $\{x = 0\}$ is a closed subset of $\mathbb{R}^2$. Let

$$x_0 := \sup\{x \in [0,1] \mid \gamma([0,x]) \subset L_0\}.$$

There is a sequence $x_n \in [0, x_0)$ such that $x_n \to x_0$. Since γ is continuous and L_0 is closed, this shows that $\gamma(x_0) \in L_0$. If $x_0 = 1$ then $\gamma([0,1]) \subset L_0$ and this shows $(*)$. Assume $x_0 < 1$ and let $\gamma(x_0) = (0, y_0)$. Consider the open subset $B_{y_0/2}(0,y_0) \bigcap \mathcal{C}_0$. There is a $\delta > 0$ such that

$$\gamma(x_0 - \delta, x_0 + \delta) \subset B_{y_0/2}(0,y_0) \bigcap \mathcal{C}_0\,.$$



Since x_0 is the supremum, there is a $x_0 < t < x_0 + \delta$ such that $\gamma(t) \notin L_0$. This means that $\gamma(t) = (1/n, y)$ since $\gamma(t) \in B_{y_0/2}(0, y_0)$. Since $(x_0 - \delta, x_0 + \delta)$ is connected, the image under γ is connected. This shows that in $B_{y_0/2}(0, y_0) \bigcap \mathcal{C}_0$, the two sets $L_0 \bigcap B_{y_0/2}(0, y_0)$ and $L_{1/n} \bigcap B_{y_0/2}(0, y_0)$ are in the same connected component. But this is clearly a contradiction.

In the above example, there are two path components, but only one connected component.

7.4 Exercises

7.4.1. Show that any interval $[a, b]$ is connected.

7.4.2. Show that S^1 is not homeomorphic to $[0, 1]$.

7.4.3. Let X be a topological space and let $x \in X$. Show that the subspace $\{x\}$ is connected.

7.4.4. Let X be a topological space and let U_i, for $i \in I$, be a collection of subspaces of X. If each U_i is connected and $\bigcap_{i\in I} U_i \neq \emptyset$ then $\bigcup_{i\in I} U_i$ is connected.

7.4.5. Use a "straight line" path to show that $\mathbb{R}^n$ is path connected.

7.4.6. Let X be path connected and let $f : X \to Y$ be a continuous map. Then $f(X)$, with the subspace topology from Y, is path connected.

7.4.7. Let A and B be topological spaces. Show that $A \times B$ is connected if and only if A and B are connected.

7.4.8. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) \setminus (A \times B)$ is connected.

7.4.9. Let $\mathbb{R} \setminus \mathbb{Q}$ denote the set of all irrational numbers. Determine whether the following sets are connected:

(a) $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$,

(b) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$,

(c) $\mathbb{R}^2 \setminus \Big((\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})\Big) \subset \mathbb{R}^2$,

(d) $X = \{(x, y) \mid y = 0\} \cup \{(x, y) \mid x > 0 \text{ and } y = \frac{1}{x}\} \subset \mathbb{R}^2$.

7.4.10. Let $n > 1$. Show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$.

7.4.11. Show that $GL_n(\mathbb{C})$ is path connected. (HINT: Find a path joining a matrix A with I.)

7.4.12. In this exercise we will show that

$$G := GL_n(\mathbb{R})^+ := \{A \in GL_n(\mathbb{R}) \mid \det(A) > 0\}$$

is path connected.

(a) Let $A \in G$. Show that there is a matrix $B \in G$ such that A and B can be joined by a path (of course, the path has to be in G) and $B_{11} \neq 0$.

(b) Show that there is a matrix $C \in G$ of the type

$$\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}$$

such that B and C can be joined by a path.

(c) Show that in $GL_2(\mathbb{R})^+$ the matrices $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ can be joined to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(d) If $\lambda < 0$ show that $\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}$ can be joined to $\begin{bmatrix} -\lambda & 0 \\ 0 & D' \end{bmatrix}$. Clearly, $D' \in G$.

(e) If $\lambda > 0$ show that $\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}$ can be joined to $\begin{bmatrix} 1 & 0 \\ 0 & E \end{bmatrix}$. Clearly, $E \in GL_{n-1}(\mathbb{R})^+$.

(f) Complete the proof, that G is path connected, using induction, the base case being $n = 1$ where $GL_1(\mathbb{R})^+ = \mathbb{R}_{>0}$.

7.4.13. Use the previous exercise to show that $SL_n(\mathbb{R})$ is connected.

Chapter 8

Compactness

In this chapter we will study the properties of compact topological spaces. Compactness is a generalization of being closed and bounded in $\mathbb{R}^n$ (in the standard topology).

8.1 Compact topological spaces

Definition 8.1.1 (Hausdorff)**.** *A topological space is called Hausdorff if for any two points x, y there are open subsets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \bigcap U_y = \emptyset$.*

If X is Hausdorff and $Y \subset X$ is a subspace, then it follows immediately that Y is Hausdorff. Similarly, if X_i are Hausdorff spaces then so is their product. These easy assertions are left as exercises, see Exercise 8.4.1.

From now on, unless mentioned otherwise, X will be a Hausdorff topological space.

An open cover for a topological space X is a collection $\{U_i, i \in I\}$ of open subsets such that $X = \bigcup_{i\in I} U_i$. We say that an open cover has a finite subcover if there is a finite subset $J \subset I$ such that $X = \bigcup_{j\in J} U_j$.

Definition 8.1.2 (Compact)**.** *Let X be a Hausdorff topological space. We say X is compact if for every open cover of X, there is a finite subcover.*

Theorem 8.1.3. $[0, 1]$ *is compact.*

Proof. Suppose we are given an open cover U_i, $i \in I$ of $[0, 1]$. We need to show that this has a finite subcover. Define

$$S := \{x \in [0, 1] \mid [0, x] \text{ is covered by finitely many } U_i\}\,.$$

Then $S \neq \emptyset$ since $0 \in S$. There is an open subset U_{j_0} which contains 0. Thus, U_{j_0} contains $[0, x)$ for some $x > 0$. So, for example, $x/2 \in S$. Let $x_0 := \sup_{x\in S} x$. Clearly, $x_0 > 0$. Let us first show that $x_0 \in S$. There is some U_j such that $x_0 \in U_j$. Since U_j is open, there is an $\epsilon > 0$ such that

$$(x_0 - \epsilon, x_0 + \epsilon) \bigcap [0,1] \subset U_j \,.$$

There is a sequence of points $x_n \in S$ such that $x_n \to x_0$. There is some n such that $x_0 - \epsilon < x_n < x_0$. Then $[0, x_n]$ can be covered by finitely many U_i's and $[x_n, x_0] \subset U_j$. Thus, $[0, x_0]$ is covered by these finitely many U_i's and U_j.

If $x_0 = 1$ then we are done, as this shows that there is a finite subcover. If $x_0 < 1$ then we get a contradiction as follows. Since $x_0 \in U_j$, there is $\epsilon > 0$ such that $(x_0, x_0 + \epsilon) \subset U_j$. This shows that $[0, x_0 + \epsilon/2]$ can be covered by finitely many U_i's, which contradicts that x_0 is the supremum. □

8.2 Tube Lemma and products of compact spaces

Next we want to show that the product of two compact topological spaces is compact. We need the following important lemma for this.

Lemma 8.2.1 (Tube Lemma)**.** *Let X be a compact topological space. Let Y be any topological space. Suppose we are given an open subset $W \subset X \times Y$ such that $X \times y \subset W$. Then there is an open neighborhood V of y such that $X \times V \subset W$.*

Proof. Recall from Lemma 4.3.3 it follows that $X \times y$ is homeomorphic to X and so is compact. For every $x \in X$, there are open subset $x \in U_x \subset X$ and $y \in V_x \subset Y$ such that $(x, y) \in U_x \times V_x \subset Y$. Since X is compact and U_x cover X, there is a finite subcover, say by U_{x_i}. Let $V = \bigcap_{i=1}^r V_{x_i}$. Then

$$X \times y = \Big(\bigcup_{i=1}^r U_{x_i}\Big) \times y \subset \Big(\bigcup_{i=1}^r U_{x_i}\Big) \times V \subset \bigcup_{i=1}^r \Big(U_{x_i} \times V_{x_i}\Big) \subset W \,.$$

This proves that $X \times V \subset W$. □

Theorem 8.2.2. *Let X and Y be compact topological spaces. Then $X \times Y$ with the product topology is compact.*

Proof. By Exercise 8.4.1 the space $X \times Y$ is Hausdorff. Suppose we are given an open cover W_i, $i \in I$ for $X \times Y$. Let $y \in Y$. Then there is a finite subset $J_y \subset I$ such that

$$X \times y \subset \bigcup_{i \in J_y} W_i \,.$$

By Lemma 8.2.1, there is an open $V_y \subset Y$ such that $y \in V_y$ and $X \times V_y$ is contained in $\bigcup_{i \in J_y} W_i$. Thus, for every $y \in Y$, there is an open set $V_y \subset Y$ containing y such that $X \times V_y$ is contained in a finite subcover of the W_i's. Since Y is compact, finitely many of these V_y will cover Y. Thus, $X \times Y$ is covered by finitely many $X \times V_y$'s and each of these is covered by finitely many W_i's. Thus, we have found a finite subcover, which proves that $X \times Y$ is compact. □

Corollary 8.2.3. *$[0,1]^n$ is compact.*

Proposition 8.2.4. *Closed subspace of a compact space is compact.*

Proof. Let X be compact and suppose that $Y \subset X$ is a closed subspace. Let V_i be an open cover for Y. By the definition of subspace topology, for each i there is a U_i such that $V_i = U_i \bigcap Y$. Thus,

$$X = \Big(X \setminus Y\Big) \bigcup \bigcup_i U_i$$

is an open cover for X. Since X is compact, this has a finite subcover. Thus,

$$Y \subset X = \Big(X \setminus Y\Big) \bigcup \bigcup_{i=1}^{r} U_i\,.$$

Intersecting both sides with Y we see that Y is covered by finitely many V_i's. □

Proposition 8.2.5. *Let X be a topological space and let $Y \subset X$ be compact subspace. Then Y is closed in X.*

Proof. We will show that $U := X \setminus Y$ is open.

Let $x \in U, y \in Y$. Since X is Hausdorff, there are open subsets $V_x \subset X$ and $W_y \subset X$ such that $x \in V_y$ and $y \in W_y$ and $V_y \bigcap W_y = \emptyset$. Since $Y \subset \bigcup_{y \in Y} W_y$ and Y is compact, there is a finite subcover. Thus, $Y \subset \bigcup_{i=1}^r W_{y_i}$. Now consider the finite intersection $\bigcap_{i=1}^r V_{y_i}$. This is open since it is a finite intersection and it contains x. Note that

$$Y \bigcap \bigcap_{i=1}^{r} V_{y_i} \subset \Big(\bigcup_{i=1}^{r} W_{y_i}\Big) \bigcap \Big(\bigcap_{i=1}^{r} V_{y_i}\Big) = \emptyset\,.$$

This is because if there is a point $t \in \Big(\bigcup_{i=1}^r W_{y_i}\Big) \bigcap \Big(\bigcap_{i=1}^r V_{y_i}\Big)$ then $t \in W_{y_j}$ for some j, and then $t \in V_{y_j}$, which is not possible. Thus, $\Big(\bigcap_{i=1}^r V_{y_i}\Big)$ is an open set which contains x and is contained in U, which shows that U is open. □

Corollary 8.2.6. *Let $Y \subset \mathbb{R}^n$. Then Y is compact iff Y is closed and bounded.*

Proof. Assume that Y is compact. Then Proposition 8.2.5 shows that Y is closed. Since $Y \subset \mathbb{R}^n = \bigcup_{n \geqslant 1} B_{(0,0,\ldots,0)}(n)$ and Y is assumed to be compact, there is a finite subcover. This shows that Y is bounded.

Conversely assume that Y is closed and bounded. Then $Y \subset [0,x]^n$ for some x sufficiently large. Since $[0,x]$ is compact, as it is homeomorphic to $[0,1]$, and the product of compact spaces is compact, this shows that Y is a closed subspace of a compact space. Thus, Y is compact by Proposition 8.2.4. □

Proposition 8.2.7. *Let $f : X \to Y$ be a continuous map. Let $Z \subset X$ be a compact subspace. Then $f(Z) \subset Y$ is a compact subspace.*

Proof. Let $f(Z) \subset \bigcup_{i\in I} U_i$ be an open cover. Then $Z \subset \bigcup_{i\in I} f^{-1}(U_i)$ is an open cover for Z. Thus, there is a finite subcover $Z \subset \bigcup_{j=1}^{r} f^{-1}(U_{i_j})$. It follows that $f(Z) \subset \bigcup_{j=1}^{r} U_{i_j}$. Thus, $f(Z)$ is compact. □

Proposition 8.2.8. *Let $f : X \to Y$ be a bijective continuous map. If X is compact then f is a homeomorphism.*

Proof. Let $U \subset X$ be open. We need to show that $f(U)$ is open. The set $Z := X \setminus U$ is closed in X and so is compact. Thus, $f(Z)$ is compact and since Y is Hausdorff, $f(Z)$ is closed. Since f is a bijection, it follows that $f(U) = Y \setminus f(Z)$ which is open. □

We will not prove the following important Theorem. For a proof see [Mun00, Theorem 37.3].

Theorem 8.2.9 (Tychonoff)**.** *An arbitrary product of compact topological spaces is compact.*

8.3 Compact metric spaces

Theorem 8.3.1. *Let X be a metric space. Then X is compact iff every sequence has a convergent subsequence.*

Proof. First assume that X is compact and let $x_n \in X$ be a sequence. Let $S = \{x_n\}_{n\geqslant 1}$. If S is a finite set, then there is a $y \in Y_0$ such that there are infinitely many n such that $x_n = y$ and so we are done. Let us assume that S is infinite and we may further assume that the x_n's are distinct. Let $Y = \overline{\{x_n\}}_{n\geqslant 1}$. Then Y is a closed subspace of a compact space, and so is compact. If $x \in Y \setminus S$, then we are done, as it is clear that there is a subsequence converging to x. So let us assume that $Y = S$. Notice that all the x_n are distinct. If there is some x_j such that there is a subsequence converging to x_j, then we are done. Otherwise, this means that for each j there is an $\epsilon_j > 0$ such that $B_{\epsilon_j}(x_j) \bigcap Y$ contains only x_j. Then these form an open cover for Y which has no subcover, contradicting the compactness of Y.

Next let us assume that every sequence has a convergent subsequence and show that X is compact. Let U_i be an open cover for X. We claim that there is a $\delta > 0$, such that any ball $B_\delta(x)$, of radius δ, is contained in U_i for some i. Let us assume that this is not the case. Then for every $n > 0$ there is a ball $B_{1/n}(x_n)$ which is not contained in any U_i. The sequence x_n has a convergent subsequence, say $x_{n_j} \to x_0$. Now x_0 is in some U_{i_0} and so there is a $\delta > 0$ such that $B_\delta(x_0) \subset U_{i_0}$. Since $x_{n_j} \to x_0$, we get $x_{n_j} \in B_{\delta/2}(x_0)$ for $j \gg 0$. If $\frac{1}{n_j} < \delta/2$ then $B_{1/n_j}(x_{n_j}) \subset B_\delta(x_0) \subset U_{i_0}$, which is a contradiction. This proves the claim.

We use this claim to show that X is covered by finitely many closed balls $\overline{B_{\delta/2}(x)}$. Since each $\overline{B_{\delta/2}(x)} \subset B_\delta(x) \subset U_i$ for some i, it will follow that finitely many U_i's cover X. Choose any $y_1 \in X$ and define $X_1 := \overline{B_{\delta/2}(y_1)}$. Assume that we have defined X_n. If $X_n = X$ then stop, or else choose $y_{n+1} \in X \setminus X_n$ and define $X_{n+1} := X_n \bigcup \overline{B_{\delta/2}(y_{n+1})}$. If this process does not stop in finitely many steps, then we would have produced a sequence of point y_n such that $d(y_i, y_j) \geqslant \delta$ for all $i \neq j$. This sequence cannot have a convergent subsequence, which contradicts our assumption. □

Lemma 8.3.2 (Lebesgue number lemma)**.** *Let X be a compact metric space. Suppose we are given an open cover U_i. Then there is a $\delta > 0$, such that for every x, the ball $B_\delta(x)$ is contained in one of the U_i's.*

Proof. Since X is compact, it is covered by finitely many of the U_i's. Let us call these $U_1, U_2, \ldots, U_r$. Let C_i denote $X \setminus U_i$. Define a function

$$f(x) := \sum_{i=1}^{r} d_{C_i}(x)\,.$$

By Lemma 6.3.5, each $d_{C_i}(x)$ is continuous and so f is continuous. Since X is compact, the image is a compact subset. If $0 \in f(X)$ then it follows that each $d_{C_i}(x) = 0$, that is, $x \in C_i$ for all i, which is not possible. Thus, $0 \notin f(X)$. Since $f(X)$ is closed and $0 \notin f(X)$ it follows that there is $\delta > 0$ such that $\delta < f(X)$. It follows that at least one of the $d_{C_i}(x) > \delta/r$. Thus, for any x, there is an i such that the ball $B_{\delta/r}(x) \subset U_i$. □

Theorem 8.3.3 (Uniform continuity)**.** *Let X be a compact metric space and let $f : X \to \mathbb{R}$ be a continuous function. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$.*

Proof. Apply the previous Lemma to the cover

$$X = \bigcup_{t \in \mathbb{R}} f^{-1}(B_{\epsilon/2}(t))\,.$$

There is a $\delta > 0$ such that for every x, there is a t such that the ball $B_\delta(x) \subset f^{-1}(B_{\epsilon/2}(t))$. In particular, if $y \in B_\delta(x)$ then $|f(y) - t| < \epsilon/2$. Since this holds for $x \in B_\delta(x)$ we get

$$|f(x) - f(y)| \leqslant |f(x) - t| + |t - f(y)| < \epsilon\,.$$

This proves the theorem. □

8.4 Exercises

8.4.1. Let X_i, $i \in I$ be Hausdorff topological spaces. Show that $\prod_{i\in I} X_i$ is Hausdorff. Let X be a Hausdorff topological space and let $Y \subset X$ be a subspace. Show that Y is Hausdorff.

8.4.2. Show that S^n is compact.

8.4.3. Show that a finite union of compact subspaces of X is compact.

8.4.4. Show that if Y is compact, then the projection map $\pi_1 : X \times Y \to X$ is a closed map. (HINT: For a closed subset $Z \subset X \times Y$ show that the $X \setminus \pi_1(Z)$ is open.)

8.4.5. Let X and Y be topological spaces and Y be compact. Then $f : X \to Y$ is continuous if and only if the graph of f,

$$G_f = \{(x, f(x))|x \in X\},$$

is closed in $X \times Y$.

8.4.6. Let $p : X \to Y$ be a closed continuous map such that $p^{-1}(y)$ is compact for each $y \in Y$. Show that if Y is compact, then X is compact.

8.4.7. Show that orthogonal groups $O(n)$ are compact. Show that the unitary groups $U(n)$ are compact.

Chapter 9

Local compactness

A very important class of topological spaces is the class of locally compact topological spaces. This is obvious since $\mathbb{R}^n$ is locally compact, and open and closed subspaces of locally compact spaces are locally compact. These spaces are important not just because a lot of them occur "naturally" but also because one can prove many interesting theorems on these spaces. One such theorem is the Riesz Representation Theorem, see Rudin's Real and Complex Analysis, Chapter 2. We will, however, limit ourselves with the (somewhat boring) topic of one point compactification, which is the main result of this chapter.

9.1 Locally compact topological spaces

Definition 9.1.1. *A topological space X is called locally compact if for every $x \in X$, there is an open set U such that $x \in U$ and $\overline{U}$ is compact.*

Lemma 9.1.2. *$\mathbb{R}^n$ is locally compact.*

Proof. Obvious. $\square$

Proposition 9.1.3. *X is locally compact iff given any neighborhood W of x, there is an open set V such that $\bar{V}$ is compact and $x \in V \subset \bar{V} \subset W$.*

Proof. It is obvious that if this condition is satisfied then X is locally compact. Let us prove the converse.

First recall the proof of Proposition 8.2.5. We proved that if Y is a compact subset and $x \notin Y$, then there are open sets C and D such that $Y \subset C$, $x \in D$ and $C \bigcap D = \emptyset$. Let U be a neighborhood of x such that $\bar{U}$ is compact. Let $Y := \bar{U} \setminus W \bigcap U$. Since Y is a closed subspace of a compact space, it follows that Y is compact. Since $x \notin Y$, there are open sets $C \supset Y$ and $x \in D$ as mentioned. Intersecting D with U we may assume that $D \subset U$. It is clear that $\bar{D} \bigcap Y = \emptyset$. Now $\bar{D} \subset \bar{U}$ and $\bar{D} \bigcap Y = \emptyset$ implies that

$\bar{D} \subset (\bar{U} \setminus Y) = W \bigcap U$. Thus,

$$x \in D \subset \bar{D} \subset W \bigcap U \subset W\,.$$

Since $\bar{D} \subset \bar{U}$, it follows that $\bar{D}$ is compact. $\square$

9.2 One point compactification

Given a locally compact topological space which is not compact, we can compactify it in many ways. For example, take the interval $I := (-1, 1)$. This is a locally compact topological space. This is contained in $[-1, 1]$ which is compact. On the other hand, note that I is homeomorphic to $\mathbb{R}$ and via the stereographic projection $\mathbb{R}$ is homeomorphic to $S^1 \setminus \{(0, 1)\}$. Thus, there is a continuous inclusion $(-1, 1) \subset S^1$ and since S^1 is compact, we may view S^1 also as a compactification of I. However, as we have already seen, S^1 is not homeomorphic to $[-1, 1]$ and so these are two different compactifications. Given a locally compact topological space, we describe a "natural" way to compactify it.

Theorem 9.2.1 (One point compactification)**.** *Let X be a locally compact topological space which is not compact. Then there is a compact topological space $\hat{X}$ such that*

(1) $X \subset \hat{X}$ is an open subspace,

(2) $\hat{X}$ is compact and Hausdorff,

(3) $\hat{X} \setminus X$ is a point.

If Y is another topological space satisfying the above conditions, then Y is homeomorphic to $\hat{X}$.

Proof. Let $\hat{X}$ be the set $X \sqcup \{p\}$. For open subsets of $\hat{X}$ take those subsets $U \subset \hat{X}$ which satisfy any one of the following conditions

(1) U is $\emptyset$ or $\hat{X}$,

(2) $p \notin U$ and U is open in X,

(3) $p \in U$ and $\hat{X} \setminus U (= X \setminus U) \subset X$ is compact.

Let us check that this collection, call it $\mathcal{T}$, satisfies the conditions for being a topology on $\hat{X}$. Obviously, the empty set and $\hat{X}$ are in $\mathcal{T}$.

Let U_i be a finite collection in $\mathcal{T}$. First assume that $p \in \bigcap_i U_i$. In this case, for $\bigcap_i U_i$ to be open we need to check that

$$X \setminus \bigcap_i U_i = \bigcup_i X \setminus U_i$$

is compact. But this being a finite union of compact sets is compact. Next assume that $p \notin \bigcap_i U_i$. Then

$$\bigcap_i U_i = \bigcap_i \left(X \bigcap U_i \right).$$

If $p \notin U_i$, then $X \bigcap U_i = U_i$. However, if $p \in U_i$ then $X \bigcap U_i = U_i \setminus \{p\} \subset X$. Let Z denote $X \setminus U_i = X \setminus \left(U_i \setminus \{p\} \right)$. Since Z is compact, it is closed and it follows that $X \setminus Z = U_i \setminus \{p\} = X \bigcap U_i$ is open in X. It follows that each $X \setminus U_i$ is open in X. Since a finite intersection of open sets is open it follows that $\bigcap_i U_i$ is open in X and so is in $\mathcal{T}$.

Let U_i be an arbitrary collection of elements in $\mathcal{T}$. If $p \notin \bigcup_i U_i$ then $p \notin U_i$ and each U_i is open in X. Thus, $\bigcup_i U_i$ is open in X and so it is in $\mathcal{T}$. Next consider the case when $p \in \bigcup_i U_i$. This means that $p \in U_j$ for some j. Now

$$X \setminus \bigcup_i U_i = \bigcap_i X \setminus U_i \,.$$

Notice that $X \setminus U_i$ is always closed in X, since if $p \in U_i$ then $X \setminus U_i$ is compact and hence closed, and if $p \notin U_i$ then U_i is open in X and so $X \setminus U_i$ is closed. Since $X \setminus U_j$ is compact, and a closed subspace of a compact space is compact, it follows that $\bigcup_i U_i$ is in $\mathcal{T}$.

It is clear that $\hat{X} \setminus X$ is a point. Let us check that $\hat{X}$ is Hausdorff. If $x, y \in X$, then there are open sets U_x, U_y in X such that U_x contains x, U_y contains y and $U_x \bigcap U_y = \emptyset$. Clearly U_x and U_y are open in $\hat{X}$. Let $x \in X$. Since X is locally compact, there is a neighborhood U of x such that $\bar{U} \subset X$ is compact. Let $V := \hat{X} \setminus \bar{U}$. Then V is an open neighborhood of p which does not meet U. This shows that $\hat{X}$ is Hausdorff.

Let us check that $\hat{X}$ is compact. Suppose we are given an open cover U_i of $\hat{X}$. There is a j such that $p \in U_j$. Then $\hat{X} \setminus U_j$ is compact. There is a finite subcollection of U_i which covers $\hat{X} \setminus U_j$. This subcollection along with U_j covers all of $\hat{X}$. Thus, $\hat{X}$ is compact.

Let us check that $X \subset \hat{X}$ is a subspace. Every open subset of X is already open in the subspace topology. We need to check that if U is open in $\hat{X}$ then $U \bigcap X$ is open in X. If $p \notin U$ then this is clear. If $p \in U$, then as we have already seen, $X \bigcap U = X \setminus Z$, where $Z = X \setminus U$ is compact. Thus, $X \bigcap U$ is open in X.

Suppose Y is another topological space which satisfies these three conditions. Let $y = Y \setminus X$. Define a map $\Phi : \hat{X} \to Y$ by defining it to be identity on X and $\Phi(p) = y$. If $V \subset Y \setminus \{y\}$ then clearly $\Phi^{-1}(V)$ is open in $\hat{X}$. Let V be an open subset of Y which contains y. Then $Y \setminus V$ is a compact subspace of X. Thus, $\Phi^{-1}(Y \setminus V) = \hat{X} \setminus \Phi^{-1}(V)$ is a compact subspace of X, since Φ is the identity on X. Since $p \in \Phi^{-1}(V)$, it follows that $\Phi^{-1}(V)$ is open in $\hat{X}$. Thus, Φ is continuous. It now follows from Proposition 8.2.8 that Φ is a homeomorphism. $\square$

The one point compactification of a locally compact topological space has the following universal (see section 10.1 for what we mean by "universal") property.

Proposition 9.2.2. *Let X be a locally compact topological space which is not compact and let T be a compact topological space containing X as an open subspace. Then there is a unique continuous map $\Phi : T \to \hat{X}$ which is the identity on X and sends $T \setminus X$ to p.*

Proof. Let Φ be as in the statement of the proposition. If $V \subset \hat{X} \setminus \{p\}$ then clearly $\Phi^{-1}(V)$ is open in T. Let V be an open subset of $\hat{X}$ which contains p. Then $\hat{X} \setminus V$ is a compact subspace of X. Thus, $\Phi^{-1}(\hat{X} \setminus V) = T \setminus \Phi^{-1}(V)$ is a compact subspace of X. Thus, it is closed in T. It follows that $\Phi^{-1}(V)$ is open in T. Thus, Φ is continuous. □

9.3 Metric spaces which are not locally compact

There is an important class of spaces which are not locally compact. Let X be an infinite dimensional Hilbert space. We claim that X is not locally compact. Since X is a metric space, if it were locally compact, then there will be a neighborhood of 0 whose closure is compact. Thus, there is an $r > 0$ such that

$$D(0, r) := \{v \in H \mid \|v\| \leqslant r\}$$

is compact. Clearly, the linear map $H \to H$ given by multiplication by $1/r$ is continuous, thus, $D(0, 1)$ is also compact. Let $e_1, e_2, \ldots$ be an orthonormal set in H. Since $D(0, 1)$ is a compact metric space, it follows from Theorem 8.3.1 that every sequence has a convergent subsequence. However, the sequence $\{e_i\}$ has no Cauchy subsequence. This shows that H is not locally compact.

9.4 Exercises

9.4.1. Show that the rationals $\mathbb{Q}$ are not locally compact.

9.4.2. Let $X_1, \ldots, X_n$ be locally compact topological spaces. Show that $X_1 \times \ldots \times X_n$ is locally compact.

9.4.3. Let $\{X_\alpha\}$ be an indexed family of nonempty locally compact spaces. What condition should we put on $X'_\alpha s$ so that the product $\prod_\alpha X_\alpha$ is locally compact?

9.4.4. Show that closed subspace of a locally compact space is locally compact.

9.4.5. Let X be a locally compact space. Show that if $f : X \to Y$ is continuous and open, then $f(X)$ is locally compact.

9.4.6. If $f : X \to Y$ is a homeomorphism of locally compact Hausdorff spaces, show that f extends to a homeomorphism of their one-point compactifications.

9.4.7. Show that one point compactification of $\mathbb{Z}_+ := \{n \in \mathbb{Z} \mid n > 0\}$, is homeomorphic with the subspace $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

9.4.8. Show that the one point compactification of $\mathbb{R}^n$ is S^n.

9.4.9. Let $X = \{0\} \cup \{n + 1/n \mid n \geqslant 3\}$. Let $f : \mathbb{R} \to S^1$ denote the map $x \mapsto e^{2\pi i x}$. Show that the restriction $f|_X : X \to f(X)$ is a bijective continuous map of locally compact topological spaces, but f is not a homeomorphism.

Chapter 10

Quotient topology

10.1 Example of a universal property

Let us recall the following from group theory. Let G be a group and let H be a subgroup. Then we have the set G/H of left cosets of H in G. We also have a natural map $G \to G/H$, given by $g \mapsto gH$. One of the questions we address is when we can give a group structure to the set G/H, so that the natural map $\pi : G \to G/H$ becomes a group homomorphism. We know that this is possible iff H is a normal subgroup.

Further, if H is normal, then the map $\pi : G \to G/H$ has a certain "universal" property, which is the following. If $f : G \to G'$ is a group homomorphism such that $H \subset \mathrm{Ker}(f)$, then this group homomorphism factors uniquely through π, that is, there is a unique group homomorphism $\tilde{f}$ such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & G' \\ & G/H \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

We will describe a similar construction in topology.

10.2 Quotient topology

Let X be a set (without a topology for the time being) and let $\sim$ be an equivalence relation on X. Let Y be the set of equivalence classes $X/\sim$ and let $\pi : X \to Y$ denote the natural map which sends x to its equivalence class. The map π has the property that if $f : X \to Z$ is any map which is constant on each equivalence class, then f factors through π. This means that there exists $\tilde{f} : Y \to Z$ which makes the following diagram commute

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Now let us assume that X also has a topology. Suppose we want to give Y a topology so that π becomes a continuous map. This can be easily done, for example, give Y the trivial topology. However, we ask a more difficult question. Give Y a topology such that for every $f : X \to Z$ which is continuous and constant on equivalence classes, then induced map $\tilde{f}$ is also continuous. Apriori, it is not clear if such a topology exists on Y. In fact, such a topology exists and it is unique.

We define U to be open in Y if the set $\pi^{-1}(U)$ is open in X. With minimal effort one should be able to check that this defines a topology on Y and we leave this to the reader. Let us denote this topological space by Y_q.

Proposition 10.2.1. *$\pi : X \to Y_q$ is continuous.*

Proof. We need to check that if U is open in Y_q, then $\pi^{-1}(U)$ is open in X. But this is true by the definition of being open in Y_q. □

Proposition 10.2.2. *Let $f : X \to Z$ be a continuous map which is constant on equivalence classes. Then there is a unique continuous map $\tilde{f} : Y_q \to Z$ such that $f = \tilde{f} \circ \pi$.*

Proof. Let $f : X \to Z$ be a continuous map which is constant on equivalence classes. Let U be an open subset of Z. We need to check that $\tilde{f}^{-1}(U)$ is open in Y_q. By definition, this is true iff $\pi^{-1}(\tilde{f}^{-1}(U))$ is open in X. Since $\pi^{-1}(\tilde{f}^{-1}(U)) = f^{-1}(U)$ and f is continuous, it follows that $\tilde{f}^{-1}(U)$ is open in Y and that $\tilde{f}$ is continuous. □

Proposition 10.2.3. *Y_q is the unique topology on Y which has this universal property.*

Proof. Let Y_t be another topology on Y such that the natural map $\pi : X \to Y_t$ is continuous and has the required universal property. To emphasize the topology, let us denote this map by $\pi_t : X \to Y_t$. Let us consider the diagram

$$\begin{tikzcd} X \arrow[rr, "\pi_t"] \arrow[dr, "\pi"'] & & Y_t \\ & Y_q \arrow[ur, dashed, "\tilde{\pi}_t"'] & \end{tikzcd}$$

Note that $\tilde{\pi}_t$ is forced to be the identity map at the level of points, this is because the underlying set of both Y_q and Y_t is just the set of equivalence classes. Since $\tilde{\pi}_t$ is continuous, it follows that every open subset of Y_t is open in Y_q.

Next we consider the diagram

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\pi_t"'] & & Y_q \\ & Y_t \arrow[ur, dashed, "\tilde{\pi}"'] & \end{tikzcd}$$

Since π_t has the universal property, it follows that $\tilde{\pi}$, which is just the identity map, is continuous. This shows that every open subset in Y_q is open in Y_t, which proves that both the topologies are the same. $\square$

10.3 Grassmannians

As an application of quotient topology, let us construct the space of Grassmannians. First we need some preliminaries about topological groups.

Definition 10.3.1. *Let G be a group along with a topology. Let $G \times G$ have the product topology. We say that G is a topological group if the following two maps are continuous:*

1. $m : G \times G \to G \quad (x, y) \mapsto xy$
2. $i : G \to G \quad g \mapsto g^{-1}$

We remark that since $i^2 = Id$ it follows that i is bijective and a homeomorphism.

The main example of a topological group we have in mind is the group $GL(n, \mathbb{R})$. Let us check that this is a topological group. Let us check that group multiplication is a continuous map. Note that the two projections from $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$ are continuous. Further, the coordinate functions on $GL(n, \mathbb{R})$ are continuous. Thus, we see that the coordinate functions on $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$ given by $(A, B) \mapsto A_{ij}$ and $(A, B) \mapsto B_{ij}$ are continuous. To show that m is continuous, it suffices to show that the coordinate functions of m are continuous. But these coordinate functions are polynomials in terms of the A_{ij} and B_{ij}. Thus, it follows that m is continuous. Similarly, we can easily check that i is continuous.

The same proof as above shows that $GL(n, \mathbb{C})$ is a topological group.

Let G be a topological group. One easily checks that if $H \subset G$ is a subgroup, then H is a topological group with the subspace topology from G. Using this, it follows that the subgroups of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$, for example, $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $U(n)$, $SU(n)$ are all topological groups.

Lemma 10.3.2. *Let G be a topological group. Then the following hold:*

(1) The translation maps L_a, R_a are homeomorphisms

(2) *If $\{U\}$ is the collection of open sets containing the identity $e \in G$, then for any $a \in G$, the collection $\{aU\} = \{L_a(U)\} = \{Ua\} = \{R_a(U)\}$ is the collection of open subsets containing a.*

(3) *Let V be an open subset such that $e \in V$. Then there is an open subset U such that $e \in U$ and $U^2 := m(U,U)$ is contained in V.*

(4) *Let V be an open subset containing $x \in G$. Then there is an open set U containing $e \in G$ such that UxU is contained in V.*

Proof. Let $a \times G \subset G \times G$ have the subspace topology. Consider the composition $a \times G \hookrightarrow G \times G \xrightarrow{m} G$. This composite is L_a and is continuous as it is the composite of two continuous functions. Recall from Lemma 4.3.3 that the space $a \times G$ with the subspace topology is homeomorphic to G. This shows that L_a is continuous. Similarly, we see that $L_{a^{-1}}$ is continuous. As L_a and $L_{a^{-1}}$ are inverses of each other, it follows that L_a is a homeomorphism. Similarly, it follows that R_a is a homeomorphism. This proves (1).

(2) follows easily using (1).

Using the continuity of m, and since $(e,e) \in m^{-1}(V)$, it follows that there is a basic open subset $W_1 \times W_2$ containing (e,e) and contained in $m^{-1}(V)$. Letting $U = W_1 \cap W_2$ proves (3).

Consider the composite $G \times x \times G \subset G \times G \times G \to G$ given by $(a,x,b) \mapsto axb$. This is continuous. Again, the subspace topology on $G \times x \times G$ is the product topology on $G \times G$. If we denote the above map by ϕ, then it follows that there are open sets W_1 and W_2 containing e such that $W_1 \times x \times W_2 \subset \phi^{-1}(V)$. Letting $U = W_1 \cap W_2$ proves (4). $\square$

Lemma 10.3.3. *Let $H \subset G$ be a closed subgroup. Let $x \notin H$. There is an open set U containing e such that $UxU \cap H = \emptyset$.*

Proof. Since H is closed and $x \notin H$, there is an open set V such that $x \in V$ and $V \cap H = \emptyset$. Point (4) of the previous Lemma shows that there is an open set U such that $UxU \subset V$. Thus, $UxU \cap H = \emptyset$. $\square$

Proposition 10.3.4. *Let $H \subset G$ be a closed subgroup. Then G/H with the quotient topology is a Hausdorff space.*

Proof. Note that to talk about the quotient topology, we need an equivalence relation on G. It is clear that H defines an equivalence relation on G, namely, $x \sim y$ iff $x^{-1}y \in H$.

Suppose we are given two distinct elements of G/H, say, xH and yH. Since these are not equal, it follows that $x^{-1}y \notin H$. From the previous lemma it follows that there is an open set U containing e such that $Ux^{-1}yU \cap H = \emptyset$. Replacing U by $U \cap U^{-1}$ if necessary ($U^{-1} = i(U)$ is open and contains e) we may assume that $U = U^{-1}$. Thus, $U^{-1}x^{-1}yU \cap H = \emptyset$. One checks easily that $xUH \cap yUH = \emptyset$.

Let $\pi : G \to G/H$ denote the canonical map. We claim that $\pi(xUH)$ is an open subset. From the definition, we need to check that $\pi^{-1}(\pi(xUH))$ is an open subset. For any subset $A \subset G$ we have that $\pi^{-1}(\pi(A)) = \cup_{h\in H} Ah = AH$. In particular, $\pi^{-1}(\pi(xUH)) = xUHH = xUH$. Also note that $xUH = \cup_{h\in H} xUh$. Since translations are homeomorphisms, it follows easily that xUh is an open set. Thus, their union is also open. Thus, $\pi(xUH)$ is open and contains xH. Similarly, $\pi(yUH)$ is open and contains yH.

Finally we claim that $\pi(xUH) \cap \pi(yUH) = \emptyset$. Since π is surjective, for this it suffices to show that $\pi^{-1}(\pi(xUH) \cap \pi(yUH))$ is empty. But

$$\pi^{-1}(\pi(xUH) \cap \pi(yUH)) = \pi^{-1}(\pi(xUH)) \cap \pi^{-1}(\pi(yUH)) = xUH \cap yUH = \emptyset\,.$$

Thus, it follows that G/H is Hausdorff. □

We apply the above discussion to the group $G = GL(n, \mathbb{R})$ and the subgroup P (often called the Parabolic subgroup)

$$P = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \qquad A \in GL(r, \mathbb{R}), C \in GL(n-r, \mathbb{R})\,.$$

Clearly, P is a closed subgroup. We get the Hausdorff topological space G/P, which is often called the Grassmannian of r planes in $\mathbb{R}^n$. The name is motivated by the fact that the points in G/P are in bijection with the set of r-dimensional subspaces of $\mathbb{R}^n$. Let $\mathscr{S}$ denote the set of r-dimensional subspaces of $\mathbb{R}^n$. Let us check this. Define a map

$$\Phi : G \to \mathscr{S}$$

as follows. Given $A \in GL(n, \mathbb{R})$, let $\Phi(A)$ be the span of the first r column vectors of A. Given an r-dimensional subspace $V \subset \mathbb{R}^n$, choose a basis for V, say $v_1, \dots, v_r$. Extend this to a basis for $\mathbb{R}^n$, $v_1, \dots, v_n$. The matrix $A := [v_1|v_2|\dots|v_n]$ obtained by writing the v_i as column vectors is in $GL(n, \mathbb{R})$ and clearly $\Phi(A) = V$.

We claim that $\Phi(A) = \Phi(B)$ iff there is $T \in P$ such that $A = BT$. Let $A = [v_1|v_2|\dots|v_n]$ and let $B = [w_1|w_2|\dots|w_n]$. Then the v_i and w_i form a basis for $\mathbb{R}^n$. Thus, writing v_i in terms of the w_i we see that there is a matrix T such that $A = BT$. Taking determinant it is clear that $T \in GL(n, \mathbb{R})$. Let $1 \leqslant i \leqslant r$. Since v_i is in the span of $\langle w_1, \dots, w_r\rangle$, it follows that $T \in P$. This shows that the map Φ factors as

$$\begin{tikzcd} G \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & \mathscr{S} \\ & G/P \arrow[ur, "\Phi_0"'] & \end{tikzcd}$$

where Φ_0 is a bijection. Thus, using Φ_0 we can put a topology on $\mathscr{S}$. The space G/P is often denoted $Gr(n,r)$.

Theorem 10.3.5. *$Gr(n,r)$ is a Hausdorff, compact and path connected topological space.*

Proof. We have already seen that G/P is Hausdorff and so $Gr(n,r)$ is Hausdorff. To see compactness, consider the composite map $O(n,\mathbb{R}) \subset GL(n,\mathbb{R}) \xrightarrow{\pi} Gr(n,r)$. We claim that this composite map is surjective. Let $V \subset \mathbb{R}^n$ be an r-dimensional subspace. Then we can find an orthonormal basis for V, $v_1,\ldots,v_r$ and extend it to an orthonormal basis for $\mathbb{R}^n$, $v_1,\ldots,v_n$. The matrix $A = [v_1|v_2|\ldots|v_n]$ is in $O(n,\mathbb{R})$ and $\Phi(A) = V$. Thus, it follows that $Gr(n,r)$ is compact. If $det(A) = -1$ then replacing v_n by $-v_n$ we may further assume that $A \in SO(n,\mathbb{R})$. This shows that $SO(n,\mathbb{R})$ surjects onto $Gr(n,r)$. In the exercises it is proved that $SO(n,\mathbb{R})$ is connected. This proves that $Gr(n,\mathbb{R})$ is connected.

It remains to show that $Gr(n,r)$ is path connected. For that we first make a general observation. Let X be a connected space such that every point $x \in X$ has an open neighbourhood U with U path connected. Then we claim that X is path connected. To see this, fix a point $x_0 \in X$ and consider the set W containing those points which can be connected to x_0 using a path. Since $x_0 \in W$, it follows that W is nonempty. We claim W is open. Let $x \in W$. There is an open set U containing x such that U is path connected. Given any $y \in U$, we can find a path from y to x to x_0. This shows that y can be joined to x_0 using a path. Thus, $U \subset W$. This shows that W is open. Next we claim that W is closed. Suppose $x \notin W$. There is an open set U containing x such that U is path connected. We claim that $U \cap W$ is empty. If not, there is $y \in U \cap W$. Then y can be joined to x using a path, and y can be joined to x_0 using a path. It follows that x can be joined to x_0 using a path. It follows that $x \in W$, a contradiction. Thus, $U \cap W$ is empty and so $X \setminus W$ is also open. If $X \setminus W$ is nonempty, then this shows that X can be written as the disjoint union of two nonempty open subsets, contradicting the connectedness of X. It follows that W is all of X. Thus, every point of X can be connected to x_0, and so X is path connected.

We will use the idea in the preceding para to show that $Gr(n,r)$ is path connected. Since $Gr(n,r)$ is connected, it suffices to show that every point xP has a open neighbourhood which is path connected. For $A \in GL(n,\mathbb{R})$ consider left translation $L_A : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$. It is clear that $\pi \circ L_A$ is constant on equivalence classes, and so the map descends to give a commutative square in which all maps are continuous.

$$\begin{tikzcd} G \arrow[r, "L_A"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/P \arrow[r, "L_A"] & G/P \end{tikzcd}$$

Further, the set theoretic inverse of L_A is $L_{A^{-1}}$. It follows that L_A induces a homeomorphism of G/P. We will show that the coset $eP \in G/P$ has a neighbourhood U which is path connected. Then for any coset AP, the open set $L_A(U)$ is path connected and contains AP, which will prove the assertion.

Let $V \subset G$ be the set of matrices of the type

$$V := \begin{bmatrix} I_{r\times r} & 0 \\ * & I_{(n-r)\times(n-r)} \end{bmatrix}.$$

Clearly, V is homeomorphic to $\mathbb{R}^{r\times(n-r)}$. We claim that $\pi(V)$ is an open subset of $Gr(n,r)$. By definition we need to show that $\pi^{-1}(\pi(V)) = VP$ is open in G. We leave it to the reader to check that VP is the set of matrices $A \in G$ such that the $r\times r$ minor $(a_{ij})_{1\leqslant i,j\leqslant r}$ has nonzero determinant. Clearly, this is an open subset of G. Thus, it follows that $\pi(V)$ is open. As V is path connected, it follows that $\pi(V)$ is path connected. This completes the proof that $Gr(n,r)$ is path connected. □

Remark 10.3.6. We continue with the notation in the above proof. Since $V \cap P = Id$, it follows easily that π is bijective on V. We claim that $\pi : V \to \pi(V)$ is a homeomorphism. In this remark we will prove this claim. We have already observed that VP is an open subset of G. We claim that the map $V \times P \to VP$ is a homeomorphism. By looking at the coordinates, it is clear that this map is continuous. Let us construct the inverse. Let

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in VP.$$

Then $det(A) \neq 0$. We have an equality

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

It is clear that the maps

$$X \mapsto \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \qquad X \mapsto \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

are continuous. It follows that the inverse $VP \to V \times P$ is continuous. This proves that $V \times P \to VP$ is a homeomorphism. In particular, if U is an open subset of V then UP is open in VP and so in G. This shows that $\pi^{-1}(\pi(U))$ is open in G, that is, $\pi(U)$ is open in G/P. Thus, the map $\pi : V \to \pi(V)$ is an open map. This proves that the restriction of π is a homeomorphism from V to $\pi(V)$. This remark provides open subsets homeomorphic to $\mathbb{R}^n$ inside G/P which we can use to give it a manifold structure.

10.4 Exercises

10.4.1. Show that if G is a locally compact topological group and H is a closed subgroup, then G/H is locally compact (and Hausdorff).

10.4.2. Let G be a topological group. Let $H \subset G$ be a subgroup which is connected. Assume that G/H is connected in the quotient topology. Show that G is connected. (HINT: Use an idea which we used while showing that if X and Y are connected then $X \times Y$ is connected.)

10.4.3. $SO(n+1)$ acts on S^n in a natural way, what is this action? Show that the map

$$SO(n+1) \times S^n \to S^n$$

which defines the action is continuous. Show that this action is transitive. Now show that $SO(n+1)/SO(n)$ with the quotient topology is homeomorphic to S^n.

10.4.4. Use the preceding exercise and induction to show that $SO(n)$ is connected.

10.4.5. $U(n)$ acts on S^{2n-1} in a natural way, what is this action? (HINT: Identify $S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_i |z_i|^2 = 1\}$) Modify the above exercise to show that $U(n)$ is connected. Similarly, show that $SU(n)$ is connected.

10.4.6. Let X be the topological space $\mathbb{R}^{n+1} \setminus \{0\}$. The multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ acts on X by $\lambda \cdot (a_0, \ldots, a_n) := (\lambda a_0, \ldots, \lambda a_n)$. Define an equivalence relation on X by setting $x \sim y$ iff x and y are in the same orbit (check that this defines an equivalence relation). The space $X/\sim$ with the quotient topology is denoted $\mathbb{P}^n_{\mathbb{R}}$ and often referred to as the projective space of lines in $\mathbb{R}^{n+1}$. Observe that the points of $\mathbb{P}^n_{\mathbb{R}}$ are in bijection with the set of lines in $\mathbb{R}^{n+1}$. Show that $\mathbb{P}^n_{\mathbb{R}}$ is compact and path connected.

10.4.7. Let X be the topological space $\mathbb{C}^{n+1} \setminus \{0\}$. The multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ acts on X by $\lambda \cdot (a_0, \ldots, a_n) := (\lambda a_0, \ldots, \lambda a_n)$. Define an equivalence relation on X by setting $x \sim y$ iff x and y are in the same orbit (check that this defines an equivalence relation). The space $X/\sim$ with the quotient topology is denoted $\mathbb{P}^n_{\mathbb{C}}$ and often referred to as the projective space of (complex) lines in $\mathbb{C}^{n+1}$. Show that $\mathbb{P}^n_{\mathbb{C}}$ is compact and path connected.

10.4.8. In this exercise by k we will mean the field $\mathbb{R}$ or $\mathbb{C}$. Consider the group $G = GL(n+1, k)$ with the standard topology. Let H denote the "parabolic" subgroup of G consisting of invertible matrices whose first column is $(\lambda, 0 \ldots, 0)^t$, where $\lambda \in k^\times$. Show that G/H is homeomorphic to $\mathbb{P}^n_k$.

10.4.9 (Glueing open subsets)**.** This extremely important construction is used to construct all sorts of objects in mathematics: smooth manifolds, real analytics manifolds, complex manifolds, algebraic varieties, schemes etc.

Let I be a set. Let X_i, for each $i \in I$, be a collection of topological spaces. Assume that for each pair (i, j) we have the following:

(1) Open sets $X_{ij} \subset X_i$ such that $X_{ii} = X_i$

(2) Homeomorphisms $\varphi_{ij} : X_{ij} \to X_{ji}$ such that

   (a) $\varphi_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$,

   (b) On $X_{ij} \cap X_{ik}$ we have $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$.

Using the above data put an equivalence relation $\sim$ on the set $X := \bigsqcup_{i \in I} X_i$ as follows. For $x \in X_i$ and $y \in X_j$ define $x \sim y$ iff $x \in X_{ij}$ and $y = \varphi_{ij}(x)$. Let Y be the set $X/\sim$ equipped with the quotient topology. Recall that a subset $V \subset X$ is open iff $V \cap X_i$ is open in X_i for all i.

(i) Consider the canonical map $\psi_i : X_i \to Y$ which is the composite $X_i \to X \to Y$. Show that the image of ψ_i is an open subset and $\psi_i : X_i \to \psi_i(X_i)$ is a homeomorphism.

(ii) Let $V \subset Y$ be a subset such that $V \subset \psi_i(X_i)$. Show that V is open in Y iff $\psi_i^{-1}(V)$ is open in X_i.

(iii) Let $V \subset Y$ be a subset such that $V \subset \psi_i(X_i)$. Let V have the subspace topology from Y and let $g : V \to Z$ be a map to a topological space Z. Show that g is continuous iff $g \circ \psi_i : \psi_i^{-1}(V) \to Z$ is continuous.

Chapter 11

Urysohn's Lemma and Applications

Given a topological space, it is natural to study continuous maps from it to other topological spaces, in particular, continuous functions on the given topological space. Urysohn's Lemma says that on a normal topological space there are plenty of continuous functions, in fact, sufficiently many to separate disjoint closed subsets. Urysohn's Metrization Theorem gives a necessary condition for the topology to arise from a metric. Obviously, metric spaces being more "natural" than an abstract topological space, such a theorem is very desirable.

11.1 Normal spaces and Urysohn's Lemma

Definition 11.1.1. *A topological space X is called normal if for every two disjoint closed subsets A and B, there are open subsets U and V such that $A \subset U$, $B \subset V$ and $U \bigcap V = \emptyset$.*

Lemma 11.1.2. *Let X be a normal topological space. Let $A \subset U$ be sets such that A is closed and U is open. Then there is an open subset V such that $A \subset V \subset \bar{V} \subset U$.*

Proof. Let $B := X \setminus U$. Since X is normal, there are open subsets V and W such that $A \subset V$, $B \subset W$ and $V \bigcap W = \emptyset$. This shows that $A \subset V \subset X \setminus W \subset U$. Since $X \setminus W$ is closed we have

$$A \subset V \subset \bar{V} \subset X \setminus W \subset U\,.$$

This completes the proof of the Lemma. □

Proposition 11.1.3. *A metric space is normal.*

Proof. Let A and B be closed subsets. Recall the functions d_A and d_B. For each $a \in A$, let $\epsilon_a := d_B(a)/4$ and for each $b \in B$, let $\epsilon_b := d_A(b)/4$. Define

$$U := \bigcup_{a \in A} B(a, \epsilon_a), \qquad V := \bigcup_{b \in B} B(b, \epsilon_b)\,.$$

Clearly $A \subset U$ and $B \subset V$. We claim that $U \bigcap V = \emptyset$. If not, let $x \in U \bigcap V$. There are $a \in A$ and $b \in B$ such that $x \in B(a, \epsilon_a) \bigcap B(b, \epsilon_b)$. We may assume that $\epsilon_a \leqslant \epsilon_b$. Then using triangle inequality we get

$$d(a,b) \leqslant d(a,x) + d(b,x) \leqslant \epsilon_a + \epsilon_b \leqslant 2\epsilon_b = d_A(b)/2,$$

which is a contradiction. □

Proposition 11.1.4. *Let X be a metric space and let A and B be disjoint closed subsets. Then there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Let

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)}\,.$$

Now apply Exercise 6.4.2. □

The next Theorem is what Munkres describes as a "***deep***" result !! Indeed, we agree with the assessment of Munkres and the reader is encouraged to look up the description of a "deep" result, as mentioned in Munkres, before Urysohn's Lemma.

Theorem 11.1.5 (Urysohn's Lemma)**.** *Let X be a normal topological space and let A and B be disjoint closed subsets. Then there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

Proof. Suppose we have a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Then for each $q \in [0,1) \cap \mathbb{Q}$ we get the subset $U_q = f^{-1}([0,q))$ which is open in X and contained in $X \setminus B$. Note that if $q_1, q_2 \in [0,1) \cap \mathbb{Q}$ and $q_1 < q_2$ then $U_{q_1} \subset U_{q_2}$. As $f(\bar{U}_{q_1}) \subset \overline{f(U_{q_1})} \subset [0, q_1]$ we get that

$$U_{q_1} \subset \bar{U}_{q_1} \subset U_{q_2}\,.$$

Motivated by this, to prove the Theorem, we shall try to find such a sequence of open subsets of X and define f using these.

Define $U_1 = X \setminus B$. Since X is normal and A, B are disjoint closed subsets, with $A \subset U_1$, by applying Lemma 11.1.2, we may find U_0 such that

$$A \subset U_0 \subset \bar{U}_0 \subset U_1\,.$$

For every $p \in \mathbb{Q}$ we will define an open subset U_p, such that this collection satisfies

$$(*) \qquad p < t \qquad \Longrightarrow \qquad \bar{U}_p \subset U_t \, .$$

Step 1: Define open sets U_p for $p \in \mathbb{Q} \bigcap [0,1]$ which satisfy condition $(*)$.

Since $\mathbb{Q} \bigcap [0,1]$ is countable, we can index its elements by $\mathbb{N}$. Thus,

$$\mathbb{Q} \bigcap [0,1] = \{q_n\}_{n \geqslant 1} \, .$$

We may also choose $q_1 = 0$ and $q_2 = 1$. We have defined sets U_{q_1} and U_{q_2} which satisfy condition $(*)$. Let us assume that we have defined sets U_{q_r} which satisfy $(*)$ and then define $U_{q_{r+1}}$. Consider the set of rational numbers

$$\{q_1, q_2, \ldots, q_r, q_{r+1}\} \, .$$

We know that $q_1 < q_{r+1} < q_2$. There are unique $1 \leqslant i, j \leqslant r$ such that $q_i < q_{r+1} < q_j$. The sets U_{q_i} and U_{q_j} have already been defined and satisfy

$$\bar{U}_{q_i} \subset U_{q_j} \, .$$

Using Lemma 11.1.2, we can find $U_{q_{r+1}}$ such that

$$\bar{U}_{q_i} \subset U_{q_{r+1}} \subset \bar{U}_{q_{r+1}} \subset U_{q_j} \, .$$

It is clear that for all $x, y \in \{q_1, q_2, \ldots, q_r, q_{r+1}\}$ we have

$$(*) \qquad x < y \qquad \Longrightarrow \qquad \bar{U}_x \subset U_y \, .$$

Proceeding inductively we construct U_p for every $p \in \mathbb{Q} \bigcap [0,1]$.

Step 2: Define U_p for all $p \in \mathbb{Q}$ by setting $U_p = \emptyset$ for $p < 0$, and $U_p = X$ for $p > 1$.

Step 3: Define a function $f : X \to \mathbb{R}$ as follows:

$$f(x) := \inf\{p \in \mathbb{Q} \mid x \in U_p\} \, .$$

We need to check the following:

(1) $f(X) \subset [0,1]$,

(2) $f(A) = 0$,

(3) $f(B) = 1$,

(4) f is continuous.

For (1) note that for every $x \in X$, the set $\{p \in \mathbb{Q} \mid x \in U_p\}$ contains $(1, \infty)$, since if $p > 1$ then $U_p = X$. This shows that $f(x) \leqslant 1$ for all $x \in X$. Similarly, note that if $x \in U_p$, then $U_p \neq \emptyset$ and so $p \geqslant 0$. This shows that $f(x) \geqslant 0$ for all $x \in X$. This proves (1).

If $x \in A$, then $x \in U_0$ and so clearly $f(x) \leqslant 0$. But we also know that $f(x) \geqslant 0$, thus, $f(x) = 0$. This proves (2).

Let $x \in B$. If $f(x) < 1$ then there is $p \in \mathbb{Q}$ such that $f(x) < p < 1$ and $x \in U_p$. However, $x \in U_p \subset \bar{U}_p \subset U_1$. This shows that $x \in U_1 = X \setminus B$ and $x \in B$, which is a contradiction. Thus, $f(x) \geqslant 1$ and since $f(x) \in [0, 1]$ it follows that $f(B) = 1$. This proves (3).

Now let us show that f is continuous. It suffices to show that for any two rationals c, d such that $c < d$, we have $f^{-1}(c, d)$ is open in X. Let $x \in f^{-1}(c, d)$. There are rationals p, q such that

$$c < p < f(x) < q < d$$

We claim that

$$x \in U_q \setminus \bar{U}_p \qquad \text{and} \qquad U_q \setminus \bar{U}_p \subset f^{-1}(c, d)$$

As $U_q \setminus \bar{U}_p$ is open, it will follow that $f^{-1}(c, d)$ is open. Since $f(x) < q$, there is a $t \in \mathbb{Q}$ such that $f(x) < t < q$ and $x \in U_t \subset \bar{U}_t \subset U_q$. Since $p < f(x)$ there is $s \in \mathbb{Q}$ such that $p < s < f(x)$ and $x \notin U_s$. As $\bar{U}_p \subset U_s$ and $x \notin U_s$, it follows that $x \notin \bar{U}_p$. This shows that $x \in U_q \setminus \bar{U}_p$.

Now we prove the second part of the claim, that $U_q \setminus \bar{U}_p \subset f^{-1}(c, d)$. Suppose $y \in U_q$, then clearly $f(y) \leqslant q < d$. If $y \notin \bar{U}_p$ then $y \notin U_p$ and so $p \leqslant f(y)$. Thus, if $y \in U_q \setminus \bar{U}_p$ then we see that $f(y) \in [p, q] \subset (c, d)$. This shows that $U_q \setminus \bar{U}_p \subset f^{-1}(c, d)$. This proves that f is continuous. $\square$

Corollary 11.1.6 (Moral of Urysohn's Lemma)**.** *In a normal topological space, continuous functions can separate disjoint closed sets.*

Theorem 11.1.7 (Tietze Extension Theorem)**.** *Let X be a normal topological space and let A be a closed subset.*

1. *Let $f : A \to [-r, r]$ be a continuous function. Then we can extend f to a continuous function $X \to [-r, r]$.*

2. *Let $f : A \to \mathbb{R}$ be a continuous function. Then we can extend f to a continuous function $X \to \mathbb{R}$.*

Proof. Let I^r denote the interval $[-r, r]$. Divide I^r into three parts

$$I^r = I_1^r \bigcup I_2^r \bigcup I_3^r,$$

where $I_1^r = [-r, -r/3]$, $I_2^r = [-r/3, r/3]$ and $I_3^r = [r/3, r]$. Define a function $g_1 : X \to [-r, r]$ as follows. The subsets $f^{-1}(I_1^r)$ and $f^{-1}(I_3^r)$ are disjoint closed subsets of A and so disjoint closed subsets of X. By Urysohn's Lemma, there is a continuous function

$$g : X \to [-r/3, r/3],$$

such that $g(f^{-1}(I_1^r)) = -r/3$ and $g(f^{-1}(I_3^r)) = r/3$. Let us denote this function $g_1(x)$.

Next let us check that the function $g_1(x)$ satisfies, for $x \in A$,

$$\|f(x) - g_1(x)\|_\infty \leqslant 2r/3\,.$$

If $x \in f^{-1}(I_1^r)$, then $-r \leqslant f(x) \leqslant -r/3$ and $g_1(x) = -r/3$. Clearly

$$\|f(x) - g_1(x)\|_\infty \leqslant 2r/3\,.$$

Similarly, if $x \in f^{-1}(I_3^r)$. If $x \in f^{-1}(I_2^r)$, then both $f(x), g(x) \in [-r/3, r/3]$, so $\|f(x) - g_1(x)\|_\infty \leqslant 2r/3$ in this case too. Thus, in all cases we see that $\|f(x) - g_1(x)\|_\infty \leqslant 2r/3$. Thus, we have a function $g_1 : X \to [-r/3, r/3]$, such that on the set A,

$$f(x) - g_1(x) : X \to [-2r/3, 2r/3].$$

Divide $[-2r/3, 2r/3]$ into three equal parts as earlier and repeat the above construction with $f(x)$ replaced by $f(x) - g_1(x)$ on A. Doing this we get a function $g_2 : X \to [-2r/3^2, 2r/3^2]$ such that on the set A,

$$f(x) - g_1(x) - g_2(x) : X \to \Big[-\frac{2^2 r}{3^2}, \frac{2^2 r}{3^2}\Big]\,.$$

In particular, this means that for $x \in A$

$$\|f(x) - g_1(x) - g_2(x)\|_\infty \leqslant \frac{2^2 r}{3^2}.$$

In this way, we can construct a sequence of continuous functions

$$g_n : X \to [-2^{n-1}r/3^n, 2^{n-1}r/3^n]\,,$$

such that on A we have,

$$\|f(x) - g_1(x) - g_2(x) - \cdots - g_n(x)\|_\infty \leqslant \frac{2^n r}{3^n}\,.$$

If we define

$$F_n(x) := \sum_{i=1}^{n} g_i(x),$$

then clearly F_n converge uniformly to a function F. Applying Theorem 6.3.2 we see that F is continuous. It is clear that $F(x)$ agrees with $f(x)$ on the set A. Also

$$\|F(x)\| \leqslant \sum_{n\geqslant 1} \frac{2^{n-1}r}{3^n} = r\,,$$

which implies that $F : X \to [-r, r]$. This proves the first part of the theorem.

To prove the second part of the theorem, let $f : A \to \mathbb{R}$ be continuous. Let $\phi : \mathbb{R} \to (-1, 1)$ be a homeomorphism. Let $\tilde{f} := \phi \circ f$. Then

$$\tilde{f} : A \to (-1, 1) \subset [-1, 1]$$

and so using the previous part we may extend it to a continuous function $\tilde{F} : X \to [-1, 1]$ such that $\tilde{F} = \tilde{f}$ on the set A. Let

$$D := \tilde{F}^{-1}(-1) \bigcup \tilde{F}^{-1}(1)\,.$$

This is a closed subset which is disjoint from A. Thus, there is a function $h : X \to [0, 1]$ such that $h(D) = 0$ and $h(A) = 1$. The function $\tilde{F}(x)h(x)$ has image in $(-1, 1)$ and on A is equal to $\tilde{f}$. Thus, the function $\phi^{-1} \circ (\tilde{F}h)$ is the required function. □

11.2 Second countability

Definition 11.2.1. *A topological space is called second countable if it has a basis of countable cardinality.*

Proposition 11.2.2. *Let X be a metric space. Then X is second countable iff X has a countable dense subset.*

Proof. Suppose there is a countable basis $\mathscr{B}$. For each $U \in \mathscr{B}$ choose an element $x_U \in U$. We claim that the collection $\{x_U\}$ is dense in X. Let V be any open subset, then there is a basic open set $U \subset V$, and so $x_U \in V$. Thus, $\{x_U\} \cap V \neq \emptyset$, which proves $\{x_U\}$ is dense. Clearly this set is countable since $\mathscr{B}$ is countable.

Conversely, suppose that there is a countable dense set S. Let

$$\mathscr{B} := \{B(s, 1/n) \mid s \in S, n \geqslant 1\}\,.$$

Clearly $\mathscr{B}$ is a countable collection. We claim that $\mathscr{B}$ is a basis. Let U be any open subset and let $x \in U$. Then there is $n > 0$ such that $B(x, 1/n) \subset U$. Let $s \in B(x, 1/4n)$, then one checks easily that $B(s, 1/4n) \subset U$ and it is obvious that $x \in B(x, 1/4n)$. This shows that $\mathscr{B}$ is a basis. □

Corollary 11.2.3. *The Hilbert space*

$$l^2 := \{(a_1, a_2, \ldots) \mid a_j \in \mathbb{C}, \sum |a_j|^2 < \infty\}$$

is second countable.

Proof. By the preceding proposition, it suffices to prove that there is a countable dense set. Consider the collection S of those sequences such that only finitely many a_j's are nonzero and all the $a_j \in \mathbb{Q} + i\mathbb{Q}$. Suppose $\mathbf{b} = (b_1, b_2, \ldots) \in l^2$. Then there is $k \gg 0$ such that $\sum_{j \geqslant k} |b_j|^2 < \epsilon^2/4$. Find $a_j \in \mathbb{Q} + i\mathbb{Q}$ so that $\sum_{j=1}^{k-1} |a_j - b_j|^2 < \epsilon^2/4$. Letting $\mathbf{a} = (a_1, a_2, \ldots, a_{k-1}, 0, 0 \ldots)$, we see that

$$\|\mathbf{a} - \mathbf{b}\|_2^2 < \epsilon^2/2 < \epsilon^2 .$$

This shows that $S \bigcap B(\mathbf{b}, \epsilon) \neq \emptyset$. □

Corollary 11.2.4. *A compact metric space is second countable.*

Proof. For each $n \geqslant 1$ consider the open cover $X = \bigcup_{x \in X} B(x, 1/n)$. This has a finite subcover. Let S_n be the set containing the centers of these finitely many balls. Let $S = \bigcup_{n \geqslant 1} S_n$. Let $B(x, r)$ be any open subset. We claim that $B(x, r) \bigcap S \neq \emptyset$. Choose $n \gg 0$ such that $1/n < r/4$. Since $\bigcup_{s \in S_n} B(s, 1/n)$ is a cover of X (by the definition of S_n), it follows that there is $s \in S_n$ such that $B(x, r/4) \bigcap B(s, 1/n) \neq \emptyset$. From this one checks that $s \in B(x, r)$. □

11.3 Regular spaces

Definition 11.3.1. *A topological space X is called regular if for a point x and a closed subset A with $x \notin A$, there are open sets U and V such that $x \in U$, $A \subset V$ and $U \bigcap V = \emptyset$.*

Theorem 11.3.2. *A regular and second countable space is normal.*

Proof. Let $\mathscr{B}$ denote a countable basis for X. Let C and D be two closed subsets of X. For $x \in D$, by regularity, we can find a basic open set V_x such that $x \in V_x$ and $\bar{V}_x \bigcap C = \emptyset$. Since the basis is countable, we may index the collection of basic open sets $\{V_x\}$ by positive integers. Thus, we have found V_n such that $D \subset \bigcup_n V_n$ and $\bar{V}_n \bigcap C = \emptyset$. Define $W_k = \bigcup_{n=1}^{k} V_n$. Then

(1) $W_1 \subset W_2 \subset \cdots$,

(2) $D \subset \bigcup_n W_n$,

(3) $\bar{W}_n \bigcap C = \emptyset$.

Similarly, we can find open sets U_n such that

(1) $U_1 \subset U_2 \subset \cdots$,

(2) $C \subset \bigcup_n U_n$,

(3) $\bar{U}_n \bigcap D = \emptyset$.

Since D does not meet any of the $\bar{U}_k$, it follows that

$$D \subset \bigcup_{n \geqslant 1} (W_n \setminus \bar{U}_n) .$$

Similarly,

$$C \subset \bigcup_{n \geqslant 1} (U_n \setminus \bar{W}_n) .$$

We claim that

$$\Big(\bigcup_{n \geqslant 1} (U_n \setminus \bar{W}_n) \Big) \bigcap \Big(\bigcup_{n \geqslant 1} (W_n \setminus \bar{U}_n) \Big) = \emptyset .$$

If not, then there are x, j, k such that

$$x \in (U_j \setminus \bar{W}_j) \bigcap (W_k \setminus \bar{U}_k) .$$

If $j \leqslant k$, then we get a contradiction since $U_j \subset U_k \subset \bar{U}_k$. If $j \geqslant k$, then we get a contradiction since $W_k \subset W_j \subset \bar{W}_k$. □

11.4 Metrizable spaces

Lemma 11.4.1. *Let (X, d) be a metric space. Define d' on $X \times X$ as follows*

$$d'(x, y) = \min\{d(x, y), 1\} .$$

Then d' is a metric on X.

Proof. It is clear that $d'(x, x) = \min\{d(x, x), 1\} = 0$. If $d'(x, y) = 0$ then clearly $d(x, y) = 0$ and so $x = y$. Similarly, it is clear that $d'(x, y) = d'(y, x)$. Suppose $x, y, z \in X$ and $d(x, y) \geqslant 1$ or $d(y, z) \geqslant 1$. Then clearly

$$d'(x, z) \leqslant 1 \leqslant d'(x, y) + d'(y, z) .$$

If $d(x, y) < 1$ and $d(y, z) < 1$ then $d'(x, y) = d(x, y)$ and $d'(y, z) = d(y, z)$.

(1) If $d(x, y) + d(y, z) < 1$ then $d(x, z) \leqslant d(x, y) + d(y, z) < 1$. Thus,

$$d'(x, z) = d(x, z) \leqslant d(x, y) + d(y, z) = d'(x, y) + d'(y, z).$$

(2) Finally, if $d(x,y)+d(y,z) \geqslant 1$ then

$$d'(x,z) \leqslant 1 \leqslant d(x,y)+d(y,z) = d'(x,y)+d'(y,z)\,.$$

This shows that d' is a metric on X. $\square$

Since the ϵ balls are the same when $\epsilon < 1$, the metric d' induces the same topology on X as the metric d.

Let us now take $X = \mathbb{R}$ with the standard metric. Let

$$d'(x,y) := \min\{|x-y|, 1\}\,.$$

This modified metric induces the standard topology on $\mathbb{R}$, as remarked above. Let $\mathbb{R}^{\mathbb{N}} := \prod_{n\geqslant 1}\mathbb{R}$. Define the following metric on $\mathbb{R}^{\mathbb{N}}$

$$D(\mathbf{x},\mathbf{y}) := \sup_{i\in\mathbb{N}}\left\{\frac{d'(x_i,y_i)}{i}\right\}\,.$$

Lemma 11.4.2. *The function D is a metric.*

Proof. It is clear that $D(\mathbf{x},\mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$. It is also clear that $D(\mathbf{x},\mathbf{y}) = D(\mathbf{y},\mathbf{x})$. For $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^{\mathbb{N}}$ it is clear that for every $i \in \mathbb{N}$

$$\frac{d'(x_i,z_i)}{i} \leqslant \frac{d'(x_i,y_i)}{i} + \frac{d'(y_i,z_i)}{i}\,.$$

From this the triangle inequality for D follows by taking supremum. Here we use $\sup\{a_i + b_i\} \leqslant \sup\{a_i\} + \sup\{b_i\}$. $\square$

Theorem 11.4.3. *The metric D induces the product topology on $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let us first show that every open subset in the product topology is open in the topology induced by the metric. A basis for the product topology on $\mathbb{R}^{\mathbb{N}}$ is given by sets

$$U(\mathbf{x},k,\epsilon) = \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} \mid d'(x_j,y_j) < \epsilon \qquad \forall\ 1 \leqslant j \leqslant k\}.$$

It suffices to show that there is a $\delta > 0$ such that

$$\{\mathbf{y} \mid D(\mathbf{x},\mathbf{y}) < \delta\} \subset U(\mathbf{x},k,\epsilon).$$

Now $D(\mathbf{x},\mathbf{y}) < \epsilon/k$ implies that

$$\frac{d'(x_j,y_j)}{j} \leqslant \sup_{j\in\mathbb{N}}\left\{\frac{d'(x_j,y_j)}{j}\right\} < \epsilon/k\,.$$

If $j \leqslant k$ then this implies that $d'(x_j, y_j) < \epsilon j/k < \epsilon$. Thus,

$$\{\mathbf{y} \mid D(\mathbf{x}, \mathbf{y}) < \epsilon/k\} \subset U(\mathbf{x}, k, \epsilon).$$

This shows that every open subset in the product topology is open in the topology induced by the metric D.

Next let us show that every open subset in the topology induced by the metric is open in the product topology. Fix an open subset $\{\mathbf{z} \mid D(\mathbf{x}, \mathbf{z}) < \epsilon\}$. Choose k so that $1/k < \epsilon/2$ and let $\mathbf{y} \in U(\mathbf{x}, k, \epsilon/2)$. Then

$$\frac{d'(x_j, y_j)}{j} < d'(x_j, y_j) < \epsilon/2 \qquad\qquad 1 \leqslant j \leqslant k,$$

and for $j > k$

$$\frac{d'(x_j, y_j)}{j} \leqslant \frac{1}{j} < \frac{1}{k} < \epsilon/2\,.$$

From this it is clear that

$$D(\mathbf{x}, \mathbf{y}) := \sup_{i \in \mathbb{N}} \left\{ \frac{d'(x_i, y_i)}{i} \right\} \leqslant \epsilon/2 < \epsilon\,.$$

Thus, $U(\mathbf{x}, k, \epsilon/2) \subset \{\mathbf{z} \mid D(\mathbf{x}, \mathbf{z}) < \epsilon < 1\}$. This shows that every open subset in the topology induced by the metric is open in the product topology. This proves the theorem. □

Theorem 11.4.4 (Urysohn's Metrization Theorem)**.** *A regular topological space with a countable basis is metrizable.*

Proof. In view of the fact that $\mathbb{R}^{\mathbb{N}}$ with the product topology is metrizable, it suffices to give a continuous map $f : X \to \mathbb{R}^{\mathbb{N}}$ which is a bijection onto its image and a homeomorphism between X and $f(X)$ ($f(X)$ being given the subspace topology from $\mathbb{R}^{\mathbb{N}}$).

Let $\mathscr{B}$ be a countable basis for the topology on X. For every pair of basic open sets V and U such that $\bar{V} \subset U$, by Urysohn's lemma (since regular and second countable implies normal), there is a continuous function $f_{V,U} : X \to [0, 1]$ such that $f_{V,U}(\bar{V}) = 1$ and $f_{V,U}(X \setminus U) = 0$. This gives a countable collection of continuous functions and we can use these to define a continuous function

$$F : X \to \prod_{(V,U)} \mathbb{R}\,, \qquad\qquad x \mapsto \prod_{(V,U)} f_{V,U}(x)\,.$$

Let $x, y \in X$. Since X is Hausdorff there is a basic open set V such that $x \in V$ and $y \notin \bar{V}$. Since X is normal, there is a basic open set W such that $x \in W \subset \bar{W} \subset V$. By Urysohn's lemma there is a function $f_{W,V} : X \to [0, 1]$ such that $f_{W,V}(\bar{W}) = 1$ and

$f_{W,V}(X \setminus V) = 0$. This implies that $f_{W,V}(x) = 1$ and $f_{W,V}(y) = 0$. This shows that the function $F : X \to \mathbb{R}^{\mathbb{N}}$ is an inclusion.

It only remains to show that if $U \subset X$ is open, then $F(U)$ is open in $F(X)$ in the subspace topology. To do this it suffice to show that if $x \in U$, then there is an open set $Y \subset \mathbb{R}^{\mathbb{N}}$ such that $F(x) \in Y \bigcap F(X) \subset F(U)$. For this find basic open sets W, W' such that $x \in W \subset \bar{W} \subset W' \subset U$. Now consider the projection map $\pi_{W,W'}$,

$$\pi_{W,W'} : \prod_{(V,U)} \mathbb{R} \to \mathbb{R}$$

and define

$$Y := \pi^{-1}_{W,W'}(0, \infty).$$

Since $\pi_{W,W'} \circ F(x) = f_{W,W'}(x) = 1$, this shows that $F(x) \in Y \bigcap F(X)$. Suppose $F(y) \in Y$, then this means that $f_{W,W'}(y) > 0$, which shows that $y \in W' \subset U$. This shows that $Y \bigcap F(X) \subset F(U)$, which completes the proof of the theorem. $\square$

11.5 Exercises

11.5.1. Show that a closed subspace of a normal space is normal.

11.5.2. Show that if $\prod X_\alpha$ is normal then so is X_α.

11.5.3. Show that if $\prod X_\alpha$ is regular then so is X_α.

11.5.4. Show that every locally compact Hausdorff space is regular.

11.5.5. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

11.5.6. Let X be a locally compact hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable, then X has a countable basis?

11.5.7. Let X be a compact Hausdorff space that is union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable.

11.5.8. Show that the Tietze extension theorem implies the Urysohn lemma.

11.5.9. Let X and Y be normal and second countable spaces. Show that the same is true for $X \times Y$.

Chapter 12

Covering maps and Lifting Theorems

12.1 Covering maps

Definition 12.1.1 (Evenly covered neighborhood)**.** *Let $f : X \to Y$ be a continuous map. We say V is evenly covered by f if $f^{-1}(V) = \bigsqcup_i U_i$, where each $U_i \subset X$ is open and $f|_{U_i} : U_i \to V$ is a homeomorphism for every i.*

Let us understand the above definition by means of some examples.

1. Consider the map $f : \mathbb{R} \to S^1$ given by $f(x) = e^{2\pi i x}$. Inside S^1 consider open subsets (for $a, b \in \mathbb{R}$)
$$V_{a,b} := \{(\cos 2\pi\theta, \sin 2\pi\theta) \mid a < \theta < b\}$$
If $b - a < 1$ then the subsets $V_{a,b}$ are evenly covered by the map f. In fact, in this case
$$f^{-1}(V_{a,b}) = \bigsqcup_{n\in\mathbb{Z}} (a + n, b + n)$$
and the restriction of f from $(a + n, b + n) \to V_{a,b}$ is clearly a homeomorphism.

2. If $b-a > 1$ then the subset $V_{a,b}$ is not evenly covered. In fact, in this case $f^{-1}(V_{a,b}) = \mathbb{R}$. Since $\mathbb{R}$ is connected, if we write $\mathbb{R} = \bigsqcup_i U_i$, with each U_i open and nonempty, then this forces that the indexing set contains only one element and $\mathbb{R} = U_1$. Clearly $f : \mathbb{R} \to V_{(a,b)}$ is not bijective.

3. Consider the restriction of f to $\mathbb{R}_{>0}$. Clearly this map is still surjective. However, the point $1 \in S^1$ does not have a neighborhood which is evenly covered, which we now show. It is clear that an open subset of an evenly covered open set is also evenly

covered. Thus, if 1 had a neighborhood which is evenly covered, then there would be an $0 < \epsilon < 1/4$ such that $V_{-\epsilon,\epsilon}$ is evenly covered. Note that

$$f^{-1}(V_{-\epsilon,\epsilon}) = (0,\epsilon) \sqcup \bigsqcup_{n \in \mathbb{Z}_{>0}} (-\epsilon + n, \epsilon + n)$$

If we can write $f^{-1}(V_{-\epsilon,\epsilon})$ as a disjoint union of open subsets U_i such that each U_i is homeomorphic to $V_{-\epsilon,\epsilon}$, then it is forced that each U_i is connected. Thus, one of the U_i has to be $(0,\epsilon)$ and f restricted to this is not even surjective. This shows that 1 does not have a neighborhood which is evenly covered.

Definition 12.1.2 (Covering maps)**.** *A continuous map $f : X \to Y$ is called a covering map if every $y \in Y$ has an open neighborhood which is evenly covered by f.*

(1) The map $f : \mathbb{R} \to S^1$ given by $f(x) = e^{2\pi i x}$ is a covering map.

(2) The map $\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto e^z$ is a covering map.

(3) The restriction of the above map $f : \mathbb{R}_{>0} \to S^1$ is not a covering map, as 1 does not have a neighborhood which is evenly covered.

(4) The map $z \mapsto z^n$ from $\mathbb{C}^\times \to \mathbb{C}^\times$ is a covering map.

(5) If $f_i : X_i \to Y_i$ are covering maps then so is $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$.

(6) Let $f : X \to Y$ be a covering map. Let $Z \subset Y$. Then $f : f^{-1}(Z) \to Z$ is a covering map.

Proposition 12.1.3. *Let $f : X \to Y$ be a covering. Then f is open.*

Proof. Let $x \in X$ and let W be an open set containing x. Let $V \subset Y$ be an evenly covered neighborhood of $f(x)$. Then $f^{-1}(V) = \bigsqcup_i U_i$ and $f|_{U_i} : U_i \to V$ is a homeomorphism. Suppose $x \in U_j$, then $x \in U_j \cap W$. Now

$$f(U_j \cap W) = f|_{U_j}(U_j \cap W) \subset V\,.$$

This shows that $f(U_j \cap W)$ is an open subset of V which is contained in $f(W)$ and contains $f(x)$. Since V is open in Y, this shows that every point in $f(W)$ has an open neighborhood contained inside $f(W)$, that is, $f(W)$ is open. □

12.2 Lifting Theorems

Theorem 12.2.1 (Lifting paths to covers)**.** *Let $f : X \to Y$ be a covering map. Let $g : [0,1] \to Y$ be a continuous map. Let $g(0) = y_0$ and let $x_0 \in f^{-1}(y_0)$. Then there is a unique map $\tilde{g} : [0,1] \to X$ such that $\tilde{g}(0) = x_0$ and $f \circ \tilde{g} = g$.*

Proof. Let $Y = \bigcup_{y\in Y} V_y$ be an open cover, where V_y is an evenly covered neighborhood of y. Then $[0,1] = \bigcup_{y\in Y} g^{-1}(V_y)$. By Lemma 8.3.2 there is a $\delta > 0$ such that for every $t \in [0,1]$ we have $g([t-\delta, t+\delta]\bigcap[0,1]) \subset V_y$, for some $y \in Y$. Choose $n \gg 0$ such that $1/n < \delta$ and write

$$[0,1] = \bigcup_{k=0}^{n-1}[\frac{k}{n}, \frac{k+1}{n}]\,.$$

Note $g([0,1/n]) \subset V$ for an evenly covered neighborhood V_0. Since $y_0 = g(0)$ it follows that $y_0 \in V_0$. Write

$$f^{-1}(V_0) = \bigsqcup_i U_{0i}\,.$$

Since $x_0 \in f^{-1}(V_0)$, there is a j such that $x_0 \in U_{0j}$. We also know that $f : U_{0j} \to V_0$ is a homeomorphism. Let $s_0 : V_0 \to U_{0j}$ denote the inverse of this map. Clearly, $s_0(y_0) = x_0$. Define $\tilde{g} : [0,1/n] \to U_{0j}$ by

$$\tilde{g}(t) = s_0(g(t)).$$

Clearly, $\tilde{g}$ satisfies $\tilde{g}(0) = x_0$ and $f \circ \tilde{g} = g$ on $[0,1/n]$.

Let $x_1 := \tilde{g}(1/n)$ and $y_1 = g(1/n) = f(\tilde{g}(1/n))$. There is an open set V_1 which is evenly covered and such that $g([1/n,2/n]) \subset V_1$. Proceeding in the same way as above, write

$$f^{-1}(V_1) = \bigsqcup_i U_{1i}\,.$$

Since $x_1 \in f^{-1}(V_1)$, there is a j such that $x_1 \in U_{1j}$. We also know that $f : U_{1j} \to V_1$ is a homeomorphism. Let $s_1 : V_1 \to U_{1j}$ denote the inverse of this map. Clearly, $s_1(y_1) = x_1$. Define $\tilde{g}_1 : [1/n,2/n] \to U_{1j}$ by

$$\tilde{g}_1(t) = s_1(g(t))\,.$$

Clearly, $\tilde{g}_1$ satisfies $\tilde{g}_1(1/n) = x_1$ and $f \circ \tilde{g} = g$ on $[1/n,2/n]$. The maps $\tilde{g}$ and $\tilde{g}_1$ agree on $1/n$, and so by Theorem 5.3.1 we get a continuous map $\tilde{g} : [0,2/n] \to X$ which lifts g and $\tilde{g}(0) = x_0$. Proceeding in this fashion we can find a lift $\tilde{g}$ on the whole of $[0,1]$.

Next let us prove that the lift is unique. Suppose there are two lifts of g which satisfy $\tilde{g}(0) = x_0 = \tilde{h}(0)$. Let

$$t_0 := \sup\,\{x \in [0,1] \mid \tilde{g}(t) = \tilde{h}(t), \qquad 0 \leqslant t \leqslant x\}\,.$$

If $t_0 = 1$ then there is nothing to prove. Assume $t_0 < 1$. By continuity, since X is Hausdorff, we have $\tilde{g}(t_0) = \tilde{h}(t_0)$. Let V be an evenly covered open set containing $g(t_0)$. There is a $\delta > 0$ such that $g([t_0, t_0+\delta]) \subset V$. Then

$$f^{-1}(V) = \bigsqcup_i U_i\,.$$

There is a j such that $\tilde{t}_0 := \tilde{g}(t_0) = \tilde{h}(t_0) \in U_j$. Let T denote the connected component of t_0 in $f^{-1}(V)$. It is clear that T is completely contained in U_j. It follows that

$$\tilde{g}([t_0, t_0+\delta]) \subset U_j, \qquad \tilde{h}([t_0, t_0+\delta]) \subset U_j\,.$$

Let us view $\tilde{g}, \tilde{h} : [t_0, t_0+\delta] \to U_j$. We know that $f : U_j \to V$ is a homeomorphism. Let s denote the inverse. Then on $[t_0, t_0+\delta]$,

$$f \circ \tilde{g} = f \circ \tilde{h}\,.$$

Applying s to both sides we get,

$$\tilde{g} = s \circ f \circ \tilde{g} = s \circ f \circ \tilde{h} = \tilde{h}$$

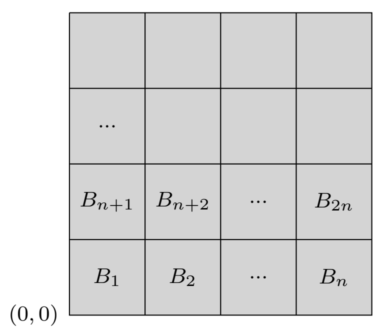
on $[t_0, t_0+\delta]$. This shows that $\tilde{g}$ and $\tilde{h}$ agree on $[0, t_0+\delta]$, which is a contradiction. This completes the proof of the theorem. □

Theorem 12.2.2 (Lifting homotopies to covers)**.** *Let $I := [0,1]$. Let $f : X \to Y$ be a covering map. Let $F : I \times I \to Y$ be a continuous map. Let $y_0 := F(0,0)$ and let $x_0 \in f^{-1}(y_0)$. Then there is unique a map $\tilde{F} : I \times I \to X$ such that $\tilde{F}(0,0) = x_0$ and $f \circ \tilde{F} = F$.*

Proof. The proof is very similar to that of the previous theorem. Cover Y by evenly covered open sets and pull this back using F to get an open cover of $I \times I$. Denote by $S_\delta(a,b)$ the set

$$\{(x,y) \in I \times I \mid |x-a| \leqslant \delta, |y-b| \leqslant \delta\}\,.$$

Now using Theorem 8.3.2 find a $\delta > 0$ such that for any $(a,b) \in I \times I$ the image $F(S_\delta(a,b))$ is contained in an evenly covered neighborhood. Finally choose $n \gg 0$ such that $1/n < \delta$.



Divide $I \times I$ into squares of side length $1/n$ and number them as shown in the diagram. By construction $F(B_i)$ is contained in an open set which is evenly covered. Let us first lift F on B_1. There is an open subset V_1 which is evenly covered such that

$$F(B_1) \subset V_1.$$

Write

$$f^{-1}(V_1) = \bigsqcup_i U_{1i}\,.$$

Since $y_0 \in V_1$ it follows that $x_0 \in U_{1j}$ for some j. Let $s_1 : V_1 \to U_{1j}$ denote the inverse of $f|_{U_{1j}}$. Then clearly

$$\tilde{F} := s_1 \circ F$$

lifts F on B_1 and satisfies $\tilde{F}(0,0) = x_0$.

Let $Z_l := \bigcup_{i=1}^{l} B_l$ and assume that we have lifted $F|_{Z_l}$ to $\tilde{F} : Z_l \to X$. It is clear that for every l the subspace $T := B_{l+1} \bigcap Z_l$ is connected. There is an evenly covered neighborhood V_{l+1} such that

$$F(B_{l+1}) \subset V_{l+1}.$$

Let $t_0 \in T$ be a point. Write

$$f^{-1}(V_{l+1}) = \bigsqcup_i U_{l+1,i}\,.$$

Then $\tilde{F}(t_0) \in U_{l+1,j}$ for some j. Let $s_{l+1} : V_{l+1} \to U_{l+1,j}$ denote the inverse of the restriction of f to $U_{l+1,j}$. Since T is connected, it follows that $\tilde{F}(T) \subset U_{l+1,j}$. There is a commutative diagram

$$\begin{tikzcd}
& U_{l+1,j} \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "f"] \\
T \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & V_{l+1} \arrow[u, bend right, "s_{l+1}"'] \arrow[r, hook] & Y
\end{tikzcd}$$

Since the left vertical arrow is a homeomorphism, it follows that $\tilde{F}|_T = s_{l+1} \circ F|_T$. Consider the function $h : B_{l+1} \to X$ given by $h = s_{l+1} \circ F$. It is clear that h and $\tilde{F}$ agree on T. Thus, by Theorem 5.3.1, it follows that we have defined a continuous map $\tilde{F}$ on Z_{l+1}. Proceeding in this fashion, $\tilde{F}$ can be defined on all of $I \times I$.

If possible, let $\tilde{G}$ be another lift of F such that $\tilde{G}(0,0) = x_0$. By unique lifting of paths, it follows that they agree on the sets $I \times 0$ and $0 \times I$. But now applying unique path lifting to the path $x \times I$, it follows that they agree on all of $I \times I$. $\square$

The most important application of the above theorem will be the following corollary.

Lemma 12.2.3. *Let $f : X \to Y$ be a covering map. Let $F : I \times I \to Y$ be a continuous map. Let $y_0 := F(0,0)$ and let $x_0 \in f^{-1}(y_0)$. Let $\tilde{F}$ be the unique lift such that $\tilde{F}(0,0) = x_0$. If $F(0 \times I) = y_0$ then $\tilde{F}(0 \times I) = x_0$. Similarly, if $F(1 \times I) = y_1$ then $\tilde{F}(1 \times I) = x_1$.*

Proof. Since $\tilde{F}$ is a lift of F, it follows that $\tilde{F}(0 \times I) \subset f^{-1}(y_0)$. Clearly, $f^{-1}(y_0)$ has the discrete topology. Since $0 \times I$ is connected, it follows that $\tilde{F}(0 \times I) = x_0$. The proof for $1 \times I$ is the same. $\square$

Remark 12.2.4. The meaning of this Lemma is the following. Suppose we have a family of paths $F_s(t) := F(t,s)$ inside Y such that each F_s starts at y_0 and ends at y_1. Then the unique lift $\tilde{F}$ is a family of paths $\tilde{F}_s(t) = \tilde{F}(t,s)$ which has the same property, that is, each $\tilde{F}_s$ starts at x_0 and ends at x_1.

Chapter 13

The Fundamental Group

13.1 Fundamental group

Let X be a topological space and let $x_0 \in X$.

Definition 13.1.1. *The space of loops in X based at x_0 is the set*

$$L(X, x_0) := \{\gamma : S^1 \to X \mid \gamma \text{ continuous and } \gamma(1) = x_0\}.$$

Since S^1 is homeomorphic to $[0,1]/\{0 \sim 1\}$ giving a map $\gamma : S^1 \to X$ is equivalent to giving a map $[0,1] \to X$ which takes the same value at 0 and 1. We will often use this fact. In view of this we may write,

$$L(X, x_0) = \{\gamma : [0,1] \to X \mid \gamma \text{ continuous and } \gamma(0) = \gamma(1) = x_0\}.$$

Definition 13.1.2. *Let $f, g \in L(X, x_0)$. A homotopy F between f and g is a continuous map $F : S^1 \times I \to X$ such that $F_t := F|_{S^1 \times t} \in L(X, x_0)$ for all $t \in I$, $F_0 = f$ and $F_1 = g$.*

Define a relation $\sim$ on $L(X, x_0)$ by $f \sim g$ if f and g are homotopic.

Proposition 13.1.3. *$\sim$ is an equivalence relation on $L(X, x_0)$.*

Proof. $f \sim f$ by taking $F(z,t) = f(z)$.

$f \sim g$ implies $g \sim f$ by taking $G(z,t) = F(z, 1-t)$.

Suppose $f \sim g$ and $g \sim h$. Let F be a homotopy between f and g and let G be a homotopy between g and h. Define a homotopy H between f and h by setting

$$H(z,t) := \begin{cases} F(z, 2t) & 0 \leqslant t \leqslant 1/2 \\ G(z, 2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

At $t = 1/2$ since $F(z,1) = g = G(z,0)$ it follows that H is a continuous function from $S^1 \times I$ to X. $\square$

Definition 13.1.4. *Let $\pi_1(X, x_0)$ denote the set of equivalence classes in $L(X, x_0)$ under the relation $\sim$. The equivalence class of f will be denoted by $[f]$.*

Define a binary operation $*$ on the space of loops as follows. For $f, g \in L(X, x_0)$ define $f * g$ by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leqslant t \leqslant 1/2 \\ g(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

Since $f(1) = g(0) = x_0$ it follows that $f * g \in L(X, x_0)$.

Proposition 13.1.5. *The binary operation $*$ descends to a binary operation*

$$* : \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)\,.$$

Proof. Suppose $f_1 \sim f_2$. Let F be a homotopy between f_1 and f_2. We leave it as an exercise to show that there is a homotopy H between $f_1 * g$ and $f_2 * g$ such that $H_t = F_t * g$.

Similarly, if $g_1 \sim g_2$ then $f * g_1 \sim f * g_2$. Thus,

$$[f_1 * g_1] = [f_2 * g_1] = [f_2 * g_2],$$

which proves the proposition. □

Proposition 13.1.6. *The binary operation $*$ on $\pi_1(X, x_0)$ is associative.*

Proof. Let $f, g, h \in L(X, x_0)$. We need to show that $(f * g) * h \sim f * (g * h)$. Let $G : I \to I$ be the following map.

$$G(t) := \begin{cases} 2t & 0 \leqslant t \leqslant 1/4 \\ t + 1/4 & 1/4 \leqslant t \leqslant 1/2 \\ t/2 + 1/2 & 1/2 \leqslant t \leqslant 1 \end{cases}$$

Define $H(t, s) = (f * (g * h))(sG(t) + (1 - s)t)$. Then $H(t, 0) = f * (g * h)$ and $H(t, 1) = (f * g) * h$. Further, for every s we have,

$$H(0, s) = f * (g * h)(0) = x_0 = f * (g * h)(1) = H(1, s)\,.$$

Thus, we get $H_s \in L(X, x_0)$ for every s. This shows that H is a homotopy between $f * (g * h)$ and $(f * g) * h$. □

Let $c_{x_0} : I \to X$ denote the constant map x_0.

Proposition 13.1.7. *In $\pi_1(X, x_0)$ we have $[f * c_{x_0}] = [c_{x_0} * f] = [f]$.*

Proof. Let us show that $f * c_{x_0} \sim f$. Let $G : I \to I$ be the following map.

$$G(t) := \begin{cases} 2t & 0 \leqslant t \leqslant 1/2 \\ 1 & 1/2 \leqslant t \leqslant 1 \end{cases}$$

Define $H(t,s) = f(sG(t) + (1-s)t)$. Then $H(t,0) = f$ and $H(t,1) = f * c_{x_0}$. We leave the remaining check that H is a homotopy between f and $f * c_{x_0}$ to the reader.

Similarly, we can show that $c_{x_0} * f \sim f$. Let $G : I \to I$ be the following map.

$$G(t) := \begin{cases} 1 & 0 \leqslant t \leqslant 1/2 \\ 2t-1 & 1/2 \leqslant t \leqslant 1 \end{cases}$$

Define $H(t,s) = f(sG(t) + (1-s)t)$. Then $H(t,0) = f$ and $H(t,1) = c_{x_0} * f$. We leave the remaining check that H is a homotopy between f and $c_{x_0} * f$ to the reader. $\square$

Proposition 13.1.8. *For $f \in L(X, x_0)$, there is $g \in L(X, x_0)$ such that $f*g \sim c_{x_0} \sim g*f$.*

Proof. Define $I(f)(t) = f(1-t)$. It suffices to show that $f*I(f) \sim c_{x_0}$, since by symmetry $f = I(I(f))$ and it will follow that $I(f) * f \sim c_{x_0}$. Define

$$H(t,s) := \begin{cases} f(2st) & 0 \leqslant t \leqslant 1/2 \\ f(2s-2st) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

Then $H(t,0) = c_{x_0}$, $H(t,1) = f * I(f)$ and $H(0,s) = x_0 = H(1,s)$. This completes the proof of the proposition. $\square$

The above propositions put together give the following theorem.

Theorem 13.1.9. *The set $\pi_1(X, x_0)$ is a group under the operation $*$ with identity element c_{x_0}.*

Having defined the fundamental group, we try to explore what continuous maps do to these. Let $f : X \to Y$ be a continuous map. Define a map $f_* : L(X, x_0) \to L(Y, f(x_0))$ by $f_*(\gamma) := f \circ \gamma$. If F is a homotopy between γ_1 and γ_2, then $f \circ F$ is a homotopy between $f_*(\gamma_1)$ and $f_*(\gamma_2)$. Thus, we get a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

Proposition 13.1.10. *f_* is a group homomorphism.*

Proof. $f_*(a * b) = f \circ (a * b) = (f \circ a) * (f \circ b) = f_*(a) * f_*(b)$. $\square$

Proposition 13.1.11. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then $g_* \circ f_* = (g \circ f)_*$.*

Proof. Obvious. $\square$

13.2 Fundamental groups of some spaces

13.2.1 $\pi_{\mathbf{1}}(\mathbb{R}, \mathbf{0})$. Let us compute the fundamental group of the real line. Consider the map $F : \mathbb{R} \times I \to \mathbb{R}$ given by

$$F(x,t) = (1-t)x\,.$$

This map has the following properties

1. $F(x,0) = x$,
2. $F(x,1) = 0$,
3. $F(0,t) = 0$.

This map "contracts" $\mathbb{R}$ to the point 0, leaving the point 0 intact at all times.

Proposition 13.2.2. $\pi_1(\mathbb{R}, 0) = \{1\}$.

Proof. Let $\gamma : (S^1, 1) \to (\mathbb{R}, 0)$ be a continuous map. Consider the composite

$$S^1 \times I \xrightarrow{\gamma \times Id} \mathbb{R} \times I \xrightarrow{F} \mathbb{R}\,.$$

It is easily checked the $F \circ (\gamma \times Id)$ is a homotopy between γ and the constant map 0. This proves the proposition. $\square$

13.2.3 $\pi_{\mathbf{1}}(\mathbf{D}, \mathbf{1})$. Similarly, we can compute the fundamental group of the closed disk

$$D := \{z \in \mathbb{C} \mid |z| \leqslant 1\}.$$

Consider the map $F : D \times I \to D$ given by

$$F(z,t) = (1-t)z + t\,.$$

This map has the following properties

1. $F(z,0) = z$,
2. $F(z,1) = 1$,
3. $F(1,t) = 1$.

This map "contracts" D to the point 1, leaving the point 1 intact at all times.

Proposition 13.2.4. $\pi_1(D, 1) = \{1\}$.

Proof. Let $\gamma : (S^1, 1) \to (D, 1)$ be a continuous map. Consider the composite

$$S^1 \times I \xrightarrow{\gamma \times Id} D \times I \xrightarrow{F} D\,.$$

It is easily checked the $F \circ (\gamma \times Id)$ is a homotopy between γ and the constant map 1. This proves the proposition. ☐

In the previous two examples the map $F : X \times I \to X$ had the special property that $F(x_0 \times I) = x_0$. The same proof shows that for a topological space X which admits such an F, the fundamental group is trivial. We leave this as an exercise to the reader.

13.2.5 $\boldsymbol{\pi_1(S^1, 1)}$. This example is more interesting and more difficult. Here we will use the lifting theorems.

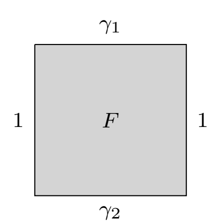
Theorem 13.2.6. *$\pi_1(S^1, 1) = \mathbb{Z}$ and a generator is given by the map $\gamma : [0, 1] \to (S^1, 1)$, $\gamma(t) = e^{2\pi i t}$.*

Proof. We will consider a loop at 1 as a continuous map $\gamma : [0, 1] \to S^1$ such that $\gamma(0) = \gamma(1) = 1$. Consider the covering map

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e^{2\pi i x}"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"] & S^1 \end{tikzcd}$$

By the lifting theorem 12.2.1, there is a unique lift of γ which makes the above diagram commute and such that $\tilde{\gamma}(0) = 0$. Obviously, $\tilde{\gamma}(1)$ is in the fiber over 1, which is exactly the set of integers. Thus, $\tilde{\gamma}(1) \in \mathbb{Z}$.

We claim that if γ_1 and γ_2 are homotopic, then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. Let F be a homotopy between γ_1 and γ_2.



It follows from Lemma 12.2.3 that $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. This shows that there is a well defined map

$$\pi_1(S^1, 1) \to \mathbb{Z} \qquad\qquad \gamma \mapsto \tilde{\gamma}(1)$$

Next let us check that this map is a group homomorphism. Let $\gamma \in \pi_1(S^1, 1)$ and let $\tilde{\gamma}$ be the unique lift starting at 0. For $m \in \mathbb{Z}$ the map

$$T_m \tilde{\gamma}(t) := \tilde{\gamma}(t) + m$$

is clearly a lift of γ starting at m. By the uniqueness of the lift, this is the unique lift of γ starting at m. Now consider two elements $a, b \in \pi_1(S^1, 1)$. Let $\tilde{a}$ and $\tilde{b}$ denote lifts of a, b starting at 0. Let m denote $\tilde{a}(1)$. Then

$$\tilde{a} * (T_m\tilde{b})$$

is a path starting at 0 and lifts $a * b$. Obviously, the end point of this path is $m + \tilde{b}(1) = \tilde{a}(1) + \tilde{b}(1)$. This proves that the map $\pi_1(S^1, 1) \to \mathbb{Z}$ is a group homomorphism.

Next let us prove that the kernel of this group homomorphism is trivial. Suppose $\tilde{\gamma}(1) = 0$, then this means that $\tilde{\gamma}$ is a loop at 0. Let $F : S^1 \times I \to \mathbb{R}$ be a homotopy between $\tilde{\gamma}$ and the constant map 0. Then $e^{2\pi i x} \circ F : S^1 \times I \to \mathbb{R} \to S^1$ gives a homotopy between γ and the constant loop at 1. This shows that γ is trivial in $\pi_1(S^1, 1)$.

Finally, it is clear that the identity map $\gamma : [0, 1] \to (S^1, 1)$,

$$\gamma(t) = e^{2\pi i t}$$

lifts to the map $\tilde{\gamma}(x) = x$. This shows that 1 is in the image of $\pi_1(S^1, 1)$, which proves surjectivity. This completes the proof of the theorem. $\square$

As an application of computing the fundamental group of S^1 we prove the fundamental theorem of algebra.

Theorem 13.2.7 (Fundamental theorem of algebra)**.** *Let $p(z) = z^n + a_1 z^{n-1} + \ldots + a_n \in \mathbb{C}[z]$ be a polynomial with $n > 1$ and $a_n \neq 0$. Then $p(z)$ has a root.*

Proof. Making the change of variable $z = Mu$, we see that this polynomial has a root iff the polynomial

$$p(u) = u^n + \frac{a_1}{M} u^{n-1} + \ldots + \frac{a_n}{M^n}$$

has a root. Choosing $M \gg 0$ we may assume that $p(z)$ is such that

$$|a_1| + |a_2| + \ldots + |a_n| < 1\,.$$

Assume that $p(z)$ does not have a root. Consider the map

$$F : S^1 \times I \to S^1 \qquad\qquad F(z, t) = \frac{p(tz)}{|p(tz)|} \frac{|p(t)|}{p(t)}\,.$$

It is easily checked that this is a homotopy between the constant loop at 1 and the loop given by $z \mapsto F(z, 1)$. Next note that

$$|z^n + t(a_1 z^{n-1} + \ldots + a_n)| > 1 - t(|a_1| + |a_2| + \ldots + |a_n|) > 1 - t \geqslant 0\,.$$

Thus, the following map is well defined.

$$G : S^1 \times I \to S^1$$

given by

$$G(z,t) = \frac{z^n + t(a_1 z^{n-1} + \ldots + a_n)}{|z^n + t(a_1 z^{n-1} + \ldots + a_n)|} \frac{|1 + t(a_1 + \ldots + a_n)|}{1 + t(a_1 + \ldots + a_n)}.$$

It is easily checked that this gives a homotopy between the loops $z \mapsto G(z,1)$ and the loop $z \mapsto z^n$. Since $G(z,1) = F(z,1)$, we get a contradiction since this shows that the constant loop is homotopic to the loop $z \mapsto z^n$. □

13.3 Dependence on base point

Let X be a path connected topological space. It is natural to ask how the fundamental group depends on the base point. For example, is it possible that $\pi_1(X, x_1)$ is abelian and $\pi_1(X, x_2)$ is not abelian? The next theorem tells us that different base points give us isomorphic groups.

Theorem 13.3.1. *Let X be a path connected topological space. Then the groups $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$ are isomorphic.*

Proof. Let $\delta : [0,1] \to X$ be a path from x_1 to x_2. Recall that $I(\delta)$ is the reverse path from x_2 to x_1. Define

$$\Phi_1(\delta) : \pi_1(X, x_1) \to \pi_1(X, x_2), \qquad \Phi_2(\delta) : \pi_1(X, x_2) \to \pi_1(X, x_1)$$

by

$$\gamma \mapsto I(\delta) * \gamma * \delta, \qquad \gamma \mapsto \delta * \gamma * I(\delta).$$

We leave it to the reader to show that both these are group homomorphisms and $\Phi_1(\delta) \circ \Phi_2(\delta) = \mathrm{Id}_{\pi_1(X,x_2)}$, $\Phi_2(\delta) \circ \Phi_1(\delta) = \mathrm{Id}_{\pi_1(X,x_1)}$. □

13.4 A lifting theorem for maps

Definition 13.4.1 (Locally path connected spaces)**.** *A topological space Y is said to be locally path connected if for every point $y \in Y$ and every open set U containing y, there is a path connected neighborhood V such that $x \in V \subset U$.*

Theorem 13.4.2 (Lifting maps to covers)**.** *Let X, Y, Z be path connected spaces and let Z be locally path connected. Let $f : X \to Y$ be a covering map and let $g : Z \to Y$ be a continuous map. Let $z_0 \in Z$, let $y_0 := g(z_0)$ and $x_0 \in f^{-1}(y_0)$. Then the map g can be lifted to $\tilde{g} : Z \to X$ such that $\tilde{g}(z_0) = x_0$ iff $g_*(\pi_1(Z, z_0)) \subset f_*(\pi_1(X, x_0))$. If a lift exists, then it is unique.*

Proof. If g can be lifted to $\tilde{g}$ then since $f \circ \tilde{g} = g$, it follows that $f_* \tilde{g}_* = g_*$, which shows that $g_*(\pi_1(Z, z_0)) \subset f_*(\pi_1(X, x_0))$. The main assertion is the converse, which we now prove.

The idea of defining the lift pointwise is simple. Since Z is path connected, for every $z \in Z$ there is a path $\gamma : [0,1] \to Z$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$. Consider the path $g \circ \gamma : [0,1] \to Y$. Let us denote by $\widetilde{g \circ \gamma}$ the unique lift of $g \circ \gamma$ starting at x_0. If a lift $\tilde{g}$ existed such that $\tilde{g}(z_0) = x_0$, then clearly, because of unique path lifting (Theorem 12.2.1), we will have

$$\tilde{g} \circ \gamma = \widetilde{g \circ \gamma}\,.$$

Evaluating at 1 this will force that $\tilde{g}(z) = \widetilde{g \circ \gamma}(1)$. Thus, it is clear that we should define

$$\tilde{g}(z) := \widetilde{g \circ \gamma}(1)\,.$$

First we need to check that if γ_1 and γ_2 are two paths from z_0 to z, then $\widetilde{g \circ \gamma_1}(1) = \widetilde{g \circ \gamma_2}(1)$. Recall that for a path γ we defined $I(\gamma)$ to be the path in the reverse direction given by $I(\gamma)(t) = \gamma(1-t)$. Clearly the path $(\gamma_1) * I(\gamma_2)$ is a loop at z_0. From this we get that

$$g_*((\gamma_1) * I(\gamma_2)) = (g \circ \gamma_1) * (g \circ I(\gamma_2))$$

is a loop at y_0. Since

$$g_*(\pi_1(Z, z_0)) \subset f_*(\pi_1(X, x_0)),$$

there is a loop h at x_0 such that $f_*(h)$ is homotopic to $(g \circ \gamma_1) * (g \circ I(\gamma_2))$. Let F be a homotopy between them such that

$$F(t,0) = f_*(h) \qquad \text{and} \qquad F(t,1) = (g \circ \gamma_1) * (g \circ I(\gamma_2))\,.$$

By Theorem 12.2.2 there is a unique lift such that $\tilde{F}(0,0) = x_0$. Now note that $\tilde{F}(t,0)$ is a lift of $f_*(h)$ and so it is simply h, that is,

$$\tilde{F}(t,0) = h(t)\,.$$

By Lemma 12.2.3, since h is a loop at x_0, it follows that $\tilde{F}(t,1)$ is a loop at x_0. Clearly, $\tilde{F}(t,1)$ is a lift of $(g \circ \gamma_1) * (g \circ I(\gamma_2))$. This forces that

- the path $t \mapsto \tilde{F}(t/2, 1)$ is the unique lift of $g \circ \gamma_1$ starting at x_0. In other words, $\tilde{F}(t/2,1) = \widetilde{g \circ \gamma_1}(t) =: x_1$.
- the path $t \mapsto \tilde{F}(1/2 + t/2, 1)$ is the unique lift of $g \circ I(\gamma_2)$ starting at $\widetilde{g \circ \gamma_1}(1)$. In other words, $\tilde{F}(1/2 + t/2, 1) = \widetilde{g \circ I(\gamma_2)}(t)$. Note that $\widetilde{g \circ I(\gamma_2)}(1) = x_0$.

Now let us make the following easy observation. Let δ be a path in Y from a to b. Let $\tilde{a}$ be a point lying over a. Let $\tilde{\delta}$ denote the unique lift of δ at $\tilde{a}$. Suppose $\tilde{\delta}$ ends at $\tilde{b}$. Then the unique lift of $I(\delta)$ starting at $\tilde{b}$ is the path $I(\tilde{\delta})$. We apply this observation to

$$\delta = g \circ I(\gamma_2),\ a = y_1,\ \tilde{a} = x_1,\ b = y_0\,.$$

Then $\tilde{b} = x_0$. Thus, the unique lift of $I(g \circ I(\gamma_2)) = g \circ \gamma_2$ starting at x_0 will end at x_1. This shows that $\widetilde{g \circ \gamma_2}(1) = \widetilde{g \circ \gamma_1}(1)$. This proves that the map $\tilde{g}$ is well defined.

Finally we need to show that the map $\tilde{g}$ is continuous. Let $z \in Z$ and let $U \subset X$ be an open subset containing $\tilde{g}(z)$. Let $y := g(z)$ and let V be an evenly covered neighborhood of y. Then $f^{-1}(V) = \bigsqcup_i U_i$ and suppose that $\tilde{g}(z) \in U_j$. Replacing V by $f(U_j \bigcap U)$ we may assume that y has an evenly covered neighborhood V such that

- $f^{-1}(V) = \bigsqcup_i U_i$,
- $\tilde{g}(z) \in U_j$,
- $U_j \subset U$.

Let W be a path connected open neighborhood of z such that $g(W) \subset V$. We claim that

$$\tilde{g}(W) \subset U_j \subset U.$$

Fix a path γ from z_0 to z. Let $t \in W$, then there is a path γ_1 from z to t contained in W. Thus, $\gamma * \gamma_1$ is a path from z_0 to t. Now $\tilde{g}(t)$ is defined to be $g \circ \widetilde{(\gamma * \gamma_1)}(1)$. Let h denote the lift of $g \circ \gamma_1$ starting at $g(z)$. Note that the image of $g \circ \gamma_1$ is contained in V. It is clear that the end of the path h is contained in the same path component of $f^{-1}(V)$ as $g(z)$, and so in U_j. Since

$$g \circ \widetilde{(\gamma * \gamma_1)} = \widetilde{g \circ \gamma} * h,$$

it follows that the end of $g \circ \widetilde{(\gamma * \gamma_1)}$ is contained in U_j. This proves that $\tilde{g}(t) \in U_j \subset U$. This proves that $\tilde{g}$ is continuous.

To prove uniqueness of the lift we will use that Z is path connected and the unique path lifting theorem (Theorem 12.2.1). Let $\tilde{h}$ be another lift such that $\tilde{h}(z_0) = x_0$. Let $\gamma : [0,1] \to Z$ be a path joining z_0 and z. Then both $\tilde{h} \circ \gamma$ and $\tilde{g} \circ \gamma$ are lifts of γ which start at x_0. Now by unique path lifting it follows that both are equal, in particular, $\tilde{h}(z) = \tilde{g}(z)$. $\square$

As an application of the lifting theorem, let us see when we can define a logarithm map.

13.4.3 **Branch of the logarithm**. Let $i : \Omega \subset \mathbb{C}^\times$ denote an open subset. The question we want to address is when there is a map log which makes the following diagram commute

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "e^z =: f(z)"] \\ \Omega \arrow[ur, dashed, "\log"] & \subset \ \mathbb{C}^\times \end{tikzcd}$$

Let $z_0 \in \Omega$ be a point and let $\tilde{z}_0$ be a point lying over z_0. Then Theorem 13.4.2 tells us that the necessary and sufficient condition for a lift to exist is that $i_*(\pi_1(\Omega, z_0)) \subset f_*(\pi_1(\mathbb{C}, \tilde{z}_0))$.

Since $\mathbb{C}$ can be "contracted" to $\tilde{z}_0$ it follows that $\pi_1(\mathbb{C}, \tilde{z}_0) = \{1\}$. It follows that the necessary and sufficient condition for a lift to exist is that $i_*(\pi_1(\Omega, z_0)) = \{1\}$. This is same as saying that every loop at z_0 in Ω is homotopic to the constant loop at z_0 in $\mathbb{C}^\times$; we emphasize that we only this homotopy to be in $\mathbb{C}^\times$ and not necessarily in Ω. If this happens, then for any point $\tilde{z}_0$ lying over $z_0 \in \mathbb{C}^\times$, there is a unique lift log such that $\log(z_0) = \tilde{z}_0$. Each such lift is called a branch of the logarithm. In particular, if $\pi_1(\Omega, z_0) = \{1\}$ then there exists a branch of the logarithm.

13.5 Applications

Proposition 13.5.1. *There is no continuous map $f : D \to S^1$ whose restriction to S^1 is the identity.*

Proof. Let us assume that such a map exists. Then $f(1) = 1$ and so we get $f_* : \pi_1(D, 1) \to \pi_1(S^1, 1)$. Let $i : S^1 \to D$ denote the inclusion. Then since $f \circ i = Id$, it follows that $(f \circ i)_* = Id_* = Id$. However, this is not possible as $(f \circ i)_* = f_* \circ i_* : \mathbb{Z} \to \{1\} \to \mathbb{Z}$ cannot be the identity. $\square$

Theorem 13.5.2. *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume that f does not have a fixed point. Define a continuous function $g : D^2 \to S^1$ as follows. Consider the line starting at x and in the direction $x - f(x)$. This line is parameterized by $x+t(x-f(x))$. Let $t_0 \geqslant 0$ be the unique number such that $x+t_0(x-f(x))$ is on S^1. It is easily checked that $g(x) = x+t_0(x-f(x))$ defines a continuous function. It is also clear that g is the identity on S^1. This contradicts the previous proposition. This shows that f has a fixed point. $\square$

Theorem 13.5.3 (Borsuk-Ulam)**.** *Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Then there is a point x such that $f(x) = f(-x)$.*

Proof. If there is no such point x, then consider the continuous map $g : S^2 \to S^1$ defined by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Notice that g satisfies $g(-x) = -g(x)$. Since the fundamental group $\pi_1(S^2, p)$ is trivial, see exercise 13.8.7, by Theorem 13.4.2, there is a map $\tilde{g}$ which makes the following diagram commute

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "x \mapsto e^{2\pi i x}"] \\ S^2 \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & S^1 \end{tikzcd}$$

Since $g(-x) = -g(x)$, it follows that for every x,

$$\tilde{g}(x) - \tilde{g}(-x) \in \frac{\mathbb{Z}}{2} \setminus \mathbb{Z}\,.$$

Since S^2 is connected, there is an odd integer q such that $\tilde{g}(x) - \tilde{g}(-x) = q/2$. We also have $\tilde{g}(-x) - \tilde{g}(x) = q/2$. Adding these we get that $q = 0$, which contradicts the fact that q is odd. $\square$

13.6 Free groups and Amalgamated Product

The aim of this section is to explain the results related to group theory that we shall need while proving the Seifert-van Kampen Theorem. The main results in the next few subsections can be described very compactly using the language of categories, in terms of final or initial objects. However, we will avoid a digression into category theory. We refer the interested reader to [Lan02, Chapter 1, Section 11,12] for this discussion.

13.6.1 **Free groups**. Let S be a set. In this section we define a pair $(F(S), f)$, where $F(S)$ is a group, $f : S \to F(S)$ is a map of sets such that the group $F(S)$ is generated by the image of f, and this pair has the following property:

(P) If G is any other group and we have a map of sets $g : S \to G$, then there is a unique group homomorphism $\tilde{g} : F(S) \to G$ such that $\tilde{g}(f(s)) = g(s)$.

We only sketch the construction and leave the details to the reader. For a detailed proof see [Lan02, Proposition 12.1].

Sketch of construction of $(F(S), f)$. For each $s \in S$ let x_s be a symbol. Consider "words" formed out of these symbols, that is,

$$x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n}\,,$$

where $n > 0$ is an integer and a_i are integers such that $a_i \neq 0$. We call the above word reduced if $s_i \neq s_{i+1}$ for all $1 \leqslant i \leqslant n-1$. For a reduced word as above, we define its length to be n. Let $F(S)$ be the set of reduced words along with a symbol e. The length of e is defined to be 0. For $n \geqslant 0$, let $F(S)_n$ denote the set of reduced words of length $\leqslant n$.

Next we define a map $m : F(S) \times F(S) \to F(S)$. In simple terms, the construction of m is as follows. Given two reduced words

$$x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n}, \quad x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}$$

we can concatenate them to get the (not necessarily reduced) word

$$x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n} x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}\,.$$

However, this word may not be reduced as it may happen that $s_n = t_1$. If this happens then we may reduce this to get a reduced word. For example, let $s, t \in S$ and consider reduced words $x_s x_t^2$ and $x_t^{-3} x_s^5$. When we concatenate these and reduce we get $x_s x_t^{-1} x_s^5$. Similarly, if we consider words $x_s x_t^2$ and $x_t^{-2} x_s^{-1}$, if we concatenate them and reduce then we get e. We proceed to do this formally.

We will define m inductively on subsets $F(S)_n \times F(S)_n \to F(S)$.

1. Define $m(e, e) := e$. This defines m on $F(S)_0 \times F(S)_0$.

2. Define $m(e, x_s^a) = m(x_s^a, e) = x_s^a$.

3. Define $m(x_s^a, x_s^b) = x_s^{a+b}$ when $a + b \neq 0$.

4. Define $m(x_s^a, x_s^{-a}) = e$ when $a + b = 0$.

5. If $s \neq t$ then define $m(x_s^a, x_t^b) = x_s^a x_t^b$. This defines m on $F(S)_1 \times F(S)_1$. Now let us assume that we have defined m on $F(S)_n \times F(S)_n$ and we shall extend this definition to $F(S)_{n+1} \times F(S)_{n+1}$.

6. Let $w := x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_{n+1}}^{a_{n+1}} \in F(S)_{n+1}$. Define $m(w, e) = m(e, w) = w$. Let $l \geqslant 1$ and let $x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l} \in F(S)_l$, where $l \leqslant n + 1$. If $s_{n+1} \neq t_1$ then define
$$m(x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_{n+1}}^{a_{n+1}}, x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}) = x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n} x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l} .$$

7. If $s_{n+1} = t_1$ and $a_n + b_1 \neq 0$ then define
$$m(x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_{n+1}}^{a_{n+1}}, x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}) = x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_{n+1}}^{a_n + b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l} .$$

8. If $s_{n+1} = t_1$ and $a_n + b_1 = 0$ then define
$$m(x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_{n+1}}^{a_{n+1}}, x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}) = m(x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n}, x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}) .$$
If $l = 1$ then the RHS is $m(x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n}, e) = x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n}$.

9. Similarly, we define $m(x_{t_1}^{b_1} x_{t_2}^{b_2} \ldots x_{t_l}^{b_l}, x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_{n+1}}^{a_{n+1}})$. This completes the definition of m.

It is easily checked that the above map makes $F(S)$ into a group with e as the identity element. The map $f : S \to F(S)$ is simply $s \mapsto x_s$.

Next let us check that the pair $(F(S), f)$ satisfies the property **(P)**. Suppose we are given another pair (G, g) where $g : S \to G$ is a map of sets. Notice that the map $\tilde{g}$, if it exists, is unique as every element of $F(S)$ is a word in the x_s and the image $\tilde{g}(x_s)$ is fixed. To prove existence, we simply define $\tilde{g}(x_{s_1}^{a_1} x_{s_2}^{a_2} \ldots x_{s_n}^{a_n}) = \tilde{g}(x_{s_1})^{a_1} \tilde{g}(x_{s_2})^{a_2} \ldots \tilde{g}(x_{s_n})^{a_n}$. It is easily checked that this is a group homomorphism.

Remark 13.6.2. We may rephrase **(P)** by saying that group homomorphisms $F(S) \to G$ are in bijective correspondence with set maps $S \to G$.

Universality. Given a group G and a group homomorphism $\psi : F(S) \to G$, we get a pair $(G, \psi \circ f)$. On the other hand, the above discussion shows that every pair (G, g) arises uniquely in this fashion, that is, there is a unique $\tilde{g} : F(S) \to G$ such that $g = \tilde{g} \circ f$. In other words, every pair "arises uniquely" from the pair $(F(S), f)$.

Suppose there is another pair (F', f') which satisfies property **(P)**. That is, if G is any other group and we have a map of sets $g : S \to G$, then there is a unique group homomorphism $\tilde{g}' : F' \to G$ such that $\tilde{g}'(f'(s)) = g(s)$. Then there is a unique isomorphism $\psi : F(S) \to F'$ such that $\psi \circ f = f'$. This is a standard argument which proceeds in the following steps:

1. Apply **(P)** to $(F(S), f)$ by taking $(G, g) = (F', f')$. Then we get a unique map $\psi : F(S) \to F'$ such that $\psi \circ f = f'$. We need to show that ψ is an isomorphism.

2. Apply **(P)** to (F', f') by taking $(G, g) = (F(S), f)$. Then we get a unique map $\psi' : F' \to F(S)$ such that $f = \psi' \circ f'$.

3. Apply **(P)** to $(F(S), f)$ by taking $(G, g) = (F(S), f)$. The map $\psi' \circ \psi : F(S) \to F(S)$ satisfies $\psi' \circ \psi \circ f = f$. However, the identity map $1_{F(S)}$ also satisfies $1_{F(S)} \circ f = f$. By uniqueness of $\tilde{g}$ in **(P)** it follows that $\psi' \circ \psi = 1_{F(S)}$.

4. Similarly, show that $\psi \circ \psi' = 1_{F'}$.

This proves that ψ is an isomorphism. Thus, if we have a pair (F', f') and if we can show that this pair has **(P)**, then it will follow that the group F' is isomorphic to $F(S)$. We will use a similar strategy while proving the Seifert-van Kampen Theorem. **The group $F(S)$ is called the free group on the set** S.

Definition 13.6.3. *Let G be a group. Let I be a set and assume that for each $i \in I$ we are given an element $g_i \in G$. We say that g_i generate G if the following happens. Given any $g \in G$ there is an $n > 0$, elements $g_{i_1}, \ldots, g_{i_n}$ and integers a_i such that $g = g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_n}^{a_n}$.*

Remark 13.6.4. Given a group G it is immediate that we can always find a set of generators for G. For example, take $I = G$ and let g_i be the element i. Of course, G may have a much smaller generating set. For example, the group $\mathbb{Z}$ is generated by the element 1.

Lemma 13.6.5. *Let G be a group. Then there is a free group $F(S)$ and a surjective group homomorphism $F(S) \to G$.*

Proof. Let g_s, for $s \in S$, be a set of generators for the group G. Consider the pair (G, g) where $g : S \to G$ is the map $s \mapsto g_s$. Applying **(P)** to $(F(S), f)$ we get a map $\tilde{g} : F(S) \to G$. As the g_s generate G, it follows that this map is surjective. □

13.6.6 **Amalgamated product of groups**. Fix a tuple (G_1, G_2, H, i_1, i_2), where G_1, G_2, H are groups and $i_j : H \to G_j$ are group homomorphisms. Consider the collection of triples (G, ϕ_1, ϕ_2) where G is a group and $\phi_i : G_i \to G$ are group homomorphisms such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$. In other words, the following diagram commutes:

$$\begin{tikzcd} & G_1 \arrow[dr, "\phi_1"] & \\ H \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & G \\ & G_2 \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

Denote the collection of such triples by $\mathscr{C}$. Given a triple (G, ϕ_1, ϕ_2), another group G' and a group homomorphism $f : G \to G'$, we can construct another triple in $\mathscr{C}$, namely $(G', f \circ \phi_1, f \circ \phi_2)$.

Theorem 13.6.7. *There is a triple $(\mathscr{G}, \psi_1, \psi_2) \in \mathscr{C}$ such that for every triple $(G, \phi_1, \phi_2) \in \mathscr{C}$ there is a unique homomorphism*

$$f_{(G,\phi_1,\phi_2)} : \mathscr{G} \to G$$

such that $(G, \phi_1, \phi_2) = (G, f_{(G,\phi_1,\phi_2)} \circ \psi_1, f_{(G,\phi_1,\phi_2)} \circ \psi_2)$.

Proof. We only give a sketch of how to proceed and leave the details to the reader. Choose sets S_i and maps $g_i : S_i \to G_i$ such that the images $g_i(S_i)$ generate G_i. Similarly, choose a set T and a map $h : T \to H$ such that $h(T)$ generates H. Then we have group homomorphisms $\pi_i : F(S_i) \to G_i$ and $\pi : F(T) \to H$, all of which are surjective.

$$\begin{tikzcd} & F(S_1) \arrow[rr, "\pi_1"] & & G_1 \\ F(T) \arrow[rr, "\pi"] & & H \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & F(S_2) \arrow[rr, "\pi_2"] & & G_2 \end{tikzcd}$$

We claim that we can find a group homomorphism $F(T) \to F(S_1)$ which creates a commutative parallelogram in the above diagram. We will repeatedly use Remark 13.6.2. To give a group homomorphism $F(T) \to F(S_1)$ it suffices to give a map of sets $T \to F(S_1)$. As π_1 is surjective, for $t \in T$, choose an arbitrary lift of $i_1 \circ h(t)$ in $F(S_1)$. This defines a map $T \to F(S_1)$ and so a group homomorphism $F(T) \to F(S_1)$. Use Remark 13.6.2 it is easily checked that the resulting parallelogram commutes. Similarly, we define a homomorphism $F(T) \to F(S_2)$ so that we have a commutative diagram

$$\begin{tikzcd}
 & F(S_1) \arrow[rr, "\pi_1"] & & G_1 \\
F(T) \arrow[ur, "j_1"] \arrow[rr, "\pi"] \arrow[dr, "j_2"'] & & H \arrow[ur, "i_1"'] \arrow[dr, "i_2"] & \\
 & F(S_2) \arrow[rr, "\pi_2"] & & G_2
\end{tikzcd}$$

Let $S = S_1 \bigsqcup S_2$ and consider the free group $F(S)$. The inclusions $S_i \to S$ give rise to group homomorphisms $\theta_i : F(S_i) \to F(S)$. Similarly, the maps $T \to S_i \to S$ gives rise to homomorphisms $\eta_i : F(T) \to F(S)$. Let J be the subset of $F(S)$ defined as

$$J := \theta_1(\mathrm{Ker}(\pi_1)) \cup \theta_2(\mathrm{Ker}(\pi_2)) \cup \{\theta_1 j_1(\alpha)\theta_2 j_2(\alpha^{-1}) \,|\, \alpha \in F(T)\}\,.$$

Let N be the smallest normal subgroup containing J. Define

$$\mathscr{G} := F(S)/N\,.$$

Define $\bar{\theta}_i$ to be the composite $F(S_i) \xrightarrow{\theta_i} F(S) \to \mathscr{G}$. It is easily checked that $\bar{\theta}_i$ factors through π_i to give a map $G_i \xrightarrow{\psi_i} \mathscr{G}$. We leave it to the reader to check that there is a commutative diagram

$$\begin{tikzcd}
 & F(S_1) \arrow[rr, "\pi_1"] & & G_1 \arrow[dr, "\psi_1"] & \\
F(T) \arrow[ur, "j_1"] \arrow[rr, "\pi"] \arrow[dr, "j_2"'] & & H \arrow[ur, "i_1"'] \arrow[dr, "i_2"] & & \mathscr{G} \\
 & F(S_2) \arrow[rr, "\pi_2"] & & G_2 \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

We leave it to the reader to check that the triple $(\mathscr{G}, \psi_1, \psi_2)$ is in $\mathscr{C}$ and has the required properties. $\square$

Definition 13.6.8. *The triple $(\mathscr{G}, \psi_1, \psi_2)$ is called the amalgamated product of G_1 and G_2 along i_1 and i_2. It is denoted $G_1 *_{H,i_1,i_2} G_2$. When there is no confusion about the maps i_1 and i_2, they are suppressed and this is simply denoted $G_1 *_H G_2$.*

In the same way as in the case of free groups, we can prove the following Proposition.

Proposition 13.6.9. *Suppose $(\mathscr{G}', \psi_1', \psi_2')$ such that for every triple $(G, \phi_1, \phi_2) \in \mathscr{C}$ there is a unique homomorphism*

$$f'_{(G,\phi_1,\phi_2)} : \mathscr{G}' \to G$$

such that $(G, \phi_1, \phi_2) = (G, f'_{(G,\phi_1,\phi_2)} \circ \psi_1', f'_{(G,\phi_1,\phi_2)} \circ \psi_2')$. Then there is a unique isomorphism of groups $\psi : \mathscr{G} \to \mathscr{G}'$ such that $\psi \circ \psi_i = \psi_i'$.

Proof. Left as an exercise. □

13.7 Seifert-van Kampen Theorem

In this section we shall prove the Seifert-van Kampen Theorem. This is an extremely useful result which enables us to compute fundamental groups of a space in terms of the fundamental groups of some subspaces. The results in this section were presented in class and written up by Kartik Patekar. The presentation and the write up closely follows [Mun00, Chapter 11], with some very minor modifications.

Theorem 13.7.1 (Seifert-van Kampen Theorem)**.** *Let X be a path connected topological space. Let U, V and $U \cap V$ be path connected subspaces of X such that $X = U \cup V$. Let $x_0 \in U \cap V$. Then the natural map*

$$\pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

is an isomorphism.

Proof. Let H be a group such that

$$\phi_1 : \pi_1(U, x_0) \to H$$

$$\phi_2 : \pi_1(V, x_0) \to H$$

are homomorphisms. Let i_1, i_2, j_1, j_2 be homomorphisms induced by inclusion map as shown in the diagram

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dashed, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

We will show that if the maps ϕ_1 and ϕ_2 are compatible, that is,

$$\phi_1 \circ i_1 = \phi_2 \circ i_2,$$

then there is a unique homomorphism

$$\Phi : \pi_1(X, x_0) \to H\,,$$

such that the above diagram commutes. This will prove that the group $\pi_1(X, x_0)$ satisfies the universal property of the amalgamated product (Proposition 13.6.9), which will prove the theorem.

The group $\pi_1(X, x_0)$ is generated by $j_1(\pi_1(U, x_0))$ and $j_2(\pi_1(V, x_0))$. Also Φ is uniquely determined on the generators $j_1(g)$ and $j_1(g)$ by the requirement that the diagram commutes. Since Φ is a homomorphism, if it exist, it is uniquely determined by the images of these generators. Thus, if Φ exists then it is unique.

To prove that Φ exist, we proceed step-wise and construct Φ. The path homotopy class of a path f in X, U, V and $U \cap V$ is denoted by $[f], [f]_U, [f]_V$ and $[f]_{U\cap V}$, respectively.

Step 1: Let $L(U, x_0)$ denote the set of loops lying entirely in U and based at x_0. Similarly, define $L(V, x_0)$. Define $\rho : L(U, x_0) \cup L(V, x_0) \to H$ as

$$\rho(f) = \begin{cases} \phi_1([f]_U) & \text{if } f \text{ is in } U \\ \phi_2([f]_V) & \text{if } f \text{ is in } V \end{cases}$$

When f is in $U \cap V$, the compatibility condition $\phi_1 \circ i_1 = \phi_2 \circ i_2$ says that $\phi_1([f]_U) = \phi_2([f]_V)$. Thus, ρ is well defined. Note that ρ satisfies the following conditions

1. If $[f]_U = [g]_U$ or if $[f]_V = [g]_V$, then $\rho(f) = \rho(g)$,
2. If both f and g are in U then $\rho(f * g) = \rho(f) \cdot \rho(g)$,
3. If both f and g are in V then $\rho(f) \cdot \rho(g) = \rho(f * g)$.

All the above statements are obvious from the definition of ρ.

Step 2: We now extend the definition of ρ from loops to paths which are completely contained in U or V. Denote the set of all paths in U by $\mathcal{P}(U)$ and the set of all paths in V by $\mathcal{P}(V)$. We want to define a function

$$\sigma : \mathcal{P}(U) \cup \mathcal{P}(V) \to H$$

which extends ρ. For this fix once and for all paths α_x from x_0 to $x \in X$, such that if x lies in U, V or $U \cap V$, then α_x lies in U, V or $U \cap V$, respectively. Choose α_{x_0} to be the constant path at x_0. We will define the function σ using the paths $\{\alpha_x\}$.

For a path γ, which lies completely in U or V, from x_1 to x_2, define

$$\sigma(\gamma) = \rho(\alpha_{x_1} * \gamma * \alpha_{x_2}^{-1})$$

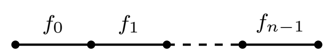
It is easy to see that σ is well defined and is an extension of ρ. It also satisfies the following two properties similar to that of ρ, namely, if both f and g are paths in U.

1. If $f \sim_U g$, then $\sigma(f) = \sigma(g)$. To see this, let f, g be paths from x_1 to x_2. Then $f \sim_U g$ implies $[\alpha_{x_1} * f * \alpha_{x_2}^{-1}]_U = [\alpha_{x_1} * g * \alpha_{x_2}^{-1}]_U$ and the rest follows by definition.

2. If f is a path from x_1 to x_2 and g is a path from x_2 to x_3, then $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$. This again follows from the definition and the fact that $\alpha_{x_1} * f * \alpha_{x_2}^{-1} * \alpha_{x_2} * g * \alpha_{x_3}^{-1} \sim_U \alpha_{x_1} * f * g * \alpha_{x_3}^{-1}$.

Similar results hold if both f and g are paths in V.

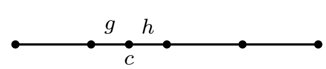
Step 3: Extend σ to $\tau : \mathcal{P}(X) \to H$. Let f be a path in X. By Lebesgue's number lemma, there exist a partition $0 = s_0 < s_1 < ... < s_n = 1$ such that $f([s_i, s_{i+1}]) \subset U$ or $f([s_i, s_{i+1}]) \subset V$. Define path f_i to be the part of f lying between s_i and s_{i+1}. Then we have $f \sim f_0 * f_1 * ... * f_{n-1}$, where each $f_i \in \mathcal{P}(U) \cup \mathcal{P}(V)$.



Define $\tau(f)$ as

$$\tau(f) = \sigma(f_0) \cdot \sigma(f_1) \cdot ... \cdot \sigma(f_{n-1})$$

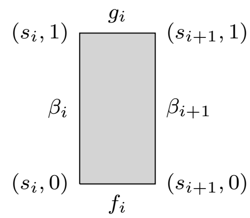
We wish to show that τ is well defined. It is sufficient to show that if a partition is refined by adding point $c \in (0, 1)$ to the partition, the image of f obtained by the above definition, and the new partition which includes c, does not change. This will prove that refinement of a partition does not change the image, and hence for any 2 given partitions P_1 and P_2, the image obtained using either partition is same as that of $P_1 \cup P_2$. This shows that $\tau(f)$ does not depend on the partition chosen.



Let $c \in (s_i, s_{i+1})$, and g, h be path obtained by restriction of f_i on sets $[s_i, c]$ and $[c, s_{i+1}]$, respectively. Since both g and h lie in either U or V, we have $\sigma(f_i) = \sigma(g * h) = \sigma(g) \cdot \sigma(h)$, which shows that τ is well defined.

Step 4: Show that if $f \sim g$, then $\tau(f) = \tau(g)$. Let F be the path homotopy between f and g. We will consider a special case first.

Special Case: There exist a partition $0 = s_0 < s_1 < ... < s_n = 1$ such that $F([s_i, s_{i+1}] \times I)$ lie in either U or V, say in U. Let β_i be the path given by $\beta_i(t) = F(s_i, t)$.



It is clear that $f_i * \beta_{i+1} \sim_U \beta_i * g_i$. From this we get that

$$\sigma(f_i) = \sigma(\beta_i) \cdot \sigma(g_i) \cdot \sigma(\beta_{i+1})^{-1}.$$

Using this result and the definition of τ, we obtain

$$\begin{aligned}\tau(f) &= \sigma(f_0) \cdot \sigma(f_1) \cdot ... \cdot \sigma(f_{n-1}) \\ &= \prod_{i=0}^{n-1} \sigma(\beta_i) \cdot \sigma(g_i) \cdot \sigma(\beta_{i+1})^{-1} \\ &= \sigma(\beta_0) \cdot \sigma(g_0) \cdot \sigma(g_1) ... \cdot \sigma(g_{n-1}) \cdot \sigma(\beta_n)^{-1} \\ &= \tau(g)\end{aligned}$$

where the last equality follows from the fact that β_0 and β_n are constant paths.

General Case: By Lebesgue's number lemma, it is always possible to find partitions $0 = s_0 < s_1 < ... < s_n = 1$ and $0 = t_0 < t_1 < ... < t_m = 1$ such that $F([s_i, s_{i+1}] \times [t_j, t_{j+1}])$ lies in either U or V. Define paths h_j in X using $h_j(s) = F(s, t_j)$, which gives $h_0 = f$ and $h_m = g$. By restricting the homotopy F we obtain that $h_0 \sim h_1 \sim ... \sim h_m$. From the special case considered above, $\tau(h_j) = \tau(h_{j+1})$, which results in $\tau(f) = \tau(g)$.

Step 5: If $f, g \in \mathcal{P}(X)$ such that $f * g$ is defined, then $\tau(f) \cdot \tau(g) = \tau(f * g)$. The proof of this is obvious from definition of τ and taking a partition containing the point 1/2.

Step 6: Restrict the above map $\tau : \mathcal{P}(X) \to H$ to $L(X, x_0) \to H$. From Step 4 it follows that this restricted map descends to a map $\Phi : \pi_1(X, x_0) \to H$. From Step 5 it follows that Φ is a group homomorphism.

Thus, the theorem is proved. $\square$

13.8 Exercises

13.8.1. Let Y be a locally path connected topological space. Let $f : X \to Y$ be a covering map. Show that X is locally path connected.

13.8.2. Let X be connected and locally path connected. Let $p : Y \to X$ be a connected cover. Let $f : Y \to Y$ be continuous such that $p \circ f = p$. Show that $f(Y)$ is both open and closed. Thus, f is surjective.

13.8.3. Let $p : Y \to X$ be a covering space such that $p(y_0) = x_0$. Show that $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is an inclusion.

13.8.4. Let $X := \mathbb{C} \setminus \mathbb{R}_{<0}$. Compute $\pi_1(X, 1)$.

13.8.5. Let X and Y be topological spaces. Let $x_0 \in X$ and $y_0 \in Y$. Let p_X and p_Y denote the projections from $X \times Y$ to X and Y. Show that the map $\pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ given by $\alpha \mapsto (p_{X*}(\alpha), p_{Y*}(\alpha))$ is an isomorphism.

13.8.6. Let G be a path connected and locally path connected topological group. Let $f : X \to G$ be a covering map. Let $x_0 \in f^{-1}(e)$. Assume that X is path connected and that $\pi_1(X, x_0) = \{1\}$. Use the lifting theorem for maps to show that there is a group structure on X which makes X into a topological group and f into a group homomorphism.

13.8.7. Let G be a group and let H and K be subgroups. We say that G is generated by H and K if every element of G can be written as $h_1k_1h_2k_2 \ldots h_rk_r$ for some $r > 0$, and $h_i \in H, k_i \in K$.

Let X be a topological space and let U and V be path connected open subsets such that $X = U \bigcup V$. Let $i : U \subset X$ and $j : V \subset X$ denote the inclusions. Assume that $x_0 \in U \cap V$ and that $U \cap V$ is path connected. Show that $\pi_1(X, x_0)$ is generated by the subgroups $i_*\pi_1(U, x_0)$ and $j_*\pi_1(V, x_0)$.

13.8.8. Use the previous problem to compute $\pi_1(S^n, p)$, where $n > 1$ and $p \in S^n$ is a point on the equator.

13.8.9. Let $f : S^1 \to S^1$ be the map $f(z) = z^k$. Compute $f_* : \pi_1(S^1, (1, 0)) \to \pi_1(S^1, (1, 0))$.

13.8.10. Let G be a topological group. Show that the group operation of G defines a group operation $\cdot$ on $\pi_1(G, e)$ as follows. Let $\alpha, \beta \in L(G, e)$. Define $(\alpha \cdot \beta)(t) := \alpha(t)\beta(t)$. Show that this descends to give $\cdot$ on the fundamental group. Show that

$$\alpha * \beta \sim \alpha \cdot \beta \sim \beta \cdot \alpha \sim \beta * \alpha.$$

Conclude that $\pi_1(G, e)$ is abelian.

13.8.11. We shall give a more formal proof of the previous exercise. Let $m : G \times G \to G$ denote the group multiplication. Show that there are group homomorphisms $i, j : \pi_1(G, e) \to \pi_1(G \times G, (e, e))$ such that

1. The map $\pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G \times G, (e, e))$ given by $(\alpha, \beta) \mapsto i(\alpha)j(\beta)$ is an isomorphism of groups.

2. $m_* \circ i = m_* \circ j = Id$

Use the above to show that $\pi_1(G, e)$ is abelian.

13.8.12. Complete the proofs of Theorem 13.6.7 and Proposition 13.6.9.

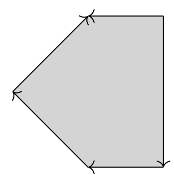
13.8.13. Let X_n be a wedge of n circles. Use the van-Kampen theorem to compute its fundamental group.

13.8.14. Compute the fundamental group of $\mathbb{P}^2_{\mathbb{R}}$ using van-Kampen theorem.

13.8.15. Compute the fundamental group of $\mathbb{P}^n_{\mathbb{C}}$ using van-Kampen theorem. Here $\mathbb{P}^n_{\mathbb{C}} := \Big(\mathbb{C}^{n+1} \setminus (0, \ldots, 0)\Big)/\mathbb{C}^\times$. (HINT: Show that $\mathbb{P}^n_{\mathbb{C}}$ can be covered by open sets homeomorphic to $\mathbb{C}^n$.)

13.8.16. Compute the fundamental group of $T^2 \vee T^2$.

13.8.17. Compute the fundamental group of the following in which the edges have been identified as given and all the vertices are identified.



13.8.18. Compute the fundamental group of $\mathbb{R}^3$ with the non-negative x, y, z axes deleted.

13.8.19. Let $X_1 = S^1 \times S^1 \times \{1\}$ and let $X_2 = S^1 \times S^1 \times \{2\}$. Compute the fundamental group of X which is obtained by identifying $S^1 \times 1 \times 1$ with $S^1 \times 1 \times 2$.

13.8.20. Use the fundamental group to show that the Mobius strip does not retract onto its boundary circle.

13.8.21. Compute the fundamental group of the union of S^2 and the straight line joining $(0, 0, 1)$ with $(0, 0, -1)$ in $\mathbb{R}^3$.

13.8.22. Let G be a group which is generated by k elements. Then there is a surjection $F_k \to G$. Let K denote the kernel. Assume that there are finitely many elements such that K is the normal subgroup generated by these. Show that there is a topological space X such that $\pi_1(X, x_0) \cong G$.

13.8.23. Let D_1 and D_2 be two disjoint discs in S^2. Cut out the interior of D_1 and D_2. Take a compact cylinder with boundary components B_1 and B_2. Identify B_1 with δD_1 and B_2 with δD_2 in such a way that the cylinder gets attached "outside" the sphere. Repeat this process with k pairs of discs and denote the resulting topological space by M_k. See the image here. Convince yourself that when $k = 1$ we get the torus. Compute the fundamental group of M_k. (HINT: Recall the computation of the fundamental group of the torus using Seifert-van Kampen Theorem.)

13.8.24. Let $n > 1$. Show that every continuous map from $\mathbb{P}^n_{\mathbb{R}} \to S^1$ is homotopic to the constant map. Show that every continuous map from $\mathbb{P}^n_{\mathbb{R}} \to S^1 \times S^1$ is homotopic to the constant map.

Chapter 14

Galois correspondence for covering maps

In this chapter we will describe a Galois correspondence for covering maps. Recall the following correspondence from field theory. Let k be a field and let $\bar{k}$ be an algebraic closure of k. Now consider the two sets

$$\{k \subset E \subset \bar{k} \mid E/k \text{ is finite}\} \longleftrightarrow \{H \subset \mathrm{Gal}(\bar{k}/k) \mid \mathrm{Gal}(\bar{k}/k)/H \text{ is finite}\}$$

The map $E \mapsto \mathrm{Gal}(\bar{k}/E)$ sets up a bijective correspondence between these two sets. The inverse of this map is given by $H \mapsto \bar{k}^H$. We will see a similar correspondence for covering maps.

14.1 Universal covers

Definition 14.1.1 (Simply connected). *A path connected topological space is called simply connected if $\pi_1(X, x) = 1$.*

Throughout this chapter X will denote a connected and locally path connected topological space.

By a pointed topological space we shall mean a topological space X along with a point $x_0 \in X$, and we will denote a pointed topological space by (X, x_0). By a morphism of pointed topological spaces $f : (X, x_0) \to (Y, y_0)$ we shall mean a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. A simply connected covering space has the following universal property.

Theorem 14.1.2. *Let (X, x_0) be a pointed topological space. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ denote a connected and simply connected cover of X. If $f : (Y, y_0) \to (X, x_0)$ is a connected* *cover, then there is unique map* $\tilde{F} : (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ *such that* $p = f \circ \tilde{F}$*. That is, the following diagram commutes*

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "f"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "p"] & (X, x_0) \end{tikzcd}.$$

Proof. Apply the lifting theorem, Theorem 13.4.2. □

Corollary 14.1.3. *Any two connected and simply connected covers of a pointed, connected and locally path connected topological space are homeomorphic by a unique homeomorphism.*

Proof. Let

$$\begin{tikzcd} & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\ (\tilde{X}_1, \tilde{x}_1) \arrow[ur, dashed, "F"] \arrow[r, "p_1"] & (X, x_0) \end{tikzcd}$$

be connected and simply connected covers of (X, x_0). Then by the previous theorem there is a unique map $F : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 \circ F = p_1$. We need to show that F is a homeomorphism. Again applying the previous theorem, there is a unique map $G : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$ such that $p_1 \circ G = p_2$. It follows that $p_2 \circ F \circ G = p_2$. But the identity morphism from $(\tilde{X}_2, \tilde{x}_2)$ to itself also satisfies the condition $p_2 \circ Id_{\tilde{X}_2} = p_2$, and by uniqueness we get that $F \circ G = Id_{\tilde{X}_2}$. Similarly, we see that $G \circ F = Id_{\tilde{X}_1}$. □

Definition 14.1.4. *Let* X *be a connected and locally path connected topological space such that it has a connected and simply connected cover* $\tilde{X}$*. Then this cover is unique up to a unique homeomorphism. Any such cover is called the universal cover of* X*.*

Remark 14.1.5. The term universal is because of the above universal property that this cover has.

14.2 Deck Transformations

Definition 14.2.1 (Automorphism group of a cover)**.** *Let* $p : Y \to X$ *be a connected cover of* X*. Define*

$$\mathrm{Aut}(Y/X) := \{f : Y \to Y \mid f \textit{ is a homeomorphism and } p \circ f = p\}.$$

Remark 14.2.2. Clearly $\mathrm{Aut}(Y/X)$ is a group under composition of maps.

Before proceeding let us recall some basic definitions related to group actions. If G acts on a set X then the action is said to be transitive if there is only one G-orbit. The action is said to be free if for every $x \in X$, the stabilizer $\mathrm{Stab}(x)$ is trivial.

Proposition 14.2.3. *Let $p : \tilde{X} \to X$ be a universal cover. For any point $x \in X$, $\mathrm{Aut}(\tilde{X}/X)$ acts on $p^{-1}(x)$ and this action is free and transitive.*

Proof. If $f \in \mathrm{Aut}(\tilde{X}/X)$ then by definition $p \circ f = p$. From this it follows that $\mathrm{Aut}(\tilde{X}/X)$ acts on $p^{-1}(x)$. Let $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$. Applying Corollary 14.1.3 to the diagram

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_2) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_1) \arrow[ur, dashed, "f"] \arrow[r, "p"] & (X, x) \end{tikzcd}$$

we see that the action is transitive. If $\tilde{x}_1 = \tilde{x}_2$, then by uniqueness in Corollary 14.1.3 we see that the action is free. $\square$

Let $p : \tilde{X} \to X$ be a universal cover. Let $V \subset X$ be a path connected open set which is evenly covered. Let $x \in V$ be a point, then we may write

$$p^{-1}(V) = \bigsqcup_{t \in p^{-1}(x)} U_t.$$

Clearly, each U_t being homeomorphic to V is path connected and so is a path component of $p^{-1}(V)$.

Proposition 14.2.4. *Let $V \subset X$ be a path connected open set which is evenly covered. The group $\mathrm{Aut}(\tilde{X}/X)$ acts freely transitive on the path components of $p^{-1}(V)$.*

Proof. Let $f \in \mathrm{Aut}(\tilde{X}/X)$. Then $f : \bigsqcup_{t \in p^{-1}(x)} U_t \to \bigsqcup_{t \in p^{-1}(x)} U_t$ is a homeomorphism since it is an automorphism of $\tilde{X}$. It follows that f takes path components to path components, and so for each $t \in p^{-1}(x)$, we see that $f : U_t \to U_{f(t)}$. As $\mathrm{Aut}(\tilde{X}/X)$ acts freely on $p^{-1}(x)$ it follows that if $f : U_t \to U_t$, then $f(t) = t$ and so f is the identity. Also since the action is transitive, it follows that for every t_1 and t_2 in $p^{-1}(x)$ there is f such that $f(t_1) = t_2$. Thus, $f(U_{t_1}) = U_{t_2}$. $\square$

Corollary 14.2.5. *Let $p : \tilde{X} \to X$ be a universal cover. Let $\mathrm{Aut}(\tilde{X}/X)$ have the discrete topology. There is a continuous map $a : \mathrm{Aut}(\tilde{X}/X) \times \tilde{X} \to \tilde{X}$ which makes the following diagram commute.*

$$\begin{tikzcd} \mathrm{Aut}(\tilde{X}/X) \times \tilde{X} \arrow[r, "a"] \arrow[d, "p_2"'] & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[r, "p"] & X, \end{tikzcd}$$

where $p_2 : \mathrm{Aut}(\tilde{X}/X) \times \tilde{X} \to \tilde{X}$ denotes the projection to $\tilde{X}$.

Proof. There is a natural map of sets $a : \mathrm{Aut}(\tilde{X}/X) \times \tilde{X} \to \tilde{X}$ given by

$$(f, t) \mapsto f(t).$$

It is a trivial check that the diagram

$$\begin{tikzcd} \mathrm{Aut}(\tilde{X}/X) \times \tilde{X} \arrow[r, "a"] \arrow[d, "p_2"'] & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[r, "p"] & X \end{tikzcd}$$

commutes. From the commutativity of the above diagram it is clear that $\mathrm{Aut}(\tilde{X}/X)$ maps the fiber $p^{-1}(x)$ to itself. We have already seen, in Proposition 14.2.4, that the action on $p^{-1}(x)$ is free and transitive.

Let us check that the map a is continuous when $\mathrm{Aut}(\tilde{X}/X)$ is given the discrete topology. Let $\mathscr{B}$ be the collection of open subset $U \subset \tilde{X}$ such that U is path connected and $p(U)$ is evenly covered. Then it is an easy check that $\mathscr{B}$ forms a basis for the topology on $\tilde{X}$. Let $x \in V \subset X$ be an evenly covered path connected open set. Then

$$p^{-1}(V) = \bigsqcup_{t \in p^{-1}(x)} U_t.$$

It suffices to show that $a^{-1}(U_t)$ is open in $\mathrm{Aut}(\tilde{X}/X) \times \tilde{X}$. Suppose $a(f, y) \in U_t$, then $f(y) \in U_t$. Let $t' = f^{-1}(t)$, then as shown in Proposition 14.2.4 we see that $f : U_{t'} \to U_t$. Thus, $y \in U_{t'}$ and clearly $a(\{f\} \times U_{t'}) \subset U_t$, that is, $\{f\} \times U_{t'} \subset a^{-1}(U_t)$. This proves that a is continuous. □

For simplicity of notation let G denote the group $\mathrm{Aut}(\tilde{X}/X)$. Since G acts on $\tilde{X}$, we can form equivalence classes of G-orbits for this action. Let $\tilde{X}/G$ denote the space of G-orbits with the quotient topology. From Proposition 14.2.3 it follows that the G-orbits are exactly of the form $p^{-1}(x)$ for $x \in X$. It follows that there is a commutative diagram

(14.2.6)
$$\begin{tikzcd} \tilde{X} \arrow[rr, "p"] \arrow[dr, "\pi"'] & & X \\ & \tilde{X}/G \arrow[ur, "q"'] & \end{tikzcd}$$

Since the G-orbits are exactly the fibers of p, we see that q is a bijection. By the universal property of the quotient topology, Proposition 10.2.2, q is continuous. We now show that q is open, which will prove that it is a homeomorphism.

Theorem 14.2.7. *The map q in equation (14.2.6) is a homeomorphism.*

Proof. We have already see that q is continuous, because of the universal property of the quotient map, and it is a bijection. Let $U \subset \tilde{X}/G$ be an open subset. Then $\pi^{-1}(U)$ is open by the definition,2 of quotient topology and $U = \pi(\pi^{-1}(U))$, as π is surjective. Thus,

$$q(U) = q \circ \pi(\pi^{-1}(U)) = p(\pi^{-1}(U)).$$

Since p is a covering map, it is open by Proposition 12.1.3. It follows that $q(U)$ is open. $\square$

Corollary 14.2.8. *π is a covering map.*

Proof. This is obvious since q identifies the maps p and π. $\square$

14.3 Covering space action

The results of the previous section are a special case of a more general situation which we now describe.

Definition 14.3.1 (Covering space action)**.** *Let G be a group acting on X such that the action satisfies the following conditions:*

(1) The action is free, that is, for all $x \in X$, the stabilizer $\mathrm{Stab}(x) = \{e\}$.

(2) For every $x \in X$, there is an open set $x \in U \subset X$ such that if $g \neq e$ then $g \cdot U \cap U = \emptyset$.

Then we say that the action of G on X is a covering space action.

Proposition 14.3.2. *Let G act on X and assume that the action is a covering space action. Let X/G denote the space of G-orbits with the quotient topology and let $\pi : X \to X/G$ denote the quotient map. Then π is a covering map.*

Proof. For any subset $V \subset X$ it is clear that

$$\pi^{-1}(\pi(V)) = \bigcup_{g \in G} g \cdot V.$$

Assume that V is open. As each $g \in G$ is a homeomorphism of X we see that $g \cdot V$ is open. By the definition of quotient topology we see that $\pi(V)$ is open, that is, π is an open map. Let $x \in X$ and let U be an open set such that $x \in U$ and for every $g \neq e$ we have $g \cdot U \cap U = \emptyset$. We get

$$\pi^{-1}(\pi(U)) = \bigsqcup_{g \in G} g \cdot U\,.$$

To show that $\pi(U)$ is an evenly covered open set it suffices to show that the restriction

$$\pi|_{g \cdot U} : g \cdot U \to \pi(U)$$

is a bijection. We know $\pi|_{g\cdot U}$ is open and continuous, if it is a bijection then it will be a homeomorphism. If this is not a bijection then there are two distinct points $x, y \in U$ which are in the same G orbit, that is, there is $g \in G$ such that $g \cdot x = y$. Since $x \neq y$ it is not possible that $g = e$. If $g \neq e$ then we know that $g \cdot U \cap U = \emptyset$, which gives a contradiction. Thus, $\pi|_{g\cdot U}$ is a bijection and this proves that the open set $\pi(U)$ is evenly covered. □

Theorem 14.3.3. *Let $\tilde{X}$ be a connected, simply connected and locally path connected topological space. Let G act on $\tilde{X}$ and assume that the action is a covering space action. Let $\pi : \tilde{X} \to \tilde{X}/G =: X$ denote the quotient map. Then there is an isomorphism $G \xrightarrow{\sim} \operatorname{Aut}(\tilde{X}/X)$.*

Proof. It is obvious that there is a map $G \to \operatorname{Aut}(\tilde{X}/X)$ since the diagram

$$\begin{tikzcd} \tilde{X} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & \tilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

commutes and g is a homeomorphism.

If $g \in G$ is in the kernel of this map then we will have that $g \in \operatorname{Stab}(\tilde{x})$ for every $\tilde{x} \in \tilde{X}$. However, since the action of G is a covering space action, each such stabilizer is trivial. This shows that $G \subset \operatorname{Aut}(\tilde{X}/X)$.

Notice that $\tilde{X}$ is a universal cover for X. By Proposition 14.2.3 we know that $\operatorname{Aut}(\tilde{X}/X)$ acts freely transitively on the fiber $\pi^{-1}(x_0)$, where $x_0 := \pi(\tilde{x})$. This means that the map

$$\operatorname{Aut}(\tilde{X}/X) \to \pi^{-1}(x_0), \qquad f \mapsto f(\tilde{x})$$

defines a bijection. Thus, to show that G is all of $\operatorname{Aut}(\tilde{X}/X)$ it suffices to show that the composition

$$G \to \operatorname{Aut}(\tilde{X}/X) \to \pi^{-1}(x_0)$$

is surjective. But it is obvious that this composite is surjective since the fiber $\pi^{-1}(x_0)$ is precisely the G orbit of $\tilde{x}$. □

Next we come to the main result of this chapter. Fix $x_0 \in X$ and define

$$\mathscr{C}(X) := \{\text{connected covers } (Y, y_0) \to (X, x_0)\}$$

and

$$\mathscr{S}(\operatorname{Aut}(\tilde{X}/X)) := \{\text{subgroups of } \operatorname{Aut}(\tilde{X}/X)\}$$

Theorem 14.3.4 (Galois correspondence for covering maps)**.** *Let $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal cover. There are bijective maps*

$$\Phi : \mathscr{C}(X) \to \mathscr{S}(\mathrm{Aut}(\tilde{X}/X)) \qquad \text{and} \qquad \Psi : \mathscr{S}(\mathrm{Aut}(\tilde{X}/X)) \to \mathscr{C}(X)$$

which are inverses of each other. These maps are given by

$$\Phi((Y, y_0) \to (X, x_0)) := \mathrm{Aut}(\tilde{X}/Y) \qquad \Psi(H) := ((\tilde{X}/H, \tilde{x}_0/H) \to (X, x_0)).$$

Proof. Before beginning with the proof we make some remarks. In the definition of Φ, in order to talk about $\mathrm{Aut}(\tilde{X}/Y)$ we need a covering map $\tilde{X} \to Y$. What the map should be is clear, using the lifting theorem there is a unique map of pointed spaces $(\tilde{X}, \tilde{x}_0) \to (Y, y_0)$. We need to show that $\tilde{X} \to Y$ is a covering map. Similarly, in the definition of Ψ, we need to show that $\tilde{X}/H \to X$ is a covering map. Once we prove these, then at least the maps Φ and Ψ will be defined. Let us begin by proving these.
Defining Φ : By Theorem 13.4.2, there is a unique $\tilde{p}$ which makes the following diagram commute.

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "f"] & \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, dashed, "\tilde{p}"] \arrow[r, "p"] & (X, x_0) \arrow[r, equal] & \tilde{X}/\mathrm{Aut}(\tilde{X}/X) \end{tikzcd}$$

We need to show that $\tilde{p}$ is a covering map. Choose $y \in Y$. Since f and p are covering maps, there is a path connected open set $V \subset X$ containing $x := f(y)$ and evenly covered by both f and p. Then

$$f^{-1}(V) = \bigsqcup_{y' \in f^{-1}(x)} U_{y'}, \qquad p^{-1}(V) = \bigsqcup_{x' \in p^{-1}(x)} W_{x'}.$$

Here $U_{y'}$ is the unique "sheet" over V which contains y', similarly, $W_{x'}$. Since $W_{x'}$ is connected, it follows that $\tilde{p}(W_{x'}) \subset U_{\tilde{p}(x')}$. We may factor $\tilde{p}$ restricted to $W_{x'}$ as the composite

$$W_{x'} \xrightarrow{p} V \xrightarrow{(f|_{U_{\tilde{p}(x')}})^{-1}} U_{\tilde{p}(x')}$$

This proves that every sheet $W_{x'}$ maps homeomorphically onto the sheet $U_{\tilde{p}(x')}$. From this it is clear that

$$\tilde{p}^{-1}(U_y) = \bigsqcup_{x' \in \tilde{p}^{-1}(y)} W_{x'},$$

and the restriction of $\tilde{p}$ to $W_{x'}$ is a homeomorphism onto U_y. It follows that $\tilde{p}$ is a covering map. Define $\Phi(f) := \mathrm{Aut}(\tilde{X}/Y)$. It is easily checked that this is a subgroup of $\mathrm{Aut}(\tilde{X}/X)$.

Defining Ψ : Let $H \subset \mathrm{Aut}(\tilde{X}/X)$. Then we have a commutative diagram

$$\begin{tikzcd} & (\tilde{X}/H, \tilde{x}_0/H) \arrow[d, "f"] & \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, "\tilde{p}"] \arrow[r, "p"] & (X, x_0) \arrow[r, equal] & \tilde{X}/\mathrm{Aut}(\tilde{X}/X) \end{tikzcd}$$

By Proposition 14.3.2, we see that the map $\tilde{p}$ is a covering map. We need to show that f is a covering map. Let $x \in V$ be a path connected open subset of X which is evenly covered. We write

$$p^{-1}(V) = \bigsqcup_{t \in p^{-1}(x)} U_t \,.$$

We can decompose the set $p^{-1}(x)$ into disjoint H-orbits, let us write

$$p^{-1}(x) = \bigsqcup_{i \in I} H \cdot t_i \,.$$

In fact, the indexing set I can be taken to be $f^{-1}(x)$. It is easy to see that (since path components are in 1-1 correspondence with elements of $p^{-1}(x)$ with compatible action of $\mathrm{Aut}(\tilde{X}/X)$)

$$\begin{aligned} p^{-1}(V) &= \bigsqcup_{i \in I} H \cdot U_{t_i} \\ &= \bigsqcup_{i \in I} \bigsqcup_{h \in H} h \cdot U_{t_i} \end{aligned}$$

If Z_1 and Z_2 are two H-invariant subsets of X which are disjoint, then Z_1/H is disjoint from Z_2/H. Since $H \cdot U_{t_i}$ are H-invariant and disjoint for different $i \in I$, it follows that

$$\tilde{p}(p^{-1}(V)) = \bigsqcup_{i \in I} \Big(H \cdot U_{t_i}\Big)/H \,.$$

Since $\tilde{p}$ is a covering map and so is open, it follows that $\Big(H \cdot U_{t_i}\Big)/H$ is an open subset of $\tilde{X}/H$. Since $\tilde{p} : U_{t_i} \to \Big(H \cdot U_{t_i}\Big)/H$ is continuous, bijective and open, we see that it is a homeomorphism. Let us also observe that since $\tilde{p}$ is surjective,

$$f^{-1}(V) = \tilde{p}(\tilde{p}^{-1}(f^{-1}(V))) = \tilde{p}(p^{-1}(V)) = \bigsqcup_{i \in I} \Big(H \cdot U_{t_i}\Big)/H \,.$$

Thus, to prove that f is a covering map it only remains to show that the restriction of f from $\Big(H \cdot U_{t_i}\Big)/H$ to V is a homeomorphism. But this is clear since we have a commutative diagram

$$\begin{tikzcd} U_{t_i} \arrow[r, "\tilde{p}"] \arrow[d, "p"'] & \big(H \cdot U_{t_i}\big)/H \arrow[dl, "f"] \\ V & \end{tikzcd}$$

in which $\tilde{p}$ and p are homeomorphisms. This proves that f is a covering map and so defines Ψ.

Let us first check that $\Phi \circ \Psi = Id$. We need to show that

$$H = \mathrm{Aut}(\tilde{X}/(\tilde{X}/H)).$$

But this is the content of Theorem 14.3.3.

Next let us check that $\Psi \circ \Phi = Id$. This is equivalent to showing that there is a commutative diagram

$$\begin{tikzcd} \tilde{X}/\mathrm{Aut}(\tilde{X}/Y) \arrow[rr, "\sim"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

It is clear that $\mathrm{Aut}(\tilde{X}/Y) \subset \mathrm{Aut}(\tilde{X}/X)$. Applying Theorem 14.2.7 to the map $\tilde{p}$ we get that there is a commutative diagram

$$\begin{tikzcd} & Y \arrow[d, "f"] & \\ \tilde{X}/\mathrm{Aut}(\tilde{X}/Y) \arrow[ur, dashed, "\sim"] \arrow[r, "p"] & X \arrow[r, equal] & \tilde{X}/\mathrm{Aut}(\tilde{X}/X) \end{tikzcd}$$

This proves that $\Psi \circ \Phi = Id$ and completes the proof of the theorem. $\square$

Theorem 14.3.5. *Let $p : \tilde{X} \to X$ be a universal cover for X. Then $\mathrm{Aut}(\tilde{X}/X)$ is isomorphic to $\pi_1(X, x_0)$.*

Proof. Let us first define a map

$$\Phi : \mathrm{Aut}(\tilde{X}/X) \to \pi_1(X, x_0).$$

Fix a point $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. Let $f \in \mathrm{Aut}(\tilde{X}/X)$, then obviously $p \circ f(\tilde{x}_0) = x_0$. Since $\tilde{X}$ is path connected, there is path γ from $\tilde{x}_0$ to $f(\tilde{x}_0)$. It is clear that $p \circ \gamma$ is a loop at x_0, and so define

$$\Phi(f) := [p \circ \gamma] \in \pi_1(X, x_0).$$

We need to show that Φ does not depend on the choice of the path γ. Suppose γ_1 is another path from $\tilde{x}_0$ to $f(\tilde{x}_0)$. Recall that $I(\gamma_1)$ denote the reverse path from $f(\tilde{x}_0)$ to

$\tilde{x}_0$. Then $\gamma * I(\gamma_1)$ is a path in $\tilde{X}$ at the point $\tilde{x}_0$. Since $\tilde{X}$ is simply connected, there is a homotopy F between the constant path $c_{\tilde{x}_0}$ and the path $\gamma * I(\gamma_1)$. Thus, $p \circ F$ is a homotopy between the constant path c_{x_0} and the path $(p \circ \gamma) * (p \circ I(\gamma_1))$. Thus, we get that in $\pi_1(X, x_0)$

$$\begin{aligned} e &= [(p \circ \gamma) * (p \circ I(\gamma_1))] \\ &= [p \circ \gamma] * [p \circ I(\gamma_1)] \\ &= [p \circ \gamma] * [I(p \circ \gamma_1)] \\ &= [p \circ \gamma] * [p \circ \gamma_1]^{-1} . \end{aligned}$$

This proves that $[p \circ \gamma_1] = [p \circ \gamma]$, proving that the map Φ is well defined.

Next we show that Φ is a group homomorphism. Let $f_1, f_2 \in \mathrm{Aut}(\tilde{X}/X)$. Let γ_1 be a path from $\tilde{x}_0$ to $f_1(\tilde{x}_0)$ and let γ_2 be a path from $\tilde{x}_0$ to $f_2(\tilde{x}_0)$. Then $f_2 \circ \gamma_1$ is a path from $f_2(\tilde{x}_0)$ to $f_2(f_1(\tilde{x}_0))$. Thus, $\gamma_2 * (f_2 \circ \gamma_1)$ is a path from $\tilde{x}_0$ to $f_2 \circ f_1(\tilde{x}_0)$. Thus, using the definition of Φ we get

$$\begin{aligned} \Phi(f_2 \circ f_1) &= [p \circ (\gamma_2 * (f_2 \circ \gamma_1))] \\ &= [p \circ \gamma_2] * [p \circ f_2 \circ \gamma_1] \\ &= [p \circ \gamma_2] * [p \circ \gamma_1] \\ &= \Phi(f_2) * \Phi(f_1) . \end{aligned}$$

This proves that Φ is a group homomorphism.

Next we show that Φ is an inclusion. Suppose $\Phi(f) = e$. Let $\tilde{x}_1 := f(\tilde{x}_0)$ and let γ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. By assumption, $p \circ \gamma$ is homotopic to the constant loop c_{x_0}. By Lemma 12.2.3 it follows that γ, which is the unique lift of $p \circ \gamma$ starting at $\tilde{x}_0$, is a loop at $\tilde{x}_0$. This shows that $\tilde{x}_1 = \tilde{x}_0$. Thus, $f(\tilde{x}_0) = \tilde{x}_0$. But by uniqueness in Theorem 14.1.2 it follows that $f = Id_{\tilde{X}}$. This proves that Φ is an inclusion.

Next we show that Φ is a surjection. Let γ be a loop at x_0. Lift γ to a path $\tilde{\gamma}$ in $\tilde{X}$ starting at $\tilde{x}_0$ and let the end point of the path be $\tilde{x}_1$. Then obviously $\tilde{x}_1 \in p^{-1}(x_0)$. Now apply Corollary 14.1.3 to get an $f \in \mathrm{Aut}(\tilde{X}/X)$ such that $f(\tilde{x}_0) = \tilde{x}_1$. Clearly, $\tilde{\gamma}$ is a path from $\tilde{x}_0$ to $f(\tilde{x}_0)$, and so $\Phi(f) = [p \circ \tilde{\gamma}] = [\gamma]$. This proves that Φ is surjective, and so the Theorem is proved. □

Remark 14.3.6. In view of the above theorem, the Galois correspondence for covering maps becomes a correspondence between the connected covers of X and subgroups of the fundamental group of X. This statement is slightly better than the one involving $\mathrm{Aut}(\tilde{X}/X)$ since the fundamental group seems more intrinsic to the space X in comparison to the group $\mathrm{Aut}(\tilde{X}/X)$. Let $H \subset \mathrm{Aut}(\tilde{X}/X)$ be a subgroup and consider the cover

$$p : (\tilde{X}/H, \tilde{x}_0/H) \to (X, x_0).$$

Show that $\Phi(H) = p_*(\pi_1(\tilde{X}/H, \tilde{x}_0/H))$.

14.4 Existence of universal cover

This section was written up by Sagnik Das. In the entire chapter we very strongly used the hypothesis that X has a simply connected cover. When is this hypothesis satisfied?

Definition 14.4.1 (Semilocally simply connected)**.** *A topological space is called semilocally simply connected if every point $x \in X$ has a neighborhood $i : U \hookrightarrow X$ such that $i_*\pi_1(U, x) \subset \pi_1(X, x)$ is trivial.*

Lemma 14.4.2. *Let U be a path-connected subset of X. Let $x \in U$ and assume $i_*\pi_1(U, x) \subset \pi_1(X, x)$ is trivial. Then for all $y \in U$, $i_*\pi_1(U, y) \subset \pi_1(X, y)$ is trivial. Further, if W is any path-connected subspace of U, then for all $y \in W$, $i_*\pi_1(W, y) \subset \pi_1(X, y)$ is trivial.*

Proof. Left as an exercise. □

It is easily checked that the set of all open subsets U which satisfy the hypothesis of Lemma 14.4.2 forms a basis for the topology on X. We denote this collection by $\mathcal{U}$.

In the following Theorem we shall use the following notation. Suppose we have two paths from x_0 to x_1. We say that they are homotopic if there is a homotopy F between them such that each F_t is a path from x_0 to x_1.

Theorem 14.4.3. *Let X be a connected, locally path connected and semilocally simply connected topological space. Then X has a simply connected cover.*

Proof. Let X be connected, locally path connected and a semilocally simply connected space. Let $\mathcal{P}$ be the set of paths of X that begin at x_0. For $\alpha, \beta \in \mathcal{P}$ define $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and α is homotopic to β. Clearly $\sim$ is an equivalence relation. For $\alpha \in \mathcal{P}$ denote by $[\alpha]$ the equivalence class of α. Define $\tilde{X}$ to be $\mathcal{P}/\sim$.

Let $\alpha \in \mathcal{P}$. Let $U \in \mathcal{U}$ be such that $\alpha(1) \in U$. Define

$$[U, \alpha] = \{[\alpha * \gamma] \mid \gamma : I \longmapsto U, \gamma(0) = \alpha(1)\} \subset \tilde{X}\,.$$

(14.4.4) It is easily checked that if $[\alpha] = [\beta]$ then $[U, \alpha] = [U, \beta]$.

We will now show that the above collection of sets $\{[U, \alpha]\}$ satisfies the hypothesis for being basis of $\tilde{X}$, that is, satisfies the condition in Proposition 2.1.3. Let $[\alpha] \in \tilde{X}$. Let U be an open set as above such that $\alpha(1) \in U$. Then $[U, \alpha]$ contains $[\alpha]$ since

$$[\alpha] = [\alpha * c_{\alpha(1)}] \in [U, \alpha]$$

where $c_{\alpha(1)} : I \longmapsto U$ is the constant path at $\alpha(1)$. This shows that this collection satisfies Proposition 2.1.3 (1).

First we show that if $[\gamma] \in [U, \alpha]$ then $[U, \alpha] = [U, \gamma]$. If $[\gamma] \in [U, \alpha]$ then $[\gamma] = [\alpha * \beta]$. Clearly, we have $[U, \alpha * \beta] \subset [U, \alpha]$. Combining this with observation (14.4.4) we see that

$[U,\gamma] \subset [U,\alpha]$. We also have that $[\gamma * \beta^{-1}] = [\alpha]$. Thus, by the same reasoning we get $[U,\alpha] \subset [U,\gamma]$. This shows that $[U,\alpha] = [U,\gamma]$.

Now let $[\gamma] \in [U,\alpha] \cap [V,\beta]$. Then by the preceding discussion we have $[U,\alpha] = [U,\gamma]$ and $[V,\gamma] = [V,\beta]$. Let W be a path-connected open subset of $U \cap V$ containing $\gamma(1)$. Note that $W \in \mathcal{U}$ by Lemma 14.4.2. Then we have $[W,\gamma] \subset [U,\gamma] \cap [V,\gamma] = [U,\alpha] \cap [V,\beta]$. This shows that this collection satisfies Proposition 2.1.3 (2). Hence $\{[U,\alpha]\}$ constitutes a basis for a topology on $\tilde{X}$.

Now we show that $p : \tilde{X} \longmapsto X$ defined by $p(\alpha) = \alpha(1)$ is continuous. Let $U \in \mathcal{U}$ and consider $p^{-1}(U) = \{[\gamma] \in \tilde{X} | \gamma(0) = x_0, \gamma(1) \in U\}$. Let $[\gamma] \in p^{-1}(U)$, we claim $[U,\gamma] \subset p^{-1}(U)$. Let $[\delta] \in [U,\gamma]$. Then we have

$$[\delta] = [\gamma * \beta] \implies \delta(1) = \beta(1) \in U$$

and hence $[U,\gamma] \subset p^{-1}(U)$. This proves that $p^{-1}(U)$ is open. As $\mathcal{U}$ forms a basis for X, and $p^{-1}(U)$ is open, hence p is continuous and we have

$$p^{-1}(U) = \cup_{\gamma \in p^{-1}(U)} [U,\gamma] .$$

Next we show that if $[U,\alpha] \cap [U,\alpha'] \neq \phi$ then $[U,\alpha] = [U,\alpha']$. Assume that the intersection is nonempty. Then we will have $[\alpha * \beta_1] = [\alpha' * \beta_2]$ for some $\beta_1, \beta_2 : I \longmapsto U$ such that $\beta_1(0) = \alpha(1)$ and $\beta_2(0) = \alpha'(1)$. First note that $\beta_1(1) = \beta_2(1)$.

Let $[\alpha * \beta] \in [U,\alpha]$, $\beta : I \longmapsto U$ is a path and $\beta(0) = \alpha(1)$. Define $\beta_3 := \beta^{-1} * \beta_1 * \beta_2^{-1}$. Now we see

$$[\alpha * \beta_1 * \beta_2^{-1} * \alpha'^{-1}] = [\alpha * \beta_1 * \beta_2^{-1} * (\beta_3^{-1} * \beta^{-1} * \beta * \beta_3) * \alpha'^{-1}]$$

Now by associativity we have

$$[\alpha * \beta_1 * \beta_2^{-1} * \alpha'^{-1}] = [\alpha * (\beta_1 * \beta_2^{-1} * \beta_3^{-1} * \beta^{-1}) * \beta * \beta_3 * \alpha'^{-1}]$$

Now we observe that $(\beta_1 * \beta_2^{-1} * \beta_3^{-1} * \beta^{-1})$ is a loop in $U \in \mathcal{U}$. Hence, in X, the loop $(\beta_1 * \beta_2^{-1} * \beta_3^{-1} * \beta^{-1})$ is homotopic to constant loop at $\alpha(1)$. Thus, we have

$$[\alpha * \beta_1 * \beta_2^{-1} * \alpha'^{-1}] = [\alpha * \beta * \beta_3 * \alpha'^{-1}] .$$

Since $[\alpha * \beta_1] = [\alpha' * \beta_2]$ we have that $[\alpha * \beta_1 * \beta_2^{-1} * \alpha'^{-1}]$ is homotopic to the constant loop. This implies $[\alpha * \beta] = [\alpha' * \beta_3^{-1}] \in [U,\alpha']$. Thus, $[U,\alpha] \subset [U,\alpha']$. Reversing the role of α and α' we will have $[U,\alpha'] \subset [U,\alpha]$ which implies $[U,\alpha'] = [U,\alpha]$. This discussion shows that

$$p^{-1}(U) = \coprod_{\gamma_i} [U,\gamma_i] .$$

Next we show $p : [U, \alpha] \longmapsto U$ is a homeomorphism. First let us show that p is bijective on $[U, \alpha]$. Clearly p is surjective as U is path connected. To check the injectivity let $p(\alpha * \beta) = p(\alpha * \beta')$. Then we have $\beta(1) = \beta'(1)$. Then $\alpha * \beta * {\beta'}^{-1} * \alpha^{-1}$ makes sense. But we see $\beta * {\beta'}^{-1}$ is a loop in U which will be homotopic to the constant loop at $\alpha(1)$. Then we have

$$[\alpha * \beta * {\beta'}^{-1} * \alpha^{-1}] = [c_{x_0}]$$

Hence we have $[\alpha * \beta] = [\alpha * \beta']$ and $p : [U, \alpha] \longmapsto U$ is injective. This proves bijectivity of p on $[U, \alpha]$.

Let A be a topological space and suppose $\mathscr{A}$ is a basis for the topology on A. Let $f : A \to B$ be a map of topological spaces such that for every $U \in \mathscr{A}$ the image $f(U)$ is open in B. Then f is an open map. If we apply this observation to the map $p : \tilde{X} \to X$, then using the above observation that for $U \in \mathcal{U}$ we have $p([U, \alpha]) = U$, which is open in X, then we see that p is an open map. Thus, the restriction of p to $[U, \alpha]$ is also an open map.

Finally we will prove that $\pi_1(\tilde{X}, c_{x_0}) = \{e\}$. Let us begin by making the following observation. Suppose $\gamma : I \to X$ is a path, then the unique lift of this path $\tilde{\gamma} : I \to \tilde{X}$ is given by $\tilde{\gamma}(s) = \gamma_s$, where $\gamma_s : I \longmapsto X$ is the path defined by $\gamma_s(t) = \gamma(st)$. Clearly, this is a lift of γ since $p \circ \tilde{\gamma} = \gamma$. We only need to check that $\tilde{\gamma}$ is continuous. Let U be an open set which contains $\gamma(s)$ and assume that $\gamma(s - \epsilon, s + \epsilon) \subset U$. The path $\gamma_s \in [U, \gamma_s]$ and we will show that $\tilde{\gamma}^{-1}([U, \gamma_s])$ contains $(s - \epsilon, s + \epsilon)$. Let $s' \in (s - \epsilon, s + \epsilon)$. Consider the path $\beta(t) = \gamma(s + t(s' - s))$. Then one checks easily that $[\gamma_{s'}] = [\gamma_s * \beta]$. This proves that $(s - \epsilon, s + \epsilon) \subset \tilde{\gamma}^{-1}([U, \gamma_s])$ and hence proves continuity.

Let $\tilde{\gamma}$ be a loop in $\tilde{X}$ based at c_{x_0}. Let $\gamma := p \circ \tilde{\gamma}$. Then γ is a loop at x_0. From the discussion in the preceding paragraph, it follows that $\tilde{\gamma}(s)$ is the path γ_s. In particular, $\tilde{\gamma}(1) = \gamma$. But since $\tilde{\gamma}$ is a loop at c_{x_0} it follows that $\tilde{\gamma}(1) = c_{x_0}$. Thus, it follows that γ and c_{x_0} are the same points in $\tilde{X}$, that is, as paths in X we have $[\gamma] = [c_{x_0}]$. This shows that $p_*(\tilde{\gamma}) = [p \circ \tilde{\gamma}] = [\gamma]$ is the identity in $\pi_1(X, x_0)$. Since p_* is an inclusion for a covering map, see Exercise 13.8.3, it follows that $\pi_1(\tilde{X}, c_{x_0})$ is trivial. $\square$

14.5 Exercises

14.5.1. Prove that the set of points $z \in \mathbb{D}^2$ for which $\mathbb{D}^2 - \{z\}$ is simply connected is precisely S^1. Hence prove that if $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a homeomorphism then $f(S^1) = S^1$.

14.5.2. Find the fundamental groups of the following spaces.

1. $\mathbb{C} - \{0\}$.
2. $\mathbb{C}^*/G$ where G is the group of homeomorphisms $\{\phi^n : n \in \mathbb{Z}\}$ where $\phi(z) = 2z$.
3. $\mathbb{C}^*/H$ where H is the group of homeomorphisms $\{\phi^n : n \in \mathbb{Z}\}$ where $\phi(z) = 2\overline{z}$.

14.5.3. Let $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid \|z_0^2\| + \|z_1\|^2 = 1\}$ be considered as a subspace of $\mathbb{C}^3$. Let q be prime to p and define $h : S^3 \to S^3$ by

$$h(z_0, z_1) = (\exp(2\pi i/p)z_0, \exp(2\pi iq/p)z_1).$$

Show that h is a homeomorphism of S^3 with $h^p = 1$. Define $G := \mathbb{Z}/p\mathbb{Z}$ action on S^3 by

$$n.(z_0, z_1) = h^n(z_0, z_1),\ n \in G.$$

Show that this action is a covering space action and hence $S^3 \to S^3/G$ is a covering map. The base space is a called the lens space $L(p, q)$.

14.5.4. We defined a map $\Phi : \mathrm{Aut}(\tilde{X}/X) \xrightarrow{\sim} \pi_1(X, x_0)$. Let $H \subset \mathrm{Aut}(\tilde{X}/X)$ be a subgroup and consider the cover

$$p : (\tilde{X}/H, \tilde{x}_0/H) \to (X, x_0).$$

Show that $\Phi(H) = p_*(\pi_1(\tilde{X}/H, \tilde{x}_0/H))$.

14.5.5. Explain how to use the Galois correspondence for covering maps to show that subgroup of a free group is free.

14.5.6. Let F_n denote the free group on n symbols. Show that $F_n \subset F_2$. (HINT: Use the map $\mathbb{R} \to S^1$ to construct a cover $Y \to S^1 \vee S^1$ such that $Y/\mathbb{Z} \cong S^1 \vee S^1$. What is $Y/n\mathbb{Z}$?)

14.5.7. Let $p : Y \to X$ be a cover. Say Y is normal if $\mathrm{Aut}(Y/X)$ acts transitively on the fiber $p^{-1}(x)$. Assume that X has a universal cover. Show that under the Galois correspondence, normal subgroups correspond to normal covers.

14.5.8. State true or false with proof. There is no space Y such that $S^1 \times Y$ is homeomorphic to $\mathbb{P}^2_{\mathbb{R}}$. Here $\mathbb{P}^2_{\mathbb{R}} := \Big(\mathbb{R}^3 \setminus (0,0,0)\Big)/\mathbb{R}^\times$ = space of lines in $\mathbb{R}^3$. (HINT: First show that $\mathbb{P}^2_{\mathbb{R}} = S^2/\{\pm 1\}$)

14.5.9. Let $\mathbb{P}^n_{\mathbb{R}} := \Big(\mathbb{R}^{n+1} \setminus (0,0,0)\Big)/\mathbb{R}^\times$ = space of lines in $\mathbb{R}^{n+1}$. Compute its fundamental group.

14.5.10. Let $p : Y \to X$ be a covering map. Suppose the fundamental group of X is isomorphic to $\mathbb{Z}$ and $p^{-1}(x_0)$ is finite, find the fundamental group of Y.

14.5.11. Prove that any two n-sheeted covers of S^1 are equivalent.

14.5.12. Prove that if the universal cover of X is compact then the fundamental group of X is finite.

Chapter 15

Homotopy

15.1 Retracts and Deformation retracts

In section 13.2 we saw examples where the topological space could be continuously deformed to a subspace. In those simple examples the subspace was a point. It turned out that the fundamental group of the space was equal to the fundamental group of the subspace. In this section we will explore this idea further, of comparing the fundamental groups of spaces which can be continuously deformed into one another.

Definition 15.1.1 (Retract)**.** *Let X be a topological space and let $i : A \subset X$ be a subspace. We say that A is a retract of X if there is a continuous map $f : X \to A$ such that $f \circ i = Id_A$.*

Proposition 15.1.2. *Let $x_0 \in A \subset X$. Then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an inclusion.*

Proof. Since $(f \circ i)_* = f_* \circ i_*$ and $f \circ i = Id_A$, this shows that

$$f_* \circ i_* = Id\,.$$

This proves that i_* is an inclusion. □

Definition 15.1.3 (Deformation retract)**.** *Let $A \subset X$ and assume that there is a map $F : X \times I \to X$ such that*

1. *$F|_{X\times 0} = Id_X$,*
2. *$F(a, t) = a$ for all $a \in A$ and $t \in I$,*
3. *$F(x, 1) \in A$ for all $x \in X$.*

Then we say that A is a deformation retract of X.

In section 13.2 we saw that a point is a deformation retract of the disk D and $\mathbb{R}$. It can be shown easily that a point is a deformation retract of $\mathbb{R}^n$. Next let us see that S^1 is a deformation retract of $X := \mathbb{R}^2 \setminus (0,0)$. Define $F : X \times I \to X$ by

$$F(x,t) = (1-t)x + t\frac{x}{\|x\|}\,.$$

Proposition 15.1.4. *Let A be a deformation retract of X. Let $x_0 \in A$. Then the inclusion $i : A \subset X$ induces an isomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$.*

Proof. Obviously a deformation retract is a retract. The map $x \mapsto F(x,1)$ is a retract from $X \to A$. In view of Proposition 15.1.2 we see that i_* is an inclusion. Let $\gamma \in \pi_1(X, x_0)$ be the class of a loop. As before, consider the composite

$$G := F \circ (\gamma, Id) : S^1 \times I \to X \times I \to X$$

Then $G|_{S^1\times 0} = \gamma$ and $G|_{S^1\times 1}$ is a loop at x_0 contained in A. Thus, there is a map $\delta : S^1 \to A$ such that $i \circ \delta = G|_{S^1\times 1}$, and G is a homotopy between $i_*\delta$ and γ. This shows that i_* is surjective. □

Corollary 15.1.5. *The fundamental group $\pi_1(\mathbb{R}^2 \setminus (0,0), 1) \cong \mathbb{Z}$.*

15.2 Homotopy and homotopy equivalence

The above discussion once again shows that utility of the idea of studying maps which can be continuously deformed into one another. In the above, the identity map of X was being continuously deformed to a retract of X to A. We now formally introduce the notion of two maps being homotopic.

Definition 15.2.1 (Homotopy)**.** *Let $f, g : X \to Y$ be two maps. If there is a map $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ then we say that f and g are homotopic.*

Proposition 15.2.2. *Consider the relation $f \sim g$ if f and g are homotopic to each other. This is an equivalence relation.*

Proof. Proof is similar to the proof of Proposition 13.1.3 and is left as an exercise. □

Definition 15.2.3 (Homotopy equivalence)**.** *Two spaces X and Y are said to be homotopy equivalent if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$.*

Proposition 15.2.4. *If $i : A \subset X$ is a deformation retract then A and X are homotopy equivalent.*

Proof. Let $F : X \times I \to X$ denote the homotopy deforming X to A. Let $g := F|_{X\times 1}$. Then $g \circ i = Id_A$ and $i \circ g \sim Id_X$. □

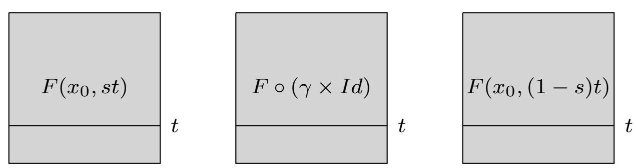
Next we want to show that if X and Y are homotopy equivalent then they have isomorphic fundamental groups. Suppose there is a point x_0 in X such that $g(f(x_0)) = x_0$, the homotopy $g \circ f \sim Id_X$ keeps x_0 fixed at all time and the homotopy $f \circ g \sim Id_Y$ keeps the point $f(x_0)$ fixed at all time, then it is easy to check that f_* and g_* are inverses of each other. However, there may not exist any such point. For the next lemma, recall the notation and isomorphism $\Phi_2(h)$ from Theorem 13.3.1.

Lemma 15.2.5. *Let $F : X \times I \to Y$ be a homotopy. Denote by F_t the restriction of F to $X \times t$. Let $x_0 \in X$ and define a path $\delta(t) := F_t(x_0)$. Then $F_{0*} = \Phi_2(\delta) \circ F_{1*}$ as maps from $\pi_1(X, x_0) \to \pi_1(Y, F_0(x_0))$.*

Proof. The claim in the lemma is that there is a commutative diagram

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "F_{1*}"] \arrow[dr, "F_{0*}"'] & \pi_1(Y,F_1(x_0)) \arrow[d, "\Phi_2(\delta)"] \\
& \pi_1(Y,F_0(x_0))
\end{tikzcd}$$

Let $\gamma : [0,1] \to X$ be a loop based at x_0. Consider the following three maps from $I \times I \to Y$.



First consider the middle box. On the horizontal line $I \times t$ this defines the map $s \mapsto F(\gamma(s), t)$. This path defines a loop at $F(x_0, t) = F_t(x_0)$, that is, its image at the end points is $F_t(x_0)$. Next consider the left box. Again, on the line $I \times t$ the path starts at $F(x_0, 0) = F_0(x_0)$ and ends at $F(x_0, t) = F_t(x_0)$. Similarly, in the right box the path starts at $F(x_0, t) = F_t(x_0)$ and ends at $F(x_0, 0) = F_0(x_0)$. It is clear that we can join these continuous maps and re-parameterize to give a map $G : I \times I \to Y$. Then

- $G(0,t) = F_0(x_0) = G(1,t)$ for all t.
- $G(s,0) \sim c_{F_0(x_0)} * (F_0 \circ \gamma) * c_{F_0(x_0)} \sim F_{0*}(\gamma)$.
- $G(s,1) \sim \delta * (F_1 \circ \gamma) * I(\delta) \sim \Phi_2(\delta)(F_{1*}(\gamma))$

This proves the lemma. □

Theorem 15.2.6. *Let X and Y be path connected spaces which are homotopy equivalent. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$.*

Proof. Before we proceed with the proof let us remark that, since X and Y are path connected, the isomorphism class of the fundamental group is independent of the base point. This theorem, therefore, states that fundamental groups of homotopy equivalent spaces are isomorphic.

Let $f : X \to Y$ and $g : Y \to X$ be maps such that $g \circ f \sim Id_X$ and $f \circ g \sim Id_Y$. By Lemma 15.2.5 we get that there is an isomorphism $\Phi_2(\delta)$ such that $(g \circ f)_* = \Phi_2(\delta) \circ Id$. This shows that $(g \circ f)_* = g_* \circ f_*$ is an isomorphism, which shows that f_* is an inclusion and g_* is a surjection. Applying the same reasoning to $f \circ g \sim Id_Y$ we see that f_* is a surjection. Thus,

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism. □

15.3 Mapping cylinder and homotopy equivalence

As subspace A being a deformation retract of X captures the idea, that X can be continuously deformed into A, in a very strong sense. Homotopy equivalence, seems to be a weaker notion of being able to deform one space into another. In this section we will see that if X and Y are homotopy equivalent, then there is a space Z such that both X and Y are deformation retracts of Z.

Definition 15.3.1 (Mapping cylinder)**.** *Let $f : X \to Y$ be a continuous map. The mapping cylinder of f, denoted M_f is the space*

$$X \times I \bigsqcup Y \big/ (x, 1) \sim f(x).$$

When we talk of the inclusion $X \hookrightarrow M_f$, unless otherwise mentioned, we will mean $x \mapsto (x, 0)/\sim$.

We will need the following result, due to Whitehead.

Lemma 15.3.2. *Let X be a topological space and let $\sim$ be an equivalence relation on X. Let Y be a locally compact topological space. Consider the obvious equivalence relation on $X \times Y$ given by $(x, y) \sim_1 (x', y')$ iff $x \sim x'$ and $y = y'$. Then the natural map $X \times Y \big/ {\sim_1} \to X/{\sim} \times Y$ is a homeomorphism.*

Proof. It is clear that the map $X \times Y \to X/{\sim} \times Y$ factors to give a bijective continuous map $q : X \times Y \big/ {\sim_1} \to X/{\sim} \times Y$. Let $V \subset X \times Y \big/ {\sim_1}$ be an open subset. Let $p : X \times Y \to X \times Y \big/ {\sim_1}$ denote the quotient map. To show that $q(V)$ is open, we need to show that for any point $(x, y) \in p^{-1}(V)$ there are open sets $U \subset X$ and $W \subset Y$ such that U is a union of equivalence classes, $(x, y) \in U \times W$ and $U \times W \subset p^{-1}(V)$.

Let $W \subset Y$ be an open set containing y such that $\bar{W}$ is compact and $x \times \bar{W} \subset p^{-1}(V)$. There is such a W since Y is locally compact and $p^{-1}(V)$ is open. Now we make the following two observations

1. Let $C(x)$ denote the equivalence class of x in X. One checks that if $(x,t) \in p^{-1}(V)$ then $C(x) \times t \subset p^{-1}(V)$. It follows that if $x \times \bar{W} \subset p^{-1}(V)$ then $C(x) \times \bar{W} \subset p^{-1}(V)$.

2. Applying tube lemma to $x \times \bar{W} \subset (X \times \bar{W}) \cap p^{-1}(V)$ we see that there is an open $U \subset X$ such that $x \in U$ and $U \times \bar{W} \subset p^{-1}(V)$.

Let
$$U' := \{a \in X \mid a \times \bar{W} \subset p^{-1}(V)\}$$
The first observation above says that U' is a union of equivalence classes and the second says that U' is open. Thus, $(x,y) \in U' \times W$, which proves the lemma. □

Proposition 15.3.3. *The inclusion $i : Y \hookrightarrow M_f$ is a deformation retract.*

Proof. Using the previous lemma, define a homotopy $F : M_f \times I \to M_f$ as follows.

- On $X \times I \times I$ define F by $F(x,t,s) = (x, s + t(1-s))/\sim$.
- On $Y \times I$ define $F(y,t) = y/\sim$.

We need to check that $F(x,1,t) = F(f(x),t)$. This is equivalent to $(x,1) \sim f(x)$, which is true. Since $F(y,t)$ is the identity on Y for every t, and $F(M_f,1) \subset Y$, the proposition is proved. □

Definition 15.3.4 (Homotopy Extension Property(HEP))**.** *Let $A \subset X$. The pair (X,A) is said to have the homotopy extension property if every continuous map $f : (X \times 0) \bigcup (A \times I) \to Y$ can be lifted to a continuous map $F : X \times I \to Y$.*

Remark 15.3.5. The space $X \times 0 \bigcup A \times I$ is homeomorphic to the mapping cylinder of $i : A \hookrightarrow X$.

Proposition 15.3.6. *(X,A) has HEP iff $X \times 0 \bigcup A \times I$ is a retract of $X \times I$.*

Proof. Assume $X \times 0 \bigcup A \times I$ is a retract of $X \times I$. Let $r : X \times I \to X \times 0 \bigcup A \times I$ denote the retract. Given $f : X \times 0 \bigcup A \times I \to Y$, the map $f \circ r$ is the required lift of f to $X \times I$.

Now assume that (X,A) has the HEP. Let $Y = X \times 0 \bigcup A \times I$ and let $f = Id_Y$. Applying HEP we get that $X \times 0 \bigcup A \times I$ is a retract of $X \times I$. □

Corollary 15.3.7. *If (X,A) has HEP then A is a closed subspace of X.*

Proof. Using the previous proposition, $X \times 0 \bigcup A \times I$ is a retract of $X \times I$. If r denotes the retract, then $X \times 0 \bigcup A \times I$ is precisely the set of points such that $r(z) = z$, and hence it is closed. Intersecting with $X \times 1$ we see that A is closed in X. □

Corollary 15.3.8. *If (X,A) has HEP then so does $(X \times Y, A \times Y)$.*

Proof. Let $r : X \times I \to X \times 0 \bigcup A \times I$ denote a retract. Then

$$r \times Id_Y : X \times I \times Y \to X \times 0 \times Y \bigcup A \times I \times Y$$

is a retract, which shows that $(X \times Y, A \times Y)$ has HEP. □

Proposition 15.3.9. *Let $f : X \to Y$ be a map. Let $X \hookrightarrow M_f$ be the inclusion $x \mapsto (x, 0)$. Then the pair $(M_f, X \sqcup Y)$ has HEP.*

Proof. We will show that $M_f \times 0 \bigcup (X \sqcup Y) \times I$ is a retract of $M_f \times I$. It is easy to see that there is a retract $r : I \times I \to I \times 0 \bigcup (0 \times I \sqcup 1 \times I)$. Taking product with X we see that there is a retract $\tilde{r} := Id_X \times r$

$$\tilde{r} : X \times I \times I \to X \times I \times 0 \bigcup (X \times 0 \times I \sqcup X \times 1 \times I)\,.$$

By Lemma 15.3.2 we have

$$M_f \times I = X \times I \times I \bigsqcup Y \times I / (x, 1, t) \sim (f(x), t)\,.$$

The space $M_f \times 0 \bigcup (X \sqcup Y) \times I$ is precisely

$$\begin{aligned} T :=& X \times I \times 0 \bigcup (X \times 0 \times I \sqcup Y \times I) / (x, 1, t) \sim (f(x), t) = \\ &= X \times I \times 0 \bigcup (X \times 0 \times I \sqcup X \times 1 \times I \cup Y \times I) / (x, 1, t) \sim (f(x), t) \end{aligned}$$

It suffices to define a map R from $X \times I \times I \sqcup Y \times I$ to T and check that $R(x, 1, t) = R(f(x), t)$. Define $R(x, s, t) := \tilde{r}(x, s, t)/\sim$ and define $R(y, t) = (y, t)/\sim$. Since $\tilde{r}$ is a retract, we see that $R(x, 1, t) = \tilde{r}(x, 1, t)/\sim = (x, 1, t)/\sim = (f(x), t)/\sim = R(f(x), t)$. Since $\tilde{r}$ is a retract, it follows that R is a retract. This proves the proposition. □

Corollary 15.3.10. *The pairs (M_f, X) and (M_f, Y) have HEP.*

Proof. Consider the retract $M_f \times 0 \bigcup (X \sqcup Y) \times I \to M_f \times 0 \bigcup Y \times I$ given by the identity on $M_f \times 0$ and $Y \times I$, and which sends (x, t) to $(x, 0)$. Combining this with the previous proposition shows that $M_f \times 0 \bigcup Y \times I$ is a retract of $M_f \times I$ and so (M_f, Y) has HEP. Similarly, one shows that (M_f, X) has HEP. □

Definition 15.3.11 (Mapping cylinder neighborhood)**.** *Let $A \subset X$. We say that A has a mapping cylinder neighborhood if there is a map $f : Z \to A$ and a map $h : M_f \to X$ such that*

1. *$h|_A = Id_A$,*
2. *$h(M_f)$ is a closed subspace,*

3. *$h(M_f \setminus Z)$ is an open subspace,*

4. *h is a homeomorphism onto its image.*

Proposition 15.3.12. *If A has a mapping cylinder neighborhood then (X, A) has HEP.*

Proof. Since h is a homeomorphism onto its image, we may identify M_f with $h(M_f)$ and view M_f as a subspace of X which contains A.

Let T denote the set $X \setminus (M_f \setminus Z)$. Then T is a closed subset of X since $M_f \setminus Z$ is open. The intersection $T \cap M_f$ is equal to Z, which is closed since both subsets are closed.

Let $f : X \times 0 \bigcup A \times I \to Y$ be given. Define $f(\alpha, t) := f(\alpha, 0)$ for $\alpha \in T$ and $t \in I$. Thus, the definition of f has been extended to $X \times 0 \bigcup T \times I \cup A \times I$. The intersection of this space with $M_f \times I$ is equal to $M_f \times 0 \bigcup Z \times I \cup A \times I$. Since $(M_f, Z \sqcup A)$ has HEP, we can extend this to a function $F : M_f \times I \to Y$. On $Z \times I$ the functions F and f agree by construction. Since $X \times I = T \times I \cup M_f \times I$, both these functions join to give a continuous function $F : X \times I \to Y$, which lifts f. □

Definition 15.3.13 (Contractible)**.** *A space is called contractible if it is homotopy equivalent to a point. This is equivalent to saying that there is $a_0 \in A$ and a map $F : A \times I \to A$ such that $F(a, 0) = a$ and $F(a, 1) = a_0$.*

Proposition 15.3.14. *If a pair (X, A) satisfies HEP and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Proof. Let $F : A \times I \to A$ be a homotopy such that $F(a, 0) = a$ and $F(a, 1)$ is constant. Extend this to $X \times 0 \bigcup A \times I$ by defining $F(x, 0) = x$. Now use HEP for (X, A) and extend this to a homotopy $F : X \times I \to X$. Since $F(A \times I) \subset A$ we get a commutative diagram

$$\begin{tikzcd} X \times I \arrow[r, "F"] \arrow[d, "q \times Id"'] & X \arrow[d, "q"] \\ X/A \times I \arrow[r, "\bar{F}"] & X/A \end{tikzcd}$$

Further, since F_1 maps A to a point we get commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{F}_1"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

This proves that $Id_{X/A} = \bar{F}_0 \sim \bar{F}_1 = q \circ g$. We also have $g \circ q = F_1 \sim F_0 = Id_X$. □

Definition 15.3.15. *Let $A \subset X$ and let $F : X \times I \to Y$ be a homotopy. We say F is a homotopy relative to A if the restriction to $A \times I$ is independent of I.*

Proposition 15.3.16. *Let (X, A) and (Y, A) satisfy HEP. Assume that $f : X \to Y$ is a homotopy equivalence such that $f|_A = Id_A$. Then there is $g : Y \to X$ such that $g \circ f \sim Id_X \operatorname{rel} A$ and $f \circ g \sim Id_Y \operatorname{rel} A$.*

Proof. Let $g : Y \to X$ be a homotopy inverse for f. This means that there is a homotopy F between $g \circ f$ and Id_X. Restricting F to $A \times I$ and using $f|_A = Id_A$, we see that $F : A \times I \to X$ is a homotopy between $g|_A$ and Id_A. Define a map $H : Y \times 0 \bigcup A \times I \to X$ by defining it to be g on $Y \times 0$ and F on $A \times I$. Clearly this is well defined since F and g agree on $A \times 0$. Using HEP for (Y, A), this extends to give a homotopy $\tilde{H} : Y \times I \to X$. Note that

- $\tilde{H}(y, 0) = H(y, 0) = g(y)$,
- $\tilde{H}(a, 1) = F(a, 1) = a$.

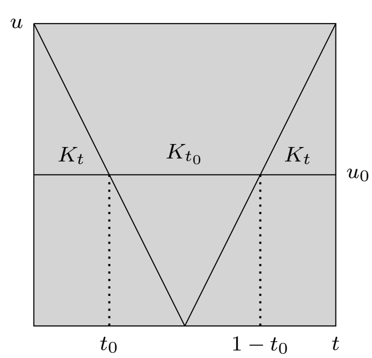
Define $g_1(y) = \tilde{H}(y, 1)$. Then $g_1|_A = Id_A$.

Let $F'(x, t) = F(x, 1-t)$. Then F' is a homotopy between Id_X and $g \circ f$. Since $\tilde{H}$ is a homotopy between g and g_1, it follows that $\tilde{H} \circ f$ is a homotopy between $g \circ f$ and $g_1 \circ f$. Denote by $F' * (\tilde{H} \circ f)$ the combined homotopy between Id_X and $g_1 \circ f$. This means the obvious, for $0 \leqslant t \leqslant 1/2$ we cover F' and for $1/2 \leqslant t \leqslant 1$ we cover $\tilde{H} \circ f$.

Let $K : A \times I \to X$ denote the restriction of $F' * (\tilde{H} \circ F)$ to $A \times I$. Let us check that

- $K(a, 0) = a$. This is true since $K(a, 0) = F'(a, 0) = a$.
- $K(a, t) = K(a, 1-t)$. To check this it suffices to assume that $0 \leqslant t \leqslant 1/2$. Then $K(a, t) = F'(a, 2t) = F(a, 1-2t)$. On the other hand $K(a, 1-t) = \tilde{H} \circ f(a, 1-2t) = \tilde{H}(a, 1-2t)$. Since by definition $\tilde{H}$ extends F, we get that $\tilde{H}(a, 1-2t) = F(a, 1-2t)$.

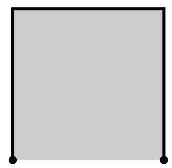
Now we shall define a homotopy of homotopies, that is, a map $\tilde{K} : A \times I \times I \to X$.



Precisely, this is given by

$$\tilde{K}(a, t, u) = \begin{cases} K(a, t) & 0 \leqslant t \leqslant \frac{1-u_0}{2} \\ K(a, \frac{1-u_0}{2}) & 0 \leqslant t \leqslant \frac{1+u_0}{2} \\ K(a, t) & \frac{1+u_0}{2} \leqslant t \leqslant 1 \end{cases}$$

For $\tilde{K}$ to be continuous, the only condition needed is that $K(a, \frac{1-u_0}{2}) = K(a, \frac{1+u_0}{2})$, which we know is true. Extend $\tilde{K}$ to $X \times I \times 0$ by $F' * (\tilde{H} \circ f)$. This is possible since on $A \times I \times 0$ these two maps agree and in fact both are equal to K. By Corollary 15.3.8 the pair $(X \times I, A \times I)$ has HEP. Thus, $\tilde{K}$ extends to a homotopy $X \times I \times I \to X$. Next we restrict this homotopy to the dark edges on the square, as in the following diagram.

$X \times$  $\xrightarrow{\tilde{K}} X$

Now we note the following.

- $\tilde{K}$ restricted to the point $X \times (0,0)$ is equal to F' restricted to $X \times 0$, which is Id_X.
- $\tilde{K}$ restricted to the point $X \times (1,0)$ is equal to $\tilde{H} \circ f$ restricted to $X \times 1$, which is $g_1 \circ f$.
- $\tilde{K}$ restricted to the path $A \times 0 \times I$ is equal to K restricted to $A \times 0$, which is Id_A.
- $\tilde{K}$ restricted to the path $A \times I \times 1$ is equal to K restricted to $A \times 0$, which is Id_A.
- $\tilde{K}$ restricted to the path $A \times 1 \times I$ is equal to K restricted to $A \times 1$, which is Id_A.

The above shows that we have found a homotopy from Id_X to $g_1 \circ f$ which is always Id_A on A. That is, $Id_X \sim g_1 \circ f$ rel A.

From the preceding sentence, or as already observed above that $g_1|_A = Id_A$, we may apply the same argument, replacing f by g_1, to $g_1 : Y \to X$. Thus, we will get $f_1 : X \to Y$ such that $Id_Y \sim f_1 \circ g_1$ rel A. Since f_1, g_1, f are all Id_A on A, we get

$$\begin{aligned} f_1 &\sim f_1 \circ (g_1 \circ f) \text{ rel } A \\ &= (f_1 \circ g_1) \circ f \\ &\sim f \text{ rel } A \end{aligned}$$

This proves that $Id_Y \sim f \circ g_1$ rel A. Thus, the proposition is proved. □

Corollary 15.3.17. *If (X, A) satisfies HEP and $i : A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.*

Proof. Apply the previous proposition with $f = i$. □

Theorem 15.3.18. *A map $f : X \to Y$ is a homotopy equivalence iff X is a deformation retract of M_f.*

Proof. If X is a deformation retract of M_f, then since Y is also a deformation retract of M_f (Proposition 15.3.3), we see that X and Y are homotopy equivalent.

Assume that $f : X \to Y$ is a homotopy equivalence. In order to apply the previous corollary we need to show that $i : X \hookrightarrow M_f$ is a homotopy equivalence. Let $j : Y \hookrightarrow M_f$ denote the inclusion, which is a homotopy equivalence. The inclusion i is homotopy equivalent to $j \circ f$. The map $F(x,t) = (x,t)/\sim$ gives a homotopy between i and $j \circ f$. Since both f and j are homotopy equivalences, it follows that i is a homotopy equivalence. It follows from the previous corollary that X is a deformation retract of M_f. $\square$

Corollary 15.3.19. *Since Y is a deformation retract of M_f, this proves that X and Y are homotopy equivalent iff there is a space Z such that both X and Y are deformation retracts of Z.*

Proof. Let $Z = M_f$. $\square$

15.4 Exercises

15.4.1. Let X be a topological space . Show that $f : S^1 \to X$ is homotopic to constant map if and only the map from f can be extended to a map $g : \mathbb{D}^2 \to X$ such that $g_{|S^1} = f$.

15.4.2. Show that there is a circle in the Mobius strip which is a deformation retract of the Mobius strip. Deduce that the Mobius strip and the cylinder are homotopy equivalent.

15.4.3. Let X be path connected and let Y be homotopy equivalent to X. Show that Y is path connected.

15.4.4. Prove that $r : \mathbb{D}^n \to S^{n-1}$ is a retract if and only if S^{n-1} is contractible.

15.4.5. Let $\alpha, \beta : I \to X$ be paths in X such that $\alpha(1) = \beta(0)$. Then given $\alpha' \sim \alpha$ relative to $\{0, 1\}$ and $\beta' \sim \beta$ relative to $\{0, 1\}$, show that

$$\alpha \star \beta \sim \alpha' \star \beta'.$$

15.4.6. Show that any path in X is homotopic to the constant path. (*This uses the fact that I is contractible.*) Use this to show that in X if there are two paths α, β such that $\alpha(0) = \beta(0)$ then they are homotopic.

15.4.7. Let $\alpha : I \to X$ be a path and define $\tau(s) = \alpha(1 - s)$ to be a path from in X. Let e_x denote the constant path at $x \in X$. Show that $\alpha \star \tau \sim e_{\alpha(0)}$ relative to $\{0, 1\}$ and $\tau \star \alpha \sim e_{\alpha(1)}$ relative to $\{0, 1\}$.

15.4.8. Recall we showed that any path from x_0 to x_1 gives an isomorphism from $\pi_1(X, x_0) \to \pi_1(X, x_1)$. Prove that two paths α, β from x_0 to x_1 give rise to the same isomorphism between $\pi_1(X, x_0) \to \pi_1(X, x_1)$ if and only if $[\beta \star \alpha^{-1}]$ is in the center of the group $\pi_1(X, x_0)$.

15.4.9. Let Y be a subspace of $\mathbb{R}^n$ and let $f, g : X \to Y$ be continuous maps. Prove that if for each $x \in X$, $f(x)$ and $g(x)$ can be joined by a straight line in Y then $f \simeq g$. Show that any two maps $f, g : X \to \mathbb{R}^n$ are homotopic.

15.4.10. Let X be any space and let $f, g : X \to S^n$ be continuous maps such that $f(x) \neq -g(x)$ for all $x \in X$. Prove that $f \simeq g$.

15.4.11. Suppose X and Y are homotopy equivalent and Y and Z are homotopy equivalent, show that X and Z are homotopy equivalent.

15.4.12. Let $f : X \to Y$ be a homotopy equivalence. Let $h : X \to Y$ be such that h is homotopic to f. Show that h is also a homotopy equivalence.

15.4.13. Show that the composite of two homotopy equivalences is a homotopy equivalence.

15.4.14. Consider the map $f : S^1 \times S^1 \to S^1 \times S^1$ which switches the factors. Show that f is not homotopic to the identity map.

15.4.15 (Higher homotopy groups)**.** Let X be a topological space and let $x_0 \in X$. Recall that a map $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subset B$. We want to define higher dimensional analogues of the fundamental group. In this exercise sheet we will prove all the statements in the section on the fundamental group in the notes. Fix an integer $n > 1$.

1. The space of n-spheres in X based at x_0 is the set
$$S(X, x_0) := \{\gamma : (I^n, \partial I^n) \to (X, x_0)\}\,.$$
Define a binary operation $*$ on $S(X, x_0)$.

2. Let $f, g \in S(X, x_0)$. A homotopy F between f and g is a continuous map $F : I^n \times I \to X$ such that $F_t := F|_{I^n \times t} \in S(X, x_0)$ for all $t \in I$, $F_0 = f$ and $F_1 = g$. Define a relation $\sim$ on $S(X, x_0)$ by $f \sim g$ if f and g are homotopic. Show that $\sim$ is an equivalence relation on $S(X, x_0)$.

3. Let $\pi_n(X, x_0)$ denote the set of equivalence classes in $S(X, x_0)$ under the relation $\sim$. The equivalence class of f will be denoted by $[f]$. Show that the binary operation $*$ descends to a binary operation
$$* : \pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)\,.$$

4. Show that the binary operation $*$ on $\pi_n(X, x_0)$ is associative.

5. Let $c_{x_0} : I \to X$ denote the constant map x_0. Show that in $\pi_n(X, x_0)$ we have $[f * c_{x_0}] = [c_{x_0} * f] = [f]$.

6. For $f \in S(X, x_0)$, show that there is $g \in S(X, x_0)$ such that $f * g \sim c_{x_0} \sim g * f$.

7. The above exercises prove that the set $\pi_n(X, x_0)$ is a group under the operation $*$ with identity element c_{x_0}. **Show that this group is abelian**.

8. Let $f : X \to Y$ be a continuous map. Define a map $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$. Show that f_* is a group homomorphism. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Show that $g_* \circ f_* = (g \circ f)_*$.

9. Let $f : Y \to X$ be a covering map. Show that $\pi_n(Y, y_0) \to \pi_n(X, f(y_0))$ is an isomorphism. Show that $\pi_n(S^1, 1) = 0$.

Chapter 16

Homology

In this chapter we will attach to each topological space certain algebraic objects (chain complexes). This will allow us to define algebraic invariants for topological spaces (homology groups). The advantage is that we may use algebraic methods (long exact sequences associated to short exact sequences of complexes) to compute homology groups.

The ideas and methods we shall encounter in this chapter have been extended enormously to various contexts, and the various objects have close interactions with each other. In this chapter we shall see the simplest, yet substantial, instance of homological algebra methods in the study of (topological/geometric) spaces. In the first section we collect together the material from homological algebra that we will need.

16.1 Complexes

Definition 16.1.1 (Complex). *A complex is a sequence $\{A_i, d_i\}$, indexed by $\mathbb{Z} \cap (a, b)$, where $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, each A_i is an abelian group and $d_i : A_i \to A_{i-1}$ is a homomorphism of abelian groups such that $d_i \circ d_{i+1} = 0$ (whenever this makes sense, see the examples below). Complexes will also be denoted by $\{A_\bullet, d_\bullet\}$.*

Usually complexes for us will be indexed by $\mathbb{Z}$, and so will be of the type

$$\cdots \to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} A_{i-2} \cdots$$

where for all i we have $d_i \circ d_{i+1} = 0$. However, in our definition we also allow finite complexes. For example, we could have

$$A_{-9} \xrightarrow{d_{-9}} A_{-8} \xrightarrow{d_{-8}} \cdots A_2 \xrightarrow{d_2} A_1 \,.$$

The above will be a complex if $d_i \circ d_{i+1} = 0$ for all $-9 \leqslant i \leqslant 1$.

Given a complex $\{A_i, d_i\}$ we have $\operatorname{Im} d_{i+1} \subset \operatorname{Ker} d_i$.

Definition 16.1.2. *We say that the complex is* **exact** *if* $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$ *for all* i *for which these terms make sense.*

For example, given abelian groups $H \subset G$, the following complex is exact

$$\cdots \to 0 \to 0 \to H \to G \to G/H \to 0 \to 0 \to \cdots$$

Lemma 16.1.3 (Snake lemma)**.** *Suppose we are given a commutative diagram as follows, such that the rows are exact.*

$$\begin{tikzcd}
 & A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & D \arrow[r, "\alpha'"] & E \arrow[r, "\beta'"] & F &
\end{tikzcd}$$

Then

(1) There is an exact complex

$$\operatorname{Ker}(f) \xrightarrow{\overline{\alpha}} \operatorname{Ker}(\mathrm{g}) \xrightarrow{\overline{\beta}} \operatorname{Ker}(h) \xrightarrow{\delta} \operatorname{Coker}(f) \xrightarrow{\overline{\alpha'}} \operatorname{Coker}(g) \xrightarrow{\overline{\beta'}} \operatorname{Coker}(h)\,. \tag{16.1.4}$$

(2) If α *is an inclusion then* $\operatorname{Ker}(f) \xrightarrow{\overline{\alpha}} \operatorname{Ker}(\mathrm{g})$ *is an inclusion.*

(3) If β' *is a surjection then* $\operatorname{Coker}(g) \xrightarrow{\overline{\beta'}} \operatorname{Coker}(h)$ *is a surjection.*

Proof. From the commutativity of the diagram it is easily checked that α maps $\operatorname{Ker}(f)$ to $\operatorname{Ker}(g)$. The induced map is denoted $\overline{\alpha}$. Similarly, one checks easily that

- β induces a map $\overline{\beta} : \operatorname{Ker}(g) \to \operatorname{Ker}(h)$ given by $b \mapsto \beta(b)$,
- α' induces a map $\overline{\alpha'} : \operatorname{Coker}(f) \to \operatorname{Coker}(g)$ given by $\overline{d} \mapsto \overline{\alpha'(d)}$,
- β' induces a map $\overline{\beta'} : \operatorname{Coker}(g) \to \operatorname{Coker}(h)$ given by $\overline{e} \mapsto \overline{\beta'(e)}$.

The map δ is defined as follows. For $c \in \operatorname{Ker}(h)$ choose a lift $b \in B$ (by a lift we mean an element which maps to c). Note that

$$\begin{aligned}
\beta'(g(b)) &= h(\beta(b)) \\
&= h(c) = 0.
\end{aligned}$$

From the exactness of the lower row, it follows that there is a unique $d \in D$ such that $g(b) = \alpha'(d)$. Define

$$\delta(c) := \overline{d} \in \operatorname{Coker}(f)\,.$$

We need to check that δ is independent of the choice of the lift b. If b_1 is another lift of c, then $b - b_1 \in \mathrm{Ker}(\beta)$. From the exactness of the top rown, there is $a \in A$ such that $\alpha(a) = b - b_1$. As earlier, there is a unique d_1 such that $g(b_1) = \alpha'(d_1)$. Note that

$$\begin{aligned} \alpha'(f(a)) = g(\alpha(a)) &= g(b) - g(b_1) \\ &= \alpha'(d) - \alpha'(d_1) \\ &= \alpha'(d - d') \,. \end{aligned}$$

Since α' is an inclusion, it follows that $f(a) = d - d_1$. Thus, $\overline{d} = \overline{d'} \in \mathrm{Coker}(f)$. This proves that δ is well defined. One easily checks that δ is a group homomorphism.

Next let us check that

$$\overline{\beta} \circ \overline{\alpha} = \delta \circ \overline{\beta} = \overline{\alpha'} \circ \delta = \overline{\beta'} \circ \overline{\alpha'} = 0.$$

That $\overline{\beta} \circ \overline{\alpha} = \overline{\beta'} \circ \overline{\alpha'} = 0$ follows easily from $\beta \circ \alpha = \beta' \circ \alpha' = 0$. Let us check that $\delta \circ \overline{\beta} = 0$. Let $b \in \mathrm{Ker}(g)$. Then by definition $\delta(\overline{\beta}(b)) = \delta(\beta(b))$. Obviously, b is a lift of $\beta(b)$. Since $g(b) = 0$, it follows from the definition of δ that $\delta \circ \overline{\beta} = 0$. Next let us check that $\overline{\alpha'} \circ \delta = 0$. Let $c \in \mathrm{Ker}(h)$ and choose a lift $b \in B$ that maps to c. Then there is $d \in D$ such that $\alpha'(d) = g(b)$ and $\delta(c) = \overline{d}$. By definition $\overline{\alpha'}(\overline{d}) = \overline{\alpha'(d)}$. Since $\overline{\alpha'(d)} = \overline{g(b)} = 0$ we see that $\overline{\alpha'} \circ \delta = 0$. This proves that the sequence of maps in (16.1.4) forms a complex. It remains to show that this complex is exact.

Suppose $b \in \mathrm{Ker}(g)$ and $\beta(b) = 0$ then by exactness of the top row we see that there is $a \in A$ such that $b = \alpha(a)$. Then $0 = g(b) = g(\alpha(a)) = \alpha'(f(a))$. Since α' is an inclusion, by the exactness of the bottom row, it follows that $f(a) = 0$. This proves that $\mathrm{Ker}(\overline{\beta}) = \mathrm{Im}(\overline{\alpha})$.

Suppose $c \in \mathrm{Ker}(h)$ and $\delta(c) = 0$. Let $b \in B$ be a lift of c. Then there is $d \in D$ such that $\alpha'(d) = g(b)$. Since $\delta(c) = \overline{d} = 0$, it follows that $d = f(a)$ for some $a \in A$. Thus, we get $\alpha'(d) = \alpha'(f(a)) = g(\alpha(a)) = g(b)$. This shows that $b - \alpha(a) \in \mathrm{Ker}(g)$. We also have $\beta(b - \alpha(a)) = \beta(b) = c$. This proves that $\mathrm{Ker}(\delta) = \mathrm{Im}(\overline{\beta})$.

Suppose $\overline{\alpha'}(\overline{d}) = \overline{\alpha'(d)} = 0$. This means that there is $b \in B$ such that $\alpha'(d) = g(b)$. Define $c := \beta(b)$. Then $h(c) = h(\beta(b)) = \beta'(g(b)) = \beta'(\alpha'(d)) = 0$. Clearly, $\delta(c) = \overline{d}$ and so this shows that $\mathrm{Ker}(\overline{\alpha'}) = \mathrm{Im}(\delta)$.

Suppose $\overline{\beta'}(\overline{e}) = \overline{\beta'(e)} = 0$. This means that there is $c \in C$ such that $\beta'(e) = h(c)$. Since β is surjective, choose a lift $b \in B$ which maps to c. Then we get $\beta'(e) = h(\beta(b)) = \beta'(g(b))$, which shows that $\beta'(e - g(b)) = 0$. Thus, by exactness of the bottom row we get that there is $d \in D$ such that $\alpha'(d) = e - g(b)$, which shows that $\overline{\alpha'}(\overline{d}) = \overline{\alpha'(d)} = \overline{e}$. This shows that $\mathrm{Ker}(\overline{\beta'}) = \mathrm{Im}(\overline{\alpha'})$. This completes the proof of the exactness of (16.1.4).

Assertions (2) and (3) are trivial and are left as an exercise. □

Definition 16.1.5 (Homology groups)**.** *Homology groups of a complex are defined to be* $H_i(\{A_\bullet, d_\bullet\}) := \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i+1}$.

The reader will recall the definition of a complex being exact from 16.1.2. A complex is exact iff all its homology groups vanish.

Definition 16.1.6 (Short exact sequence)**.** *An exact complex of the type*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a short exact sequence.

It is clear that for a short exact sequence as above, g induces an isomorphism $\bar{g} : B/f(A) \to C$.

From now on we shall denote complexes by $A_\bullet$. A **map of complexes**, denoted $f_\bullet : A_\bullet \to B_\bullet$, is a collection of maps $f_i : A_i \to B_i$ such that the diagrams

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "d_i"'] & B_i \arrow[d, "d_i"] \\ A_{i-1} \arrow[r, "f_{i-1}"] & B_{i-1} \end{tikzcd}$$

commute. One easily checks that a map of complexes gives rise to a map between the homology groups $\overline{f}_i : H_i(A_\bullet) \to H_i(B_\bullet)$, which is given by

$$\overline{f_i}(\overline{a}) := \overline{f_i(a)}\,.$$

Suppose we are given two maps of complexes

$$A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet\,,$$

then from the definition of the induced map on homology, it is clear that

$$\overline{(g \circ f)_i} = \overline{g_i} \circ \overline{f_i}\,.$$

Suppose we are given two maps of complexes $f_\bullet, g_\bullet : A_\bullet \to B_\bullet$. We say that they are **homotopic** if there are maps $F_i : A_i \to B_{i+1}$ such that

$$f_i - g_i = d_{i+1} \circ F_i + F_{i-1} \circ d_i\,.$$

Lemma 16.1.7. *If $f_\bullet$ and $g_\bullet$ are homotopic maps of complexes $A_\bullet \to B_\bullet$, then the induced maps on homology are the same.*

Proof. If $a \in A_i$ is such that $d_i(a) = 0$, then the induced map $\overline{f}_i$ is defined as $\overline{f}_i(\overline{a}) = \overline{f_i(a)}$. Thus,

$$\begin{aligned} \overline{f}_i(\overline{a}) - \overline{g}_i(\overline{a}) &= \overline{f_i(a)} - \overline{g_i(a)} \\ &= \overline{d_{i+1}(F_i(a))} + \overline{F_i(d_i(a))} = 0. \end{aligned}$$

Thus, $\overline{f}_i(\overline{a}) = \overline{g}_i(\overline{a})$. ∎

A **short exact sequence of complexes** is two maps of complexes $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to C_\bullet$, such that for each i, the following is a short exact sequence

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0\,.$$

Proposition 16.1.8 (Homology long exact sequence)**.** *Let*

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$$

denote a short exact sequence of complexes indexed by integers. Then there is a long exact homology sequence

$$\cdots \to H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \to H_{i-1}(A_\bullet) \to \cdots$$

Proof. One easily checks that there is a complex

$$\cdots \to H_i(A_\bullet) \xrightarrow{\overline{f_i}} H_i(B_\bullet) \xrightarrow{\overline{g_i}} H_i(C_\bullet) \xrightarrow{\delta_i} H_{i-1}(A_\bullet) \to \cdots$$

where $\overline{f_i}, \overline{g_i}$ are induced maps and δ_i is defined as in Lemma 16.1.4. It only remains to prove that this complex is exact.

Observe that if $A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \xrightarrow{\alpha_{i-1}} A_{i-2}$ is a complex, then we get an induced map

$$\mathrm{Coker}(\alpha_{i+1}) \xrightarrow{\overline{\alpha_i}} \mathrm{Ker}(\alpha_{i-1})\,.$$

The kernel of $\overline{\alpha_i}$ is precisely $H_i(A_\bullet)$, while the cokernel of $\overline{\alpha_i}$ is precisely $H_{i-1}(A_\bullet)$. Applying Lemma 16.1.4 to

$$\begin{tikzcd}
0 \arrow[r] & A_j \arrow[r, "f_j"] \arrow[d, "\alpha_j"'] & B_j \arrow[r, "g_j"] \arrow[d, "\beta_j"] & C_j \arrow[r] \arrow[d, "\gamma_j"] & 0 \\
0 \arrow[r] & A_{j-1} \arrow[r, "f_{j-1}"] & B_{j-1} \arrow[r, "g_{j-1}"] & C_{j-1} \arrow[r] & 0
\end{tikzcd}$$

for $j = i+1, i-1$ and using the above observation we get a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Coker}(\alpha_{i+1}) \arrow[r, "\overline{f_{i+1}}"] \arrow[d, "\overline{\alpha_i}"] & \mathrm{Coker}(\beta_{i+1}) \arrow[r, "\overline{g_{i+1}}"] \arrow[d, "\overline{\beta_i}"] & \mathrm{Coker}(\alpha_{i+1}) \arrow[r] \arrow[d, "\overline{\gamma_i}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\alpha_{i-1}) \arrow[r, "\overline{f_{i-1}}"] & \mathrm{Ker}(\beta_{i-1}) \arrow[r, "\overline{g_{i-1}}"] & \mathrm{Ker}(\gamma_{i-1}) &
\end{tikzcd}$$

Again using Lemma 16.1.4 and the above observation we get the exact sequence

$$H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \to H_{i-1}(A_\bullet) \to H_{i-1}(B_\bullet)\,.$$

The proposition now follows. ∎

16.2 Singular homology - Definition

In this section we will define a complex $\mathcal{C}_\bullet(X)$ for a topological space X. Define the **standard n-simplex** to be the topological space

$$\Delta^n := \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geqslant 0, \sum_{i=0}^n t_i = 1\}.$$

More generally by an **n-simplex** we shall mean any subset of $\mathbb{R}^l$ which is the convex hull of points $p_0, p_1, \dots, p_n$ such that $p_1 - p_0, p_2 - p_0, \dots, p_n - p_0$ are linearly independent. Denote the vertices of Δ^n by $v_0^n, v_1^n, \dots, v_n^n$, where v_i^n has 1 in the ith coordinate and 0 elsewhere. Define the ith face map $\mathfrak{f}_i^n : \Delta^n \to \Delta^{n+1}$ on the vertices of Δ^n by

$$\mathfrak{f}_i^n(v_j^n) = \begin{cases} v_j^{n+1} & j < i \\ v_{j+1}^{n+1} & j \geqslant i \end{cases} \tag{16.2.1}$$

and extend linearly to all of Δ^n. Clearly the image of $\mathfrak{f}_i^n$ is the unique face which does not contain the vertex v_i^{n+1}.

Define $\mathcal{C}_n(X)$ to be the free abelian group on all continuous maps $\sigma : \Delta^n \to X$. For $n < 0$ define $\mathcal{C}_n(X)$ to be 0. Define a map

$$d_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$$

as 0 for $n \leqslant 0$, and for $n > 0$, on the generator of this group by,

$$d_n(\sigma) := \sum_{i=0}^n (-1)^i \sigma \circ \mathfrak{f}_i^{n-1}.$$

Define d_n on all of $\mathcal{C}_n(X)$ by extending it linearly.

Lemma 16.2.2. *$d_n \circ d_{n+1} = 0$.*

Proof. It suffices to check this on the generators of the abelian group.

$$\begin{aligned} d_n \circ d_{n+1}(\sigma) &= \sum_{i=0}^{n+1} (-1)^i d_n(\sigma \circ \mathfrak{f}_i^n) \\ &= \sum_{i=0}^{n+1} (-1)^i \sum_{j=0}^{n} (-1)^j \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} \\ &= \sum_{i=0}^{n+1} \sum_{j<i} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} + \\ &\qquad \sum_{i=0}^{n+1} \sum_{j \geqslant i} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} \end{aligned}$$

Now one checks easily that if $j \geqslant i$ then $\mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} = \mathfrak{f}_{j+1}^n \circ \mathfrak{f}_i^{n-1}$.

$$= \sum_{i=0}^{n+1} \sum_{j<i} (-1)^{i+j} \sigma \circ \mathfrak{f}_i^n \circ \mathfrak{f}_j^{n-1} +$$
$$\sum_{i=0}^{n+1} \sum_{j \geqslant i} (-1)^{i+j} \sigma \circ \mathfrak{f}_{j+1}^n \circ \mathfrak{f}_i^{n-1}$$

Note in both the sums, each term is of the type $\sigma \circ \mathfrak{f}_l^n \circ \mathfrak{f}_m^{n-1}$ with $l > m$. Every such term appears exactly once in the first sum, with sign $(-1)^{l+m}$ and exactly once in the second sum, with sign $(-1)^{l-1+m}$. Thus, all terms cancel each other and the lemma is proved. □

Definition 16.2.3. *The complex $\mathcal{C}_\bullet(X)$ is called the singular chain complex of X. The homology groups of this complex are the homology groups of X and are denoted $H_n(X, \mathbb{Z})$, or simply, $H_n(X)$.*

Proposition 16.2.4. *Let X be path connected. Then $H_0(X) = \mathbb{Z}$.*

Proof. Note that $\mathcal{C}_0(X)$ is just the free abelian group on the points of X. Since $\mathcal{C}_{-1}(X) = 0$, it follows that the kernel of d_0 is all of $\mathcal{C}_0(X)$. As X is path connected, for any two points p and q, there is a path $\gamma : \Delta^1 \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. Thus, $d_1(\gamma) = [q] - [p]$. Thus, it is clear that

$$H_0(X) = \frac{\bigoplus_{x \in X} \mathbb{Z} \cdot [x]}{\langle [p] - [q] \mid p, q \in X \rangle} \cong \mathbb{Z}$$

Thus, the proposition is proved. □

Proposition 16.2.5. *Let $X = \bigsqcup_\alpha X_\alpha$ denote the path components of X. Then $H_i(X) = \bigoplus_\alpha H_i(X_\alpha)$.*

Proof. Since each Δ^n is path connected, it follows that for every $\sigma : \Delta^n \to X$, there is a unique α such that σ factors through X_α. From this it is clear that the complex $\mathcal{C}_\bullet(X) = \bigoplus_\alpha \mathcal{C}_\bullet(X_\alpha)$. The proposition follows. □

Proposition 16.2.6. *Let $X = \{p\}$. Then $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i > 0$.*

Proof. Since X is a point, for each $n \geqslant 0$, there is only one map $\Delta^n \to X$. Thus, for each $n \geqslant 0$, $\mathcal{C}_n(X) = \mathbb{Z}$. Now it is easily checked that if $n > 0$ is odd then $d_n = 0$ and if $n > 0$ is even then $d_n = Id$. Thus, the complex $\mathcal{C}_\bullet(X)$ looks like

$$\cdots \to \mathbb{Z} \xrightarrow{d_3=0} \mathbb{Z} \xrightarrow{d_2=Id} \mathbb{Z} \xrightarrow{d_1=0} \mathbb{Z} \xrightarrow{d_0=0} 0 \to 0 \cdots$$

The proposition now follows. □

16.3 Singular homology and continuous maps

It is easy to see that a continuous map induces a map at the level of homology groups. It is much more difficult to show that homotopic maps induce homotopic maps at the level of complexes, and thus, induce the same map at the level of homology groups. We will prove this in this section. The main result of this section, if two topological spaces are homotopy equivalent then their singular homology groups are isomorphic, then follows easily.

Abuse of Notation. We will often be careless about the notation for induced maps on homology. When $f_\bullet$ is a map between two complexes, we will denote the induced map by $f_*, \overline{f}_\bullet, \overline{f_i}, f_i, \ldots$. However, the reader should easily be able to figure out from the context what we mean.

Lemma 16.3.1. *Let $h : X \to Y$ be a continuous map. Then*

(1) there is an induced map $h_\bullet : \mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(Y)$, given on the generators by $h_i(\sigma) = h \circ \sigma$,

(2) there is an induced map $h_* : H_i(X) \to H_i(Y)$.

(3) if $g : Y \to Z$ is continuous, then $g_* h_* = (g \circ h)_*$.

Proof. It is easy to check that h_i defined as above is a map of complexes. From this the second assertion follows. It is clear that $g_\bullet \circ h_\bullet = (g \circ h)_\bullet$, and the third assertion follows from this. □

Remark 16.3.2. In what follows we will use the following notation. Let $\sigma : \Delta^n \to X$ and let $p_0, \ldots, p_k$ be points in Δ^n. By $\sigma|_{[p_0,\ldots,p_k]}$ we shall mean the composite map

$$\Delta^k \to \Delta^n \xrightarrow{\sigma} X,$$

where the map $\Delta^k \to \Delta^n$ is the unique map which sends $v_i^k \mapsto p_i$. For example, let $n = 2$ and let $\sigma = Id : \Delta^2 \to \Delta^2$, then $\sigma|_{[v_0^2, v_1^2]} \neq \sigma|_{[v_1^2, v_0^2]}$ as maps, although they have the same image.

Proposition 16.3.3. *Let $h, g : X \to Y$ be homotopic maps. Then they induce the same maps on homology.*

Proof. The idea is to show that the maps $h_\bullet$ and $g_\bullet$ are homotopic and then use Lemma 16.1.7. Suppose that we are given a homotopy $F : X \times I \to Y$. We will define ***prism operators*** $P_n : \mathcal{C}_n(X) \to \mathcal{C}_{n+1}(Y)$ such that

$$d_{n+1} \circ P_n + P_{n-1} \circ d_n = (F_1)_n - (F_0)_n \,, \tag{16.3.4}$$

which will prove that the two maps F_0 and F_1 induce homotopic maps on complexes. It is enough to define the P_n's on the generators and check this relation on the generators.

For points $p_0, p_1, p_2, \dots, p_r \in \mathbb{R}^l$ denote by $[p_0, p_1, \dots, p_r]$ to be the unique linear map $\Delta^r \to \mathbb{R}^l$ which sends $v_i^r \mapsto p_i$ and sends $\sum_i \alpha_i v_i^r \mapsto \sum_i \alpha_i p_i$. Let Y be a convex subset of $\mathbb{R}^l$ and let τ be a map which is defined on Y. Let $p_0, p_1, \dots, p_r \in Y$ be points. Define

$$[p_0, \dots, p_r, \tau] := \tau \circ [p_0, p_1, \dots, p_r]\,.$$

Note that

$$d_r[p_0, \dots, p_r, \tau] = \sum_{i=0}^{n} (-1)^i [p_0, \dots, \widehat{p_i}, \dots, p_r, \tau]\,.$$

For a point $p \in Y$ denote

$$\underline{p} := p \times 0 \in Y \times I \qquad \text{and} \qquad \overline{p} := p \times 1 \in Y \times I\,.$$

Define

$$\begin{aligned} P_n(\sigma) &= P_n([v_0^n, \dots, v_n^n, \sigma]) \\ &:= \sum_{i=0}^{n} (-1)^i [\underline{v_0^n}, \underline{v_1^n}, \dots, \underline{v_i^n}, \overline{v_i^n}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)]\,. \end{aligned} \tag{16.3.5}$$

Then we have

$$\begin{aligned} d_{n+1} P_n(\sigma) = &\sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i+j} [\underline{v_0^n}, \dots, \widehat{\underline{v_j^n}}, \dots, \overline{v_i^n}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)] + \\ &\sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^{i+j+1} [\underline{v_0^n}, \dots, \underline{v_i^n}, \overline{v_i^n}, \dots \widehat{\overline{v_j^n}}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)]\,. \end{aligned}$$

On the other hand $d_n(\sigma) = \sum_{j=0}^{n} (-1)^j [v_0^n, \dots, \widehat{v_j^n} \dots, v_n^n, \sigma]$. Applying the construction for P_{n-1} we see that

$$\begin{aligned} &P_{n-1}([v_0^n, \dots, \widehat{v_j^n} \dots, v_n^n, \sigma]) \\ &= \sum_{i=0}^{j-1} (-1)^i [\underline{v_0^n}, \dots, \underline{v_i^n}, \overline{v_i^n}, \dots \widehat{\overline{v_j^n}}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)] + \\ &\sum_{i=j+1}^{n} (-1)^{i-1} [\underline{v_0^n}, \dots, \widehat{\underline{v_j^n}}, \dots, \underline{v_i^n}, \overline{v_i^n}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)]\,. \end{aligned}$$

Thus, we get

$$\begin{aligned} P_{n-1}(d_n(\sigma)) = &\sum_{j=0}^{n} \sum_{i=0}^{j-1} (-1)^{i+j} [\underline{v_0^n}, \dots, \underline{v_i^n}, \overline{v_i^n}, \dots \widehat{\overline{v_j^n}}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)] + \\ &\sum_{j=0}^{n} \sum_{i=j+1}^{n} (-1)^{i+j-1} [\underline{v_0^n}, \dots, \widehat{\underline{v_j^n}}, \dots, \underline{v_i^n}, \overline{v_i^n}, \dots, \overline{v_n^n}, F \circ (\sigma \times Id_I)]\,. \end{aligned}$$

Note that each term with $j \neq i$ in $P_{n-1} \circ d_n(\sigma)$ appears with the opposite sign in $d_{n+1} \circ P_n(\sigma)$. Thus, there is a cancellation and the only terms which survive are those in $d_{n+1} \circ P_n(\sigma)$ with $j = i$. Thus, we get

$$\begin{aligned} d_{n+1}(P_n(\sigma)) + P_{n-1}(d_n(\sigma)) &= \sum_{i=0}^{n} [\underline{v_0^n}, \ldots, \underline{v_{i-1}^n}, \overline{v_i^n}, \ldots, \overline{v_n^n}, F \circ (\sigma \times Id_I)] + \\ &\quad - \sum_{i=0}^{n} [\underline{v_0^n}, \ldots, \underline{v_i^n}, \overline{v_{i+1}^n}, \ldots, \overline{v_n^n}, F \circ (\sigma \times Id_I)] \\ &= [\overline{v_0^n}, \ldots, \overline{v_n^n}, F \circ (\sigma \times Id_I)] - [\underline{v_0^n}, \ldots, \underline{v_n^n}, F \circ (\sigma \times Id_I)] \\ &= (F_1)_n(\sigma) - (F_0)_n(\sigma) \,. \end{aligned}$$

This proves that the operators P_n, which have been defined on the generators of $\mathcal{C}_n(X)$, satisfy equation (16.3.4). Thus, the proposition is proved. □

Theorem 16.3.6 (Homotopy invariance)**.** *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : H_i(X) \to H_i(Y)$ is an isomorphism.*

Proof. Let $g : Y \to X$ denote the homotopy inverse of f. Then $g \circ f \sim Id_Y$ and $f \circ g \sim Id_X$. By the preceding proposition, this shows that $g_* \circ f_* = Id$ on $H_i(X)$ and $f_* \circ g_* = Id$ on $H_i(Y)$. This proves the theorem. □

16.4 Subdividing a simplex

For points $p_0, p_1, \ldots, p_k \in \mathbb{R}^{n+1}$ we denote by $[p_0, \ldots, p_k]$ the unique linear map $\Delta^k \to \mathbb{R}^{n+1}$, which sends $v_i^k \mapsto p_i$. Notice that the image of this map is the convex hull of the points $p_0, \ldots, p_k$. In this section we will abuse notation and also denote $[p_0, \ldots, p_k]$ to be the image of this unique map. More generally, for a convex subset W, let $[p, W]$ denote the convex hull of p and W.

We define an inductive process to subdivide an n- simplex $[p_0, \ldots, p_n]$ into smaller n-simplices. In particular, here we assume that the vectors $\{p_i - p_0\}_{i>0}$ are linearly independent. For $n = 0$ we do nothing. For $n = 1$, we write

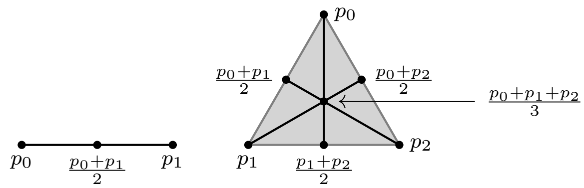
$$[p_0, p_1] = \left[\frac{p_0 + p_1}{2}, p_0\right] \bigcup \left[\frac{p_0 + p_1}{2}, p_1\right].$$

Let us assume that we know how to subdivide an $(n-1)$-simplex. Let F_i denote the face $[p_0, \ldots, \widehat{p_i}, \ldots, p_n]$. Let the subdivision of F_i be $F_i = \bigcup_\alpha W_{i,\alpha}$. Let $b(W)$ denote the point $\frac{\sum_{i=0}^n p_i}{n+1} \in W$.

Lemma 16.4.1. *There is a subdivision*

$$[p_0, \ldots, p_n] = \bigcup_{i,\alpha} [b(W), W_{i,\alpha}] \,. \tag{16.4.2}$$

Proof. Let $w \in W$. Since $b(W)$ is in the interior of W, consider the straight line $(1-t)b(W)+tw$ starting at $b(W)$. For a unique $t \geqslant 1$, it hits one of the boundary faces. Since we have assumed that each face has a subdivision, it follows that there is $W_{j,\alpha}$ such that $w \in [b(W), W_{j,\alpha}]$. $\square$



The **diameter** of a simplex $W = [p_0, \ldots, p_n]$ is defined to be the maximum distance between any two of its points. Denote the diameter by $\mathrm{diam}(W)$.

Lemma 16.4.3. *With notation as above, we have*

$$\mathrm{diam}([b(W), W_{i,\alpha}]) \leqslant \frac{n}{n+1}\mathrm{diam}(W)\,.$$

Proof. For points $v, \sum_{i=0}^{n} t_i p_i \in [p_0, \ldots, p_n]$ we have

$$\begin{aligned} \|v - \sum_{i=0}^{n} t_i p_i\| &= \|\sum_{i=0}^{n} t_i(v-p_i)\| \leqslant \sum_{i=0}^{n} t_i(\|v-p_i\|) \\ &\leqslant \max_i\{\|v-p_i\|\}(\sum_{i=0}^{n} t_i) = \max_i\{\|v-p_i\|\} \end{aligned} \tag{16.4.4}$$

Now writing $v = \sum_{j=0}^{n} s_j p_j$ and repeating the above we see that

$$\max_{i,j}\{\|p_i - p_j\|\} \leqslant \mathrm{diam}([p_0, \ldots, p_n]) \leqslant \max_{i,j}\{\|p_i - p_j\|\}\,. \tag{16.4.5}$$

When $n = 1$, the statement of the lemma is obvious, in fact, there is an equality. We will prove the lemma by induction on n. Assume that $n > 1$. Let $W_{i,\alpha}$ denote an $(n-1)$-simplex in the subdivision of the face F_i. Let us assume that $\mathrm{diam}(W_{i,\alpha}) \leqslant \frac{n-1}{n}\mathrm{diam}(F_i)$. This means that

$$\mathrm{diam}(W_{i,\alpha}) \leqslant \frac{n-1}{n}\mathrm{diam}(F_i) \leqslant \frac{n}{n+1}\mathrm{diam}(F_i) \leqslant \frac{n}{n+1}\mathrm{diam}(W)\,. \tag{16.4.6}$$

From (16.4.5) it is clear that to compute the diameter of $[b(W), W_{i,\alpha}]$, it suffices to compute the distance between the vertices. If two vertices are in $W_{i,\alpha}$ then the distance between them is bounded by $\frac{n}{n+1}\mathrm{diam}(W)$, by (16.4.6). So consider the case when one vertex is $b(W)$ and the other vertex is $w \in W_{i,\alpha}$. Doing the same computation in (16.4.4) with $v = b(W)$ we get

$$\|b(W) - w\| \leqslant \max_j\{\|b(w) - p_j\|\}\,.$$

So let us compute $\|b(W) - p_j\|$. We may write

$$b(W) = \frac{1}{n+1}p_j + \frac{n}{n+1}\frac{\sum_{l\neq j} p_l}{n}\,.$$

This shows that $b(W)$ is on the straight line joining p_j and the point $\frac{\sum_{l\neq j} p_l}{n}$ and that

$$\|b(W) - p_j\| \leqslant \frac{n}{n+1}\|p_j - \frac{\sum_{l\neq j} p_l}{n}\| \leqslant \frac{n}{n+1}\mathrm{diam}(W)\,.$$

Thus, we get that

$$\|b(W) - w\| \leqslant \frac{n}{n+1}\mathrm{diam}(W)\,.$$

By induction on n the lemma is proved. □

16.5 Barycentric subdivision

Let Y be a convex subspace of $\mathbb{R}^{l+1}$, for example, $Y = \Delta^{l+1}$. By a linear map $\sigma : \Delta^n \to Y$ we mean that σ satisfies

$$\sigma(\sum_{i=0}^{n} t_i v_i^n) = \sum_{i=0}^{n} t_i \sigma(v_i^n)\,.$$

Let $LC_n(Y)$ denote the free abelian group on linear maps $\Delta^n \xrightarrow{\sigma} Y$. It is clear that the differential $d_n : LC_n(Y) \to LC_{n-1}(Y)$. Note that $LC_0(Y)$ is simply the free abelian group on all points of Y. Let $\deg : LC_0(Y) \to \mathbb{Z}$ denote the map

$$\sum_{i=1}^{l} a_i[p_i] \mapsto \sum_{i=1}^{l} a_i\,.$$

Let $LC_\bullet(Y)$ denote the complex

$$\cdots \to LC_1(Y) \xrightarrow{d_1} LC_0(Y) \xrightarrow{d_0:=\deg} \mathbb{Z} \to 0 \to 0 \cdots$$

A linear map $\Delta^n \xrightarrow{\sigma} Y$ is completely determined by where the vertices go. Therefore, we may denote such a map by an ordered set of points $[p_0, \ldots, p_n]$, where each $p_i \in Y$. We may also write

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i [p_0, \ldots, \widehat{p_i}, \ldots, p_n]\,.$$

For a point $p \in Y$, define

$$\begin{aligned}
(\mathscr{C}_p)_{-1} : LC_{-1}(Y) &\to LC_0(Y)\\
1 &\mapsto [p]\\
(\mathscr{C}_p)_0 : LC_0(Y) &\to LC_1(Y)\\
[p_0] &\mapsto [p, p_0]\\
(\mathscr{C}_p)_k : LC_k(Y) &\to LC_{k+1}(Y)\\
[p_0, \ldots, p_k] &\mapsto [p, p_0, \ldots, p_k]
\end{aligned}$$

It is easy to see that $d_{k+1} \circ (\mathscr{C}_p)_k = Id_{LC_k(Y)} - (\mathscr{C}_p)_{k-1} \circ d_k$, or equivalently,

$$d_{k+1} \circ (\mathscr{C}_p)_k + (\mathscr{C}_p)_{k-1} \circ d_k = Id_{LC_k(Y)}\,. \tag{16.5.1}$$

Define

$$S_n = Id_{LC_n(Y)} \qquad\qquad n \leqslant 0\,.$$

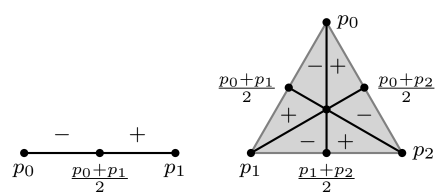
Let $n > 0$ and assume that we have constructed maps $S_i : LC_i(Y) \to LC_i(Y)$ for $0 \leqslant i \leqslant n-1$. To define S_n it suffice to define $S_n(\sigma)$ and extend linearly. For $\sigma = [p_0, \ldots, p_n]$ define $b(\sigma) := \frac{\sum_{i=0}^{n} p_i}{n+1}$. Define

$$S_n(\sigma) = (\mathscr{C}_{b(\sigma)})_{n-1}(S_{n-1}(d_n(\sigma)))\,. \tag{16.5.2}$$

Let us try to understand what this is doing in the case $n = 1$. If $\sigma = [p_0, p_1]$, then $d_1(\sigma) = p_1 - p_0$. On $LC_0(Y)$, S_0 is the identity, therefore, $S_0(d_1(\sigma)) = p_1 - p_0$. Finally, applying $(\mathscr{C}_{b(\sigma)})_0$ we get

$$S_1(\sigma) = \left[\frac{p_0 + p_1}{2}, p_1\right] - \left[\frac{p_0 + p_1}{2}, p_0\right]. \tag{16.5.3}$$

The images of the simplices occurring in $S_n(\sigma)$ along with the sign is illustrated in the following picture when $n = 1, 2$.



Proposition 16.5.4. *$S_\bullet$ is a map of complexes.*

Proof. We will prove by induction on n that $d_n \circ S_n = S_{n-1} \circ d_n$. It is clear that the square on the right in the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & LC_1(Y) \arrow[r, "d_1"] \arrow[d, "S_1"] & LC_0(Y) \arrow[r, "d_0=\deg"] \arrow[d, "S_0=Id"] & LC_{-1}(Y) \arrow[r, "0"] \arrow[d, "S_{-1}=Id"] & 0 \\
\cdots \arrow[r] & LC_1(Y) \arrow[r, "d_1"] & LC_0(Y) \arrow[r, "d_0=\deg"] & LC_{-1}(Y) \arrow[r, "0"] & 0
\end{tikzcd}$$

This shows that the induction hypothesis is true for $n = 0$. Let $n > 0$ and assume that the assertion is proved for $0 \leqslant i \leqslant n-1$. Applying d_n to (16.5.2) we get

$$d_n(S_n(\sigma)) = d_n((\mathscr{C}_{b(\sigma)})_{n-1}(S_{n-1}(d_n(\sigma))))$$

(using (16.5.1))

$$= S_{n-1}(d_n(\sigma)) - (\mathscr{C}_{b(\sigma)})_{n-2} \circ d_{n-1}(S_{n-1}(d_n(\sigma)))$$

(using induction hypothesis)

$$\begin{aligned}
&= S_{n-1}(d_n(\sigma)) - (\mathscr{C}_{b(\sigma)})_{n-2}(S_{n-1}(d_{n-1} \circ d_n(\sigma))) \\
&= S_{n-1}(d_n(\sigma))
\end{aligned}$$

Since it suffices to check on the generators of $LC_n(Y)$, this proves that $S_\bullet$ is a map of complexes. □

Proposition 16.5.5. *There are maps $T_n : LC_n(Y) \to LC_{n+1}(Y)$ such that*

$$Id_{LC_n(Y)} - S_n = T_{n-1} \circ d_n + d_{n+1} \circ T_n \,. \tag{16.5.6}$$

Proof. This is also proved by induction on n. Set $T_n = 0$ for $n \leqslant -1$. If $n \leqslant -2$ then the assertion is trivially true. For $n = -1$, we need to show that $Id_{LC_{-1}(Y)} - S_{-1} = d_0 \circ T_{-1} + T_{-2} \circ d_{-1}$. But this is also trivially true since both sides are 0. Let $n > -1$ and assume that for $-1 \leqslant i \leqslant n-1$ we have defined maps $T_i : LC_i(Y) \to LC_{i+1}(Y)$ which satisfy (16.5.6). Define

$$T_n(\sigma) = (\mathscr{C}_{b(\sigma)})_n(\sigma - T_{n-1}(d_n(\sigma)))\,. \tag{16.5.7}$$

We need to show that

$$Id_{LC_n(Y)} - S_n = T_{n-1} \circ d_n + d_{n+1} \circ T_n \,.$$

Using the definition of T_n and (16.5.1) we get

$$\begin{aligned}
d_{n+1} \circ T_n(\sigma) &= d_{n+1} \circ (\mathscr{C}_{b(\sigma)})_{n-1}(\sigma - T_{n-1}(d_n(\sigma))) \\
&= \sigma - T_{n-1}(d_n(\sigma)) - (\mathscr{C}_{b(\sigma)})_{n-1} \circ d_n(\sigma - T_{n-1}(d_n(\sigma))) \\
&= \sigma - T_{n-1}(d_n(\sigma)) - (\mathscr{C}_{b(\sigma)})_{n-1} \circ d_n(\sigma) + \\
&\qquad (\mathscr{C}_{b(\sigma)})_{n-1} \circ d_n(T_{n-1}(d_n(\sigma)))
\end{aligned}$$

(using induction hypothesis for $n-1$)

$$\begin{aligned}&= \sigma - T_{n-1}(d_n(\sigma)) - (\mathscr{C}_{b(\sigma)})_{n-1} \circ d_n(\sigma) + \\ &\qquad (\mathscr{C}_{b(\sigma)})_n(d_n(\sigma) - S_{n-1}(d_n(\sigma))) \\ &= \sigma - T_{n-1}(d_n(\sigma)) - (\mathscr{C}_{b(\sigma)})_{n-1} \circ S_{n-1}(d_n(\sigma))\end{aligned}$$

(using the definition (16.5.2) of S_n)

$$= \sigma - T_{n-1}(d_n(\sigma)) - S_n(\sigma)$$

This proves that

$$d_{n+1} \circ T_n(\sigma) + T_{n-1}(d_n(\sigma)) = \sigma - S_n(\sigma)\,.$$

Since it suffices to check the assertion on the generators of $LC_n(Y)$, this completes the proof of the proposition. □

We make the following observation from the above proof, which will be important later.

Lemma 16.5.8. *Let $n \geqslant 0$ and let $\sigma : \Delta^n \to Y$ be a linear map. Write $T_n(\sigma) = \sum_\alpha n_\alpha \sigma_\alpha$, where $\sigma_\alpha \in LC_{n+1}(Y)$. Then the image of each σ_α is contained in the image of σ.*

Proof. Let $n = 0$. Then $\sigma : \Delta^0 \to Y$ simply corresponds to a point $p_0 \in Y$. Since $T_{-1} = 0$, by the definition (16.5.7) of T it follows that $T_0(\sigma) = [p_0, p_0]$. Clearly, the image of $T_0(\sigma)$ is equal to the image of $\sigma = [p_0]$. Assume $n > 0$ and that the assertion is true for $0 \leqslant i \leqslant n-1$. Now $d_n(\sigma) = \sum_{i=0}^n (-1)^i F_i$ and so it is clear that $T_{n-1}(d_n(\alpha)) = \sum_\beta n_\beta \sigma_\beta$, where the image of σ_β is contained in one of the faces of σ. The lemma now follows from the definition (16.5.7) of T. □

Let $\mathcal{A}_\bullet(X)$ denote the complex

$$\cdots \to \mathcal{C}_1(X) \xrightarrow{d_1} \mathcal{C}_0(X) \xrightarrow{d_0 := \deg} \mathbb{Z} \to 0 \to 0 \cdots$$

Our aim is to define maps $S(X)_n : \mathcal{A}_n(X) \to \mathcal{A}_n(X)$ and $T(X)_n : \mathcal{A}_n(X) \to \mathcal{A}_{n+1}(X)$ similar to the maps S and T above. To do this, it suffices to define maps on the generators of the free abelian groups and extend linearly. Let $\sigma : \Delta^n \to X$ denote a continuous map. There are maps of complexes

$$LC_\bullet(\Delta^n) \xrightarrow{S_\bullet} LC_\bullet(\Delta^n) \xrightarrow{f_\bullet} \mathcal{A}_\bullet(\Delta^n) \xrightarrow{\sigma_\bullet} \mathcal{A}_\bullet(X)\,.$$

In the above, σ_{-1} is defined to be the identity from $\mathcal{A}_{-1}(X) \to \mathcal{A}_{-1}(X)$ and f_{-1} is also defined to be the identity. It is easily checked that with these definitions the maps are maps of complexes.

Before we define $S(X)_n$ and $T(X)_n$ we need some notation and we need to make an observation. For a convex subspace $Y \subset \mathbb{R}^l$ we have already defined maps $LC_\bullet(Y) \to LC_\bullet(Y)$. We denoted these map by $S_\bullet$ and $T_\bullet$ above, but now we shall need to emphasize the Y, and so we denote the maps constructed above by $S(Y)_n$ and $T(Y)_n$. Now consider example (16.5.3), and assume that there is a convex subspace $j : Y' \subset Y$ such that $p_0, p_1 \in Y'$. Then we may write $\sigma = [p_0, p_1]$ as $j \circ \sigma'$, where $\sigma' = [p_0, p_1] : \Delta^1 \to Y_1$. We have $j_1 : LC_1(Y') \to LC_1(Y)$. Then it is clear that

$$S(Y)_1(\sigma) = j_1(S(Y')_1(\sigma')) .$$

This is true for all n and we record this as a Lemma.

Lemma 16.5.9. *Let $j : Y' \subset Y$ be the inclusion of a convex subset. Then for all n we have $S(Y)_n(j \circ \sigma) = j_n \circ S(Y')_n(\sigma)$. Similarly, for all n we have $T(Y)_n(j \circ \sigma) = j_{n+1} \circ T(Y')_n(\sigma)$.*

Proof. The proof follows easily using the definition of $S(Y), S(Y')$, the fact that $b(\sigma) \in Y'$ and by induction on n. The case of T is similar. □

Define

$$S(X)_n = Id_{\mathcal{A}_n(X)} \qquad\qquad n \leqslant 0 .$$

For $n > 0$ and $\sigma : \Delta^n \to X$ define

$$S(X)_n(\sigma) := \sigma_n(f_n(S(\Delta^n)_n(Id_{\Delta^n}))) .$$

Proposition 16.5.10. *$S(X)_\bullet$ is a map of complexes.*

Proof. If $n \leqslant 0$ then it is clear that $d_n \circ S(X)_n = S(X)_{n-1} \circ d_n$. For $n > 0$ we have

$$d_n \circ S(X)_n(\sigma) = d_n \circ \sigma_n(f_n(S(\Delta^n)_n(Id_{\Delta^n})))$$

(using that $\sigma_\bullet, f_\bullet, S_\bullet$ are maps of complexes)

$$\begin{aligned} &= \sigma_{n-1}(f_{n-1}(S(\Delta^n)_{n-1}(d_n(Id_{\Delta^n})))) \\ &= \sigma_{n-1}(f_{n-1}(S(\Delta^n)_{n-1}(\sum_{i=0}^{n}(-1)^i \mathfrak{f}_i))) \\ &= \sum_{i=0}^{n}(-1)^i \sigma_{n-1}(f_{n-1}(S(\Delta^n)_{n-1}(\mathfrak{f}_i))) \end{aligned}$$

Now apply Lemma 16.5.9 to $Y = \Delta^n$, $Y' = \Delta^{n-1}$, $j = \mathfrak{f}_i$ and $\sigma = Id_{\Delta^{n-1}}$ we see that

$$S(\Delta^n)_{n-1}(\mathfrak{f}_i) = j_{n-1}(S(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}})) .$$

This shows that the terms in the above sum are

$$\begin{aligned} d_n \circ S(X)_n(\sigma) &= \sum_{i=0}^{n} (-1)^i \sigma_{n-1}(f_{n-1}(S(\Delta^n)_{n-1}(\mathfrak{f}_i))) \\ &= \sum_{i=0}^{n} (-1)^i \sigma_{n-1}(f_{n-1}(j_{n-1}(S(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}})))) \end{aligned}$$

On the other hand we have

$$\begin{aligned} S(X)_{n-1}(d_n(\sigma)) &= \sum_{i=0}^{n} (-1)^i S(X)_{n-1}(\sigma \circ \mathfrak{f}_i) \\ &= \sum_{i=0}^{n} (-1)^i (\sigma \circ \mathfrak{f}_i)_{n-1}(f_{n-1}(S(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}}))) \end{aligned}$$

To prove the proposition, it suffices to check that

$$\sigma_{n-1}(f_{n-1}(j_{n-1}(S(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}})))) = (\sigma \circ \mathfrak{f}_i)_{n-1}(f_{n-1}(S(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}})))$$

But this is obviously true since $j = \mathfrak{f}_i$. $\square$

Define $T(X)_n : \mathcal{A}_n(X) \to \mathcal{A}_{n+1}(X)$ by

$$T(X)_n = 0 \qquad\qquad n < 0\,.$$

For $n \geqslant 0$ and $\sigma : \Delta^n \to X$ define

$$T(X)_n(\sigma) := \sigma_{n+1}(f_{n+1}(T(\Delta^n)_n(Id_{\Delta^n})))\,.$$

Proposition 16.5.11. *The maps $T(X)_\bullet$ satisfy*

$$Id_{\mathcal{A}_n(X)} - S(X)_n = T(X)_{n-1} \circ d_n + d_{n+1} \circ T(X)_n\,. \tag{16.5.12}$$

Proof. The claim is clear for $n < 0$ since both sides are 0. For $n = 0$, we have $T(X)_0(\sigma) = \sigma_1(f_1([p_0, p_0]))$ where $p_0 = \sigma(v_0^0)$. This shows that $T(X)_0(\sigma) : \Delta^1 \to X$ is the constant map, mapping everything to the point p_0. This shows that $d_1 \circ T(X)_0 = 0$. Thus, for $n = 0$ also, both sides are 0.

Now assume that $n > 0$ and we have proved the claim for $T(X)_i$, $0 \leqslant i \leqslant n-1$. In the RHS of (16.5.12) we have

$$\begin{aligned} d_{n+1} \circ T(X)_n(\sigma) &= d_{n+1} \circ \sigma_{n+1}(f_{n+1}(T(\Delta^n)_n(Id_{\Delta^n}))) \\ &= \sigma_n(f_n(d_{n+1} \circ T(\Delta^n)_n(Id_{\Delta^n}))) \\ &= \sigma_n(f_n(Id_{\Delta^n} - S(\Delta^n)_n(Id_{\Delta^n}) - T(\Delta^n)_{n-1} \circ d_n(\sigma))) \\ &= \sigma - S(X)_n(\sigma) - \sigma_n(f_n(T(\Delta^n)_{n-1} \circ d_n(\sigma))) \end{aligned}$$

Now using Lemma 16.5.9, exactly as in the previous Proposition, we may show that

$$\begin{aligned}\sigma_n(f_n(T(\Delta^n)_{n-1}(\mathfrak{f}_i))) &= \sigma_n(f_n(j_n(T(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}})))) \\ &= (\sigma\circ\mathfrak{f}_i)_n(f_n(T(\Delta^{n-1})_{n-1}(Id_{\Delta^{n-1}}))) \\ &= T(X)_{n-1}(\sigma\circ\mathfrak{f}_i)\end{aligned}$$

This shows that

$$\sigma_n(f_n(T(\Delta^n)_{n-1}\circ d_n(\sigma))) = T(X)_{n-1}(d_n(\sigma))\,.$$

Substituting this above proves the proposition. □

Corollary 16.5.13. *The map of complexes $S(X)_\bullet : \mathcal{A}_\bullet(X) \to \mathcal{A}_\bullet(X)$ induces the identity map on homology.*

Proof. Combine the previous proposition with Lemma 16.1.7. □

16.6 Long exact sequences of singular homology

In this section we will apply the results of the previous section to prove some important properties of singular homology.

Definition 16.6.1. *For $A \subset X$ define $\mathrm{Int}(A)$ to be those points $a \in A$ for which there is a set $U \subset A$ which contains a and is open in X.*

Let U and V be subsets of X such that $X = \mathrm{Int}(U)\cup\mathrm{Int}(V)$. Let $\mathcal{A}^{\mathcal{U}}_\bullet(X)$ denote the image of the map of complexes

$$\mathcal{A}_\bullet(U)\oplus\mathcal{A}_\bullet(V)\to\mathcal{A}_\bullet(X)\,.$$

In other words, elements of $\mathcal{A}^{\mathcal{U}}_n(X)$ are sums of the type $\sum_\alpha n_\alpha\sigma_\alpha$, where each $\sigma_\alpha : \Delta^n \to X$ is a continuous map such that the image is contained in U or in V.

Lemma 16.6.2. *The inclusion $i : \mathcal{A}^{\mathcal{U}}_\bullet(X) \subset \mathcal{A}_\bullet(X)$ induces an isomorphism on homology.*

Proof. When $n < 0$, both these groups are 0 and so the assertion is trivially true. So assume that $n \geqslant 0$.

We claim that the $i_* : H_n(\mathcal{A}^{\mathcal{U}}_\bullet(X)) \to H_n(X)$ is surjective. Let $\sum_\alpha\sigma_\alpha$ be an element in $\mathcal{A}_n(X)$ which represents a homology class. For each α, we may consider the open cover $\Delta^n = \sigma_\alpha^{-1}(U)\cup\sigma_\alpha^{-1}(V)$. By Lemma 8.3.2 and Lemma 16.4.3 we can find $l \gg 0$ such that the image of each simplex in $S(X)^l_n(\sigma_\alpha)$ is contained in either U or V. We may choose an l large enough which works for all σ_α. Thus, $S(X)^l_n(\sum_\alpha\sigma_\alpha) \in \mathcal{A}^{\mathcal{U}}_n(X)$. Since $S(X)_\bullet$ is a map of complexes, it follows that $d_n(S(X)^l_n(\sum_\alpha\sigma_\alpha)) = 0$. From equation (16.5.12) it follows that

$$\sum_\alpha\sigma_\alpha - S(X)^l_n(\sum_\alpha\sigma_\alpha) = d_{n+1}\circ T(X)_n(\sum_\alpha\sigma_\alpha)\,.$$

Thus, $S(X)^l_n(\sum_\alpha \sigma_\alpha)$ defines a homology class in $H_n(\mathcal{A}^{\mathcal{U}}_\bullet(X))$ which maps to the homology class represented by $\sum_\alpha \sigma_\alpha$. This proves that i_* is surjective.

To show that i_* is an inclusion, we need to show the following. Suppose there is a homology class $\bar{\delta} \in H_n(\mathcal{A}^{\mathcal{U}}_\bullet(X))$ such that $i_*(\delta) = d_{n+1}(\sum_\alpha \sigma_\alpha)$, where $\sigma_\alpha : \Delta^{n+1} \to X$, then we can find $\sigma_\beta : \Delta^{n+1} \to X$ such that $\sum_\beta \sigma_\beta \in \mathcal{A}^{\mathcal{U}}_{n+1}(X)$ and that $\delta = d_{n+1}(\sum_\beta \sigma_\beta)$. This will show that $\bar{\delta} = 0$ in $H_n(\mathcal{A}^{\mathcal{U}}_\bullet(X))$.

From equation (16.5.12) applied to $\sum_\alpha \sigma_\alpha$ it follows that

$$\sum_\alpha \sigma_\alpha - S(X)_{n+1}(\sum_\alpha \sigma_\alpha) = T(X)_n(i_*(\delta)) + d_{n+2} \circ T(X)_{n+1}(\sum_\alpha \sigma_\alpha)$$

Let $\sigma : \Delta^n \to X$ and write $T(X)_n(\sigma) = \sum_j \sigma_j$. It follows from the definition of $T(X)_n$ and Lemma 16.5.8 that the image of each σ_j is contained in the image of σ. By definition, δ is a sum of maps whose image is contained in U or V. Thus, it is clear that $T(X)_n(i_*(\delta))$ is in the image of $\mathcal{A}^{\mathcal{U}}_{n+1}(X)$. Applying d_{n+1} to both sides of the above equation we get

$$\begin{aligned} d_{n+1}(\sum_\alpha \sigma_\alpha) &= d_{n+1}(S(X)_{n+1}(\sum_\alpha \sigma_\alpha)) + d_{n+1}(T(X)_n(i_*(\delta))) \\ &= d_{n+1}\Big(S(X)_{n+1}(\sum_\alpha \sigma_\alpha) + T(X)_n(i_*(\delta))\Big) \end{aligned} \tag{16.6.3}$$

Choose $l \gg 0$ such that $S(X)^l_{n+1}(\sum_\alpha \sigma_\alpha)$ is in the image of $\mathcal{A}^{\mathcal{U}}_{n+1}(X)$. So, for example, if $l = 1$ we are done, since in the RHS the term in the bracket is in the image of $\mathcal{A}^{\mathcal{U}}_{n+1}(X)$. The idea is to repeat the above process.

For $0 \leqslant j \leqslant l$ define $\eta_j := S(X)^j_{n+1}(\sum_\alpha \sigma_\alpha)$. Then

$$d_{n+1}(\eta_j) = d_{n+1}(S(X)^j_{n+1}(\sum_\alpha \sigma_\alpha)) = S(X)^j_n(i_*(\delta))\,.$$

It is clear that $S(X)^j_n(i_*(\delta))$ is in the image of $\mathcal{A}^{\mathcal{U}}_n(X)$ and so define $\delta_j \in \mathcal{A}^{\mathcal{U}}_n(X)$ by

$$i_*(\delta_j) := S(X)^j_n(i_*(\delta))\,.$$

Then $d_{n+1}(\eta_j) = i_*(\delta_j)$, $\delta_0 = \delta$ and $\eta_0 = \sum_\alpha \sigma_\alpha$. Equation (16.6.3) may be rewritten as

$$d_{n+1}(\eta_0) = d_{n+1}(\eta_1) + d_{n+1}(T(X)_n(i_*(\delta_0)))\,.$$

Since $d_{n+1}(\eta_j) = i_*(\delta_j)$, applying equation (16.5.12) to η_j and then applying d_{n+1} (that is, repeating the above argument replacing δ_0 by δ_j), we get for each $0 \leqslant j \leqslant l-1$,

$$d_{n+1}(\eta_j) = d_{n+1}(\eta_{j+1}) + d_{n+1}(T(X)_n(i_*(\delta_j)))\,.$$

Note that $T(X)_n(i_*(\delta_j))$ is contained in the image of $\mathcal{A}_n^{\mathcal{U}}(X)$. Adding these equations we get

$$\begin{aligned} d_{n+1}(\eta_0) &= d_{n+1}(\eta_l) + \sum_{j=0}^{l-1} d_{n+1}(T(X)_n(i_*(\delta_j))) \\ &= d_{n+1}\Big(\eta_l + \sum_{j=0}^{l-1} T(X)_n(i_*(\delta_j))\Big) \end{aligned}$$

As $\eta_l + \sum_{j=0}^{l-1} T(X)_n(i_*(\delta_j))$ is contained in the image of $\mathcal{A}_{n+1}^{\mathcal{U}}(X)$, the proof of the lemma is complete. □

Definition 16.6.4 (Reduced homology). *Denote the homology groups of the complex $\mathcal{A}_\bullet(X)$ by $\tilde{H}_n(X)$.*

Lemma 16.6.5. *For $i > 0$, $\tilde{H}_i(X) \cong H_i(X)$. For $i = 0$ there is a short exact sequence*

$$0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0\,.$$

Let X have c path components. Then $\tilde{H}_0(X) \cong \mathbb{Z}^{c-1}$.

Proof. Let $\mathbb{Z}_\bullet[-1]$ denote the complex which has $\mathbb{Z}$ in degree -1 and 0 elsewhere. All the differentials are then forced to be 0. We have a short exact sequence of complexes

$$0 \to \mathbb{Z}_\bullet[-1] \to \mathcal{A}_\bullet(X) \to \mathcal{C}_\bullet(X) \to 0\,.$$

The short exact sequence follows from the long exact homology sequence. The second assertion follows from Proposition 16.2.5. □

Theorem 16.6.6 (Mayer-Vietoris sequence). *Let U and V be subsets of X such that $X = \mathrm{Int}(U) \cup \mathrm{Int}(V)$. Then*

(a) there is a long exact sequence of reduced homology groups

$$\cdots \to \tilde{H}_n(U \cap V) \to \tilde{H}_n(U) \oplus \tilde{H}_n(V) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(U \cap V) \to \cdots\,. \tag{16.6.7}$$

(b) there is a long exact sequence of homology groups

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to H_{n-1}(U \cap V) \to \cdots\,. \tag{16.6.8}$$

Proof. We have a short exact sequence of complexes

$$0 \to \mathcal{A}_\bullet(U \cap V) \to \mathcal{A}_\bullet(U) \oplus \mathcal{A}_\bullet(V) \to \mathcal{A}_\bullet^{\mathcal{U}}(X) \to 0\,.$$

For $n \geqslant 0$ the maps

$$0 \to \mathcal{A}_n(U \cap V) \to \mathcal{A}_n(U) \oplus \mathcal{A}_n(V) \to \mathcal{A}_n^{\mathcal{U}}(X) \to 0$$

have the following description. Let us denote by f_U the inclusion of $U \cap V$ into U and by f_V the inclusion of $U \cap V$ into V. Denote by g_U and g_V the inclusions of U, V into X, respectively. The first arrow is the inclusion $\sigma \mapsto (f_U \circ \sigma, -f_V \circ \sigma)$. The second arrow is the map $(\sigma, \tau) \mapsto g_U \circ \sigma + g_V \circ \tau$. For $n = -1$ one easily checks that the maps look like

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

where the first arrow is $n \mapsto (n, -n)$ and the second arrow is $(n, m) \mapsto n + m$. From this it is easily checked that the above sequence of complexes is short exact. The first part of the theorem now follows from Proposition 16.1.8 and Lemma 16.6.2.

Let $\mathcal{C}_\bullet^{\mathcal{U}}(X)$ denote the image of $\mathcal{C}_\bullet(U) \oplus \mathcal{C}_\bullet(V) \to \mathcal{C}_\bullet(X)$. Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_\bullet[-1] \arrow[r] \arrow[d, equal] & \mathcal{A}_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d] & \mathcal{C}_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_\bullet[-1] \arrow[r] & \mathcal{A}_\bullet(X) \arrow[r] & \mathcal{C}_\bullet(X) \arrow[r] & 0
\end{tikzcd}$$

The complex $\mathbb{Z}_\bullet[-1]$ has homology only when $i = -1$ and in this case the homology is $\mathbb{Z}$. Since the degree map

$$\mathcal{A}_0^{\mathcal{U}}(X) \xrightarrow{\deg} \mathbb{Z} \qquad\qquad \mathcal{A}_0(X) \xrightarrow{\deg} \mathbb{Z}$$

is surjective, $H_{-1}(\mathcal{A}_\bullet^{\mathcal{U}}(X)) = H_{-1}(\mathcal{A}_\bullet(X)) = 0$. From the long exact homology sequence we get the following diagram when $i = 0$.

$$\begin{tikzcd}
0 \arrow[r] & H_0(\mathcal{A}_\bullet^{\mathcal{U}}(X)) \arrow[r] \arrow[d, "\wr"] & H_0(\mathcal{C}_\bullet^{\mathcal{U}}(X)) \arrow[r] \arrow[d] & H_{-1}(\mathbb{Z}_\bullet[-1]) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_0(\mathcal{A}_\bullet(X)) \arrow[r] & H_0(\mathcal{C}_\bullet(X)) \arrow[r] & H_{-1}(\mathbb{Z}_\bullet[-1]) \arrow[r] & 0
\end{tikzcd}$$

The left vertical arrows are isomorphisms is proved above. It follows from Snake Lemma that the induced map $H_i(\mathcal{C}_\bullet^{\mathcal{U}}(X)) \to H_i(\mathcal{C}_\bullet(X))$ is an isomorphism. When $i > 0$ the terms in the right column will be also be 0, in which case too we get the middle vertical arrow is an isomorphism. $\square$

Remark 16.6.9. We emphasize that, unlike in the Siefert van-Kampen Theorem, we do not require $U \cap V$ to be connected.

Proposition 16.6.10. *$H_0(S^n) = H_n(S^n) = \mathbb{Z}$. If $i \neq 0, n$ then $H_i(S^n) = 0$.*

Proof. We will prove the proposition by induction on n. Let us consider the base case $n = 1$. Let $U = S^1 \setminus \{1\}$ and $V = S^1 \setminus \{-1\}$. The spaces U and V have the same homotopy type as that of a point. Similarly, the space $U \cap V$ has the same homotopy type as a set of two points with discrete topology. Using Theorem 16.6.7 we see that there is an exact sequence

$$H_n(U) \oplus H_n(V) \to H_n(S^1) \to H_{n-1}(U \cap V)\,.$$

From Theorem 16.3.6 and Proposition 16.2.6 it follows that $H_n(S^1) = 0$ if $n > 1$. When $n = 1$ we have the exact sequence

$$\tilde{H}_1(U) \oplus \tilde{H}_1(V) \to \tilde{H}_1(S^1) \to \tilde{H}_0(U \cap V) \to \tilde{H}_0(U) \oplus \tilde{H}_0(V)\,.$$

It follows from Lemma 16.6.5 that the ends are 0 and that

$$H_1(S^1) = \tilde{H}_1(S^1) \cong \tilde{H}_0(U \cap V) \cong \mathbb{Z}$$

Since S^1 is path connected, $H_0(S^1) \cong \mathbb{Z}$. Thus, the base case for the induction is done.

Let us now assume that $n > 1$ and the proposition is true when $0 \leqslant i \leqslant n-1$. Consider the open cover of S^n where the open sets are obtained by removing the two poles. Each of the open sets has the homotopy type of a point and the intersection has the homotopy type of S^{n-1}. For $j \geqslant 1$ we have the exact sequence

$$\tilde{H}_j(U) \oplus \tilde{H}_j(V) \to \tilde{H}_j(S^n) \to \tilde{H}_{j-1}(U \cap V) \to \tilde{H}_{j-1}(U) \oplus \tilde{H}_{j-1}(V)\,.$$

Since both U and V have the homotopy type of a point, it follows that the ends in the above sequence are 0. Thus, it follows that for $j \geqslant 1$ we have

$$\tilde{H}_j(S^n) \xrightarrow{\sim} \tilde{H}_{j-1}(S^{n-1})\,.$$

Since S^n is connected, it follows that $H_0(S^n) = Z$. From this the proposition easily follows. $\square$

Corollary 16.6.11. *If $n > 0$ then the sphere S^n is not homotopy equivalent to a point.*

Corollary 16.6.12. *$\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $n = m$.*

Proof. If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ then $\mathbb{R}^n \setminus \{p\} \cong \mathbb{R}^m \setminus \{q\}$. This shows that S^n and S^m have the same homotopy type, which forces that $m = n$, by looking at homology. $\square$

16.7 Homology of pairs

Let $A \subset X$ be a subspace. Then we have an inclusion $\mathcal{C}_\bullet(A) \subset \mathcal{C}_\bullet(X)$. It is easily checked that we get a quotient complex, denoted $\mathcal{C}_\bullet(X, A)$, which sits in a short exact

$$0 \to \mathcal{C}_\bullet(A) \to \mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X, A) \to 0\,.$$

Similarly, define $\mathcal{A}_\bullet(X, A)$. The homology groups of the complex $\mathcal{C}_\bullet(X, A)$ are denoted by $H_n(X, A)$ and are known as relative homology groups. By Proposition 16.1.8 there is a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots .$$

Theorem 16.7.1 (Excision)**.** *Let $Z \subset A \subset X$ be such that $\bar{Z} \subset \mathrm{Int}(A)$. Then the inclusion $\mathcal{C}_\bullet(X - Z, A - Z) \subset \mathcal{C}_\bullet(X, A)$ induces an isomorphism on homology.*

Proof. Applying Snake Lemma (16.1.4) to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_n(A) \arrow[r] \arrow[d] & \mathcal{A}_n(X) \arrow[r] \arrow[d] & \mathcal{A}_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}_n(A) \arrow[r] & \mathcal{C}_n(X) \arrow[r] & \mathcal{C}_n(X, A) \arrow[r] & 0
\end{tikzcd}$$

we see that there is an isomorphism $\mathcal{A}_n(X, A) \xrightarrow{\sim} \mathcal{C}_n(X, A)$. Hence, it suffices to prove that $\mathcal{A}_\bullet(X - Z, A - Z) \subset \mathcal{A}_\bullet(X, A)$ induces an isomorphism on homology. Let $\mathcal{U}$ denote the cover $\{X - Z, A\}$. Then note that the image of $\mathcal{A}_\bullet(X - Z, A - Z)$ in $\mathcal{A}_\bullet(X, A)$ is precisely $\mathcal{A}^{\mathcal{U}}_\bullet(X)/\mathcal{A}_\bullet(A)$. Thus, we have an exact sequence of complexes

$$0 \to \mathcal{A}^{\mathcal{U}}_\bullet(X)/\mathcal{A}_\bullet(A) \to \mathcal{A}_\bullet(X)/\mathcal{A}_\bullet(A) \to \mathcal{A}_\bullet(X)/\mathcal{A}^{\mathcal{U}}_\bullet(X) \to 0 .$$

We already know that the inclusion $\mathcal{A}^{\mathcal{U}}_\bullet(X) \subset \mathcal{A}_\bullet(X)$ induces an isomorphism on homology. It follows from the long exact homology sequence that all homology groups of $\mathcal{A}_\bullet(X)/\mathcal{A}^{\mathcal{U}}_\bullet(X)$ are zero. It follows from the long exact homology sequence that the inclusion $\mathcal{A}^{\mathcal{U}}_\bullet(X)/\mathcal{A}_\bullet(A) \to \mathcal{A}_\bullet(X)/\mathcal{A}_\bullet(A)$ induces an isomorphism on homology. Since

$$\mathcal{A}_\bullet(X - Z, A - Z) \xrightarrow{\sim} \mathcal{A}^{\mathcal{U}}_\bullet(X)/\mathcal{A}_\bullet(A),$$

the theorem is proved. □

Corollary 16.7.2. *If $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic then $m = n$.*

Proof. If U and V are homeomorphic then there is an isomorphism $H_k(U, U - \{p\}) \cong H_k(V, V - \{q\})$. Let $Z = \mathbb{R}^m - U$ and $A = \mathbb{R}^m - \{p\}$. Then by excision we have $H_k(\mathbb{R}^m, \mathbb{R}^m - \{p\}) \cong H_k(U, U - \{p\})$. From the long exact sequence for pairs we get

$$\begin{aligned} H_k(\mathbb{R}^m - \{p\}) \to H_k(\mathbb{R}^m) \to & H_k(\mathbb{R}^m, \mathbb{R}^m - \{p\}) \to \\ & H_{k-1}(\mathbb{R}^m - \{p\}) \to H_{k-1}(\mathbb{R}^m) . \end{aligned}$$

***Case* $\mathbf{m = 1}$:**

- The map $H_0(\mathbb{R}^1 - \{p\}) \to H_0(\mathbb{R}^1)$ is clearly surjective. From this it follows that $H_0(\mathbb{R}^1, \mathbb{R} - \{p\}) = 0$.

- The kernel of the map $H_0(\mathbb{R}^1 - \{p\}) \to H_0(\mathbb{R}^1)$ is $\mathbb{Z}$. From this it follows that $H_1(\mathbb{R}^1, \mathbb{R} - \{p\}) = \mathbb{Z}$.
- $H_k(\mathbb{R}^1, \mathbb{R} - \{p\}) = 0$ for $k \geqslant 2$

***Case* $\mathbf{m > 1}$**:

- The map $H_0(\mathbb{R}^m - \{p\}) \to H_0(\mathbb{R}^m)$ is an isomorphism. From this it follows that $H_0(\mathbb{R}^m, \mathbb{R}^m - \{p\}) = H_1(\mathbb{R}^m, \mathbb{R}^m - \{p\}) = 0$.
- $H_k(\mathbb{R}^m, \mathbb{R}^m - \{p\}) = 0$ for $k \geqslant 2$ and $k \neq m$
- $H_m(\mathbb{R}^m, \mathbb{R}^m - \{p\}) = \mathbb{Z}$

Thus, we may recover m as the index for which $H_k(U, U - \{p\}) \neq 0$. This proves that $m = n$. □

If $f : X \to Y$ is a map such that $f(A) \subset B$, then we denote this by $f : (X, A) \to (Y, B)$. It is easily checked that f induces a map of complexes $f_\bullet : \mathcal{C}_\bullet(X, A) \to \mathcal{C}_\bullet(Y, B)$. It follows that f induces a map on relative homology.

Theorem 16.7.3 (Homotopy invariance for pairs)**.** *Let $f, g : (X, A) \to (Y, B)$ be maps which are homotopic through a homotopy of pairs. Then they induce the same maps on relative homology.*

Proof. Let F denote the homotopy. By definition, $F(A \times I) \subset B$. It suffices to check that the Prism operator, defined in (16.3.5) takes $\mathcal{C}_n(A) \to \mathcal{C}_{n+1}(B)$. But this is obvious from the definition. It follows that there are induced maps $\bar{P}_n : \mathcal{C}_n(X, A) \to \mathcal{C}_{n+1}(Y, B)$ such that the analogue of (16.3.4),

$$d_{n+1} \circ \bar{P}_n + \bar{P}_{n-1} \circ d_n = (\bar{F}_1)_n - (\bar{F}_0)_n \,,$$

holds. It follows that the maps $\bar{F}_{1\bullet}$ and $\bar{F}_{0\bullet}$ are homotopic. The theorem follows using Lemma 16.1.7. □

Proposition 16.7.4. *Let A be a deformation retract of V. Then the map of complexes $\mathcal{C}_\bullet(X, A) \to \mathcal{C}_\bullet(X, V)$ induces an isomorphism on relative homology.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_\bullet(A) \arrow[r] \arrow[d] & \mathcal{C}_\bullet(X) \arrow[r] \arrow[d, equal] & \mathcal{C}_\bullet(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}_\bullet(V) \arrow[r] & \mathcal{C}_\bullet(X) \arrow[r] & \mathcal{C}_\bullet(X, V) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. From the long exact sequence we get the following commutative diagram

$$\begin{tikzcd}
H_i(A) \arrow[r] \arrow[d, "\wr"] & H_i(X) \arrow[r] \arrow[d, equal] & H_i(X,A) \arrow[r] \arrow[d] & H_{i+1}(A) \arrow[r] \arrow[d, "\wr"] & H_{i+1}(X) \arrow[d, equal] \\
H_i(V) \arrow[r] & H_i(X) \arrow[r] & H_i(X,V) \arrow[r] & H_{i+1}(V) \arrow[r] & H_{i+1}(X)
\end{tikzcd}$$

An easy diagram chase now shows that the middle vertical arrow is an isomorphism. $\square$

Theorem 16.7.5. *Let A be a closed subspace of X such that there is an open neighbourhood V of A which deformation retracts onto A. Then the map $q_* : H_n(X,A) \to H_n(X/A, A/A)$ is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "1"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "6"] & H_n(X-A, V-A) \arrow[l, "2"'] \arrow[d, "5"] \\
H_n(X/A, A/A) \arrow[r, "3"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "4"']
\end{tikzcd}$$

The arrows 1 and 3 are isomorphisms because of Proposition 16.7.4. Arrows 2 and 4 are isomorphisms because of excision. Arrow 5 is an isomorphism since the pairs $(X-A, V-A)$ and $(X/A - A/A, V/A - A/A)$ are the same. It follows that arrow 6 is an isomorphism, and so is q_*. $\square$

Let X be a topological space and let A be a closed subspace. Let X/A denote the space obtained by identifying all points in A and let $q : X \to X/A$ denote the quotient map. Then we get a map of pairs $q : (X,A) \to (X/A, A/A)$. Note the A/A is a point and so from the long exact homology sequence it follows that

- there are isomorphisms $H_n(X/A) \xrightarrow{\sim} H_n(X/A, A/A)$ for $n > 1$
- Since $H_0(A/A) \to H_0(X/A)$ is an inclusion it follows that $H_1(X/A) \xrightarrow{\sim} H_1(X/A, A/A)$
- Finally, we have $H_0(X/A) \cong \mathbb{Z}^c$, where c is the number of path components of X/A. Then the image of $H_0(A/A)$ corresponds to the path component of A/A, and so we get that $H_0(X/A, A/A) \cong \mathbb{Z}^{c-1}$.

Combining the above and Theorem 16.7.5 we get the following corollary.

Corollary 16.7.6. *Let A be a closed subspace of X such that there is an open neighbourhood V of A which deformation retracts onto A. Then we have isomorphisms $H_n(X,A) \xrightarrow{\sim} H_n(X/A, A/A) \xleftarrow{\sim} H_n(X/A)$ for $n \geqslant 1$.*

As a corollary let us compute the homology groups of spheres using the above theorem.

Corollary 16.7.7. *$H_i(S^n) = 0$ if $i \neq 0, n$ and $H_0(S^n) = H_n(S^n) = \mathbb{Z}$.*

Proof. Since we know that the spheres are path connected, from the remark preceding the theorem it follows that it suffices to compute $H_i(D^n, \partial D^n)$ for $i > 0$. Here ∂D^n denotes S^{n-1} when $n > 1$, and denotes the set $\{0, 1\}$ when $n = 1$. Note that $D^n/\partial D^n = S^n$. We have

$$H_i(D^n) \to H_i(D^n, \partial D^n) \to H_{i-1}(\partial D^n) \to H_{i-1}(D^n)$$

Now proceed by induction on n. This is easy and is left to the reader. □

16.8 Relation between $\pi_1(X, x_0)$ and $H_1(X)$

In this section we want to show that there is a natural map $\pi_1(X, x_0)_{\mathrm{ab}} \to H_1(X)$ which is an isomorphism.

16.9 Exercises

16.9.1. Consider the map $f : S^1 \times S^1 \to S^1 \times S^1$ which switches the factors. Show that f is not homotopic to the constant map $c_{(1,1)}$ which maps every point to the point $(1, 1)$. Using homology to show that f is not homotopic to the identity map. See also exercise 15.4.14.

16.9.2. Let $\sigma : \Delta^1 \to X$ be a path. Let $\tilde{\sigma}$ be the path $\tilde{\sigma}(t) = \sigma(1-t)$. Show that the element $[\sigma] + [\tilde{\sigma}] \in \mathcal{C}_1(X)$ is in the image of $d_2 : \mathcal{C}_2(X) \to \mathcal{C}_1(X)$. (HINT: Show that $\sigma * \tilde{\sigma}$ is 0 in homology.)

16.9.3. Compute the homology groups of $X = \mathbb{R}^m - \{p_1, p_2, \dots, p_r\}$ for $m > 2$. (HINT: Use the pair $(\mathbb{R}^m, X)$ and excision. Or else convince yourself that X deformation retracts onto a wedge of spheres.)

16.9.4. For a continuous map $f : X \to Y$ we proved that the following diagram commutes

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\Phi(X)"'] & \pi_1(Y, y_0) \arrow[d, "\Phi(Y)"] \\
H_1(X) \arrow[r, "f_*"'] & H_1(Y)
\end{tikzcd}$$

Use the Mayer-Vietoris sequence to compute the homology groups of $\mathbb{P}^2_{\mathbb{R}}$.

16.9.5. Let G be a group and let $g \in [G, G]$ be an element. Show that $G_{\mathrm{ab}} \xrightarrow{\sim} (G/\langle g \rangle)_{\mathrm{ab}}$ is an isomorphism. Here $\langle g \rangle$ is the normal subgroup generated by g.

16.9.6. Let C denote the sphere with k handles, described in Exercise 13.8.23.

1. Use excision to compute the homology groups $H_k(C, C - \{p\})$.
2. Show that $C - \{p\}$ has the homotopy type of a wedge of $2k$ circles.
3. Compute the homology groups of C.

16.9.7. The suspension of X is $SX := X \times I/\sim$, where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$. Use the Mayer-Vietoris sequence to find the homology groups of SX in terms of the homology groups of X.

16.9.8. Give a "nice" homeomorphism $S(S^n) \xrightarrow{\sim} S^{n+1}$.

16.9.9. Fix a generator $a \in H_n(S^n)$. The map $1 \mapsto a$ gives an identification of $\mathbb{Z}$ and $H_n(S^n)$. A continuous map $f : S^n \to S^n$ induces a map on homology $f_* : H_n(S^n) \to H_n(S^n)$. Using the above identification, we get a map $\mathbb{Z} \to \mathbb{Z}$. This map is multiplication by an integer, and this integer is referred to as the degree of the map. Show that $\deg(g \circ f) = \deg(g)\deg(f)$.

16.9.10. Let $A_n : S^n \to S^n$ denote the antipodal map. Convince yourself that A_1 is rotation by 180^0. What is $\deg(A_1)$?

16.9.11. Let $\tilde{A}_n : S^n \times I \to S^n \times I$ denote the map $(x,t) \mapsto (A_n(x), 1-t)$. Show that this map induces a map on the suspension and, after identifying with the homeomorphism in exercise 16.9.8, it gives A_{n+1}. Now use the long exact sequence for the pair $(S^n \times I, S^n \times 0 \cup S^n \times 1)$ and induction on n to find $\deg(A_n)$.

16.9.12. Consider the map $S^n \times I \to S^n \times I$ given by $(x,t) \mapsto (x, 1-t)$. Show that this map induces a map on the suspension and, after identifying with the homeomorphism in exercise 16.9.8, it gives the reflection map $S^n \to S^n$

$$(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1})\,.$$

Now use the long exact sequence for the pair $(S^n \times I, S^n \times 0 \cup S^n \times 1)$ and compute the degree of the reflection map.

16.9.13. In this exercise we will compute the homology groups of $\mathbb{P}^n_{\mathbb{C}}$. Recall $\mathbb{P}^n_{\mathbb{C}} = \Big(\mathbb{C}^{n+1} \setminus (0,\ldots,0)\Big)/\mathbb{C}^*$. A point in $\mathbb{P}^n_{\mathbb{C}}$ can be represented by $[x_0, \ldots, x_n]$. For $\lambda \in \mathbb{C}^*$, $[\lambda x_0, \ldots, \lambda x_n]$ represents the same point as $[x_0, \ldots, x_n]$.

1. For points $p, q \in \mathbb{P}^n_{\mathbb{C}}$, define the unique straight line passing through p and q. Define this using coordinates and check that the line is independent of the coordinates.

2. View $\mathbb{P}^{n-1}_{\mathbb{C}}$ as sitting inside $\mathbb{P}^n_{\mathbb{C}}$ as the hyperplane $x_0 = 0$. Use the previous part to show that $U = \mathbb{P}^n_{\mathbb{C}} \setminus [1 : 0, \ldots, 0]$ deformation retracts onto $\mathbb{P}^{n-1}_{\mathbb{C}}$.

3. Show that $\mathbb{P}^1_{\mathbb{C}}$ is homeomorphic to S^2.

4. Use M-V sequence and induction to compute the homology groups of $\mathbb{P}^n_{\mathbb{C}}$. Alternatively, use the pair $(\mathbb{P}^n_{\mathbb{C}}, U)$.

Chapter 17

Some Homological Algebra

In the previous chapter we defined complexes. Throughout this chapter we shall be interested in complexes indexed by $\mathbb{Z}$.

17.1 Preliminaries

17.1.1 **Split exact sequence.** Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be a short exact sequence. We say this is split if there is a map $s : M'' \to M$ such that $\beta \circ s = \mathrm{Id}_{M''}$. If a short exact sequence is split then it is easily seen that $M \cong M' \oplus M''$. It is easy to check that a short exact sequence splits iff there is a map $s' : M \to M'$ such that $s' \circ \alpha = \mathrm{Id}_{M'}$.

17.1.2 **Cochain complexes.** In the previous chapter we introduced chain complexes. A variant of this is a cochain complex. This is a sequence of abelian groups $\{A^i, d^i\}$ where $d^n : A^n \to A^{n+1}$ are homomorphisms satisfying $d^{n+1} \circ d^n = 0$. We define the n^{th} cohomology of a cochain complex to be the group

$$H^n(\{A^\bullet, d^\bullet\}) = \frac{\mathrm{Ker}(d^n)}{\mathrm{Im}(d^{n-1})}\,.$$

Remark 17.1.3. Given a cochain complex $\{A^\bullet, d^\bullet\}$, we can view it as a chain complex by defining $B_i := A^{-i}$ and $d_i := d^{-i}$. Similarly, we can go from chain complexes to cochain complexes. We shall often use the word complex to mean a chain complex or a cochain complex, and it will be clear from the context what we mean. □

17.1.4 **Long exact cohomology sequence.** If we have a short exact sequence of cochain complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ then there is a long exact cohomology sequence

$$\ldots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \to H^{n+1}(A^\bullet) \to \ldots\,.$$

Theorem 17.1.5. *Let R be a PID, for example, $R = \mathbb{Z}$. Let M be a free R module (not necessarily finitely generated). If $N \subset M$ is a submodule then N is free.*

Proof. See [Lan02, Chapter 3, Theorem 7.3]. □

Proposition 17.1.6. *Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of abelian groups. Let N be an abelian group. Then the following two sequences are exact.*

(1) $0 \to \mathrm{Hom}(M'', N) \xrightarrow{\tilde{\beta}} \mathrm{Hom}(M, N) \xrightarrow{\tilde{\alpha}} \mathrm{Hom}(M', N)$,

(2) $M' \otimes N \xrightarrow{\alpha\otimes 1} M \otimes N \xrightarrow{\beta\otimes 1} M'' \otimes N \to 0$

Proof. The map $\tilde{\beta}$ is defined as $\tilde{\beta}(f) = f \circ \beta$. The definition of $\tilde{\alpha}$ is similar. The exactness of the first is easy and left as an exercise. This is often referred to as left exactness of $\mathrm{Hom}(-, N)$. For the second see [AM69, Proposition 2.18]. Exactness of the second sequence is often referred to as tensor product being right exact. □

17.1.7 **Shifting complexes.** Given a complex $A_\bullet$ we define the shifted complex $A[k]_\bullet$ as follows. Define

(i) $A[k]_n := A_{n+k}$

(ii) $d_{A[k],n} := (-1)^k d_{A,n}$

17.2 Universal Coefficient Theorems

17.2.1 **Homology of $A_\bullet \otimes M$.** Given a complex of abelian groups $\{A_\bullet, d_\bullet\}$ and an abelian group M we may define the following, which is obviously a complex:

$$\dots \xrightarrow{d_{n+2}\otimes 1} A_{n+1} \otimes M \xrightarrow{d_{n+1}\otimes 1} A_n \otimes M \xrightarrow{d_n\otimes 1} A_{n-1} \otimes M \xrightarrow{d_{n-1}\otimes 1} \dots$$

We denote this complex by $A_\bullet \otimes M$. In this subsection we relate the homologies of $A_\bullet \otimes M$ with the homologies of $A_\bullet$.

We begin by defining Tor groups. Let M be an abelian group. Then we can find a surjective map $F_0 \to M$, where F_0 is a free abelian group. Since subgroup of a free abelian group is free (Theorem 17.1.5), it follows that we have the following exact sequence in which the F_i are free:

$$0 \to F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} M \to 0\,.$$

Such an exact sequence is often referred to as a free resolution of M. We have the following standard facts from homological algebra:

1. Let N be any abelian group. By Proposition 17.1.6 the sequence

$$F_1 \otimes N \xrightarrow{\alpha\otimes 1} F_0 \otimes N \xrightarrow{\beta\otimes 1} M \otimes N \to 0$$

   is exact. The kernel of the map $F_1 \otimes N \xrightarrow{\alpha\otimes 1} F_0 \otimes N$ is defined to be $\mathrm{Tor}^1(M, N)$. For ease of notation we shall denote $\mathrm{Tor}^1(M, N)$ by $M * N$.

2. The group $M * N$ is independent of the free resolution we choose. If M is free, then the resolution splits, from which it easily follows that $M * N = 0$.

3. Suppose we have a short exact sequence of abelian groups $0 \to M' \to M \to M'' \to 0$ then we have a long exact sequence

$$\begin{aligned} 0 \to M' * N \to & M * N \to M'' * N \\ & \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0\,. \end{aligned}$$

Theorem 17.2.2 (Universal coefficients)**.** *Let $A_\bullet$ be a complex of free abelian groups and let M be an abelian group. Then there is a split exact sequence*

$$0 \to H_n(A_\bullet) \otimes M \xrightarrow{c} H_n(A_\bullet \otimes M) \to H_{n-1}(A_\bullet) * M \to 0\,.$$

Proof. For a complex $\mathcal{C}_\bullet$, denote the kernel of the differential d_n by $Z_n(\mathcal{C}_\bullet)$ and the image of the differential d_{n+1} by $B_n(\mathcal{C}_\bullet)$. We have inclusions $B_n(\mathcal{C}_\bullet) \subset Z_n(\mathcal{C}_\bullet) \subset \mathcal{C}_n$. For an element $z \in Z_n(\mathcal{C}_\bullet)$ we shall denote by $\bar{z}$ its image in $H_n(\mathcal{C}_\bullet)$.

For the complex $A_\bullet$, let $Z_n := \mathrm{Ker}(d_n)$ and let $B_n := \mathrm{Im}(d_{n+1})$. We shall slightly abuse notation and denote the map $A_{n+1} \to B_n$ by d_{n+1}. We have inclusions $B_n \subset Z_n \subset A_n$. Since A_n is free, using Theorem 17.1.5, we see that B_n and Z_n are free.

We have a short exact sequence $0 \to Z_n \to A_n \xrightarrow{d_n} B_{n-1} \to 0$. Since B_{n-1} is free, we have $B_{n-1} * M = 0$. Thus, tensoring the above sequence with M we get an exact sequence

$$0 \to Z_n \otimes M \to A_n \otimes M \to B_{n-1} \otimes M \to 0\,.$$

These sit in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z_{n+1} \otimes M \arrow[r] \arrow[d, "0"'] & A_{n+1} \otimes M \arrow[r, "d_{n+1}\otimes 1"] \arrow[d, "d_{n+1}\otimes 1"] & B_n \otimes M \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \otimes M \arrow[r] & A_n \otimes M \arrow[r, "d_n\otimes 1"] & B_{n-1} \otimes M \arrow[r] & 0 \end{tikzcd}$$

Define a complex $Z_\bullet \otimes M$ by taking the abelian group in degree n to be $Z_n \otimes M$ and all differentials to be 0. Similarly, we define a complex $B_\bullet \otimes M$. Putting together the preceding commutative diagrams we get a map of complexes (see 17.1.7 for the definition of $(B_\bullet \otimes M)[-1]$)

$$0 \to Z_\bullet \otimes M \to A_\bullet \otimes M \to (B_\bullet \otimes M)[-1] \to 0\,.$$

Taking homology we get an exact sequence

$$B_n \otimes M \xrightarrow{a} Z_n \otimes M \longrightarrow H_n(A_\bullet \otimes M) \longrightarrow B_{n-1} \otimes M \xrightarrow{b} Z_{n-1} \otimes M. \tag{17.2.3}$$

It is easy to check, using the definition of the connecting homomorphism, that the map a is $i_n \otimes \mathrm{Id}_M$, where $i_n : B_n \to Z_n$ is the inclusion. Similarly, the map b is $i_{n-1} \otimes \mathrm{Id}_M$. Thus, we have a short exact sequence

$$0 \to \mathrm{Coker}(a) \to H_n(A_\bullet \otimes M) \to \mathrm{Ker}(b) \to 0\,.$$

To compute the cokernel and kernel in the above, tensor the exact sequence $0 \to B_n \xrightarrow{i_n} Z_n \to H_n(A_\bullet) \to 0$ with M and use Z_n is free to get the exact sequence

$$0 \to H_n(A_\bullet) * M \to B_n \otimes M \xrightarrow{a} Z_n \otimes M \to H_n(A_\bullet) \otimes M \to 0\,. \tag{17.2.4}$$

This shows that $\mathrm{Coker}(a) = H_n(A_\bullet) \otimes M$ and $\mathrm{Ker}(b) = H_{n-1}(A_\bullet) * M$. This proves the existence of the exact sequence

$$0 \to H_n(A_\bullet) \otimes M \xrightarrow{c} H_n(A_\bullet \otimes M) \to H_{n-1}(A_\bullet) * M \to 0\,.$$

It is clear that the map c has the following description. In view of the surjectivity in (17.2.4), an element of $H_n(A_\bullet) \otimes M$ can be represented as $\sum_i \bar{z}_i \otimes m_i$, where $z_i \in Z_n$. It is clear that $\sum_i z_i \otimes m_i \in Z_n(A_\bullet \otimes M)$. It follows easily that $c(\sum_i \bar{z}_i \otimes m_i) = \sum_i \overline{z_i \otimes m_i}$. Here $\overline{z_i \otimes m_i}$ is the image of

Next we show that this sequence is split. Since B_{n-1} is free, the short exact sequence $0 \to Z_n \to A_n \to B_{n-1} \to 0$ splits. We choose a splitting $s : A_n \to Z_n$ of the inclusion $Z_n \subset A_n$. We have a commutative diagram

$$\begin{tikzcd}
A_{n+1} \otimes M \arrow[rr, "(s\circ d_{n+1})\otimes 1"] \arrow[d, "d_{n+1}\otimes 1"'] & & B_n \otimes M \arrow[d, "i_n\otimes 1"] \\
Z_n(A_\bullet \otimes M) \arrow[r, hook] \arrow[d, "a"'] & A_n \otimes M \arrow[r, "s\otimes 1"] & Z_n \otimes M \arrow[d, "b"] \\
H_n(A_\bullet \otimes M) \arrow[rr, dashed, "\bar{s}"] & & H_n(A_\bullet) \otimes M
\end{tikzcd}$$

The commutativity of the square follows because $s \circ d_{n+1}(a) = i_n \circ d_{n+1}(a)$. Note $H_n(A_\bullet \otimes M)$ is cokernel of $d_{n+1} \otimes 1$ by definition and $H_n(A_\bullet) \otimes M$ is cokernel of $i_n \otimes 1$ as tensor product is right exact. Thus, we get an induced map $\bar{s}$, which we claim is the required splitting. Consider the following diagram in which the squares commute. The square on the right is the bottom rectangle in the diagram above. The square on the left commutes as explained in the first para of the proof.

$$\begin{tikzcd}
Z_n \otimes M \arrow[r] \arrow[d, "b"'] & Z_n(A_\bullet \otimes M) \arrow[r, "s\otimes 1"] \arrow[d, "a"] & Z_n \otimes M \arrow[d, "b"] \\
H_n(A_\bullet) \otimes M \arrow[r] & H_n(A_\bullet \otimes M) \arrow[r, "\bar{s}"] & H_n(A_\bullet) \otimes M
\end{tikzcd}$$

We need to show that the composite in the bottom row is the identity. But this is clear by the surjectivity of b and as the composite in the top row is the identity. This completes the proof of the Theorem. $\square$

Corollary 17.2.5. $H_n(A_\bullet \otimes M) \cong (H_n(A_\bullet) \otimes M) \bigoplus (H_{n-1}(A_\bullet) * M)$.

17.2.6 **Cohomology of** $\mathrm{Hom}(A_\bullet, M)$**.** Given a complex of abelian groups $\{A_\bullet, d_\bullet\}$ and an abelian group M we may define the following cochain complex which we denote by $\mathrm{Hom}(A_\bullet, M)$:

$$\xrightarrow{d'_{n-2}} \mathrm{Hom}(A_{n-1}, M) \xrightarrow{d'_{n-1}} \mathrm{Hom}(A_n, M) \xrightarrow{d'_n} \mathrm{Hom}(A_{n+1}, M) \xrightarrow{d'_{n+1}} \ldots$$

Here the maps d'_n are defined as follows. If $f \in \mathrm{Hom}(A_n, M)$ then $d'_n(f) := f \circ d_{n+1}$. It is easily checked that $d'_{n+1} \circ d'_n = 0$. Thus, we get a cochain complex. The cohomology groups are

$$H^n(\mathrm{Hom}(A_\bullet, M)) := \frac{\mathrm{Ker}(d'_n)}{\mathrm{Im}(d'_{n-1})}.$$

Remark 17.2.7. For a map $g : A \to B$ and an abelian group N we define $\tilde{g} : \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N)$ by $\tilde{g}(f) = f \circ g$. Thus, comparing notations we see that $d'_n = \tilde{d}_{n+1}$. $\square$

We will prove a similar result, as the one in the preceding subsection, for the cohomology groups of the complex $\mathrm{Hom}(A_\bullet, M)$. For this we need to introduce the Ext groups. Start with a free resolution of M.

$$0 \to F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} M \to 0.$$

We have the following standard facts from homological algebra:

1. Let N be any abelian group. Recall that for a map $g : A \to B$ we define $\tilde{g} : \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N)$ by $\tilde{g}(f) = f \circ g$. By Proposition 17.1.6 the sequence

$$0 \to \mathrm{Hom}(M, N) \xrightarrow{\tilde{\beta}} \mathrm{Hom}(F_0, N) \xrightarrow{\tilde{\alpha}} \mathrm{Hom}(F_1, N)$$

   is exact. The cokernel of $\tilde{\alpha}$ is defined to be $\mathrm{Ext}^1(M, N)$.

2. The group $\mathrm{Ext}^1(M, N)$ is independent of the free resolution we choose. If M is free, then the resolution splits, from which it easily follows that $\mathrm{Ext}^1(F, N) = 0$.

3. Suppose we have a short exact sequence of abelian groups $0 \to M' \to M \to M'' \to 0$ then we have a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(M'', N) &\to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \\ &\to \mathrm{Ext}^1(M'', N) \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M', N) \to 0. \end{aligned}$$

Theorem 17.2.8 (Universal coefficients). *Let $A_\bullet$ be a complex of free abelian groups and let M be an abelian group. Then there is a split exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(A_\bullet), M) \to H^n(\mathrm{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \mathrm{Hom}(H_n(A_\bullet), M) \to 0\,.$$

Proof. For a cochain complex $\mathcal{C}^\bullet$, denote the kernel of the differential d^n by $Z^n(\mathcal{C}^\bullet)$, the image of the differential d^{n-1} by $B^n(\mathcal{C}^\bullet)$. We have inclusions $B^n(\mathcal{C}^\bullet) \subset Z^n(\mathcal{C}^\bullet) \subset \mathcal{C}^n$.

Before beginning the proof we describe the map $H^n(\mathrm{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \mathrm{Hom}(H_n(A_\bullet), M)$. Applying $\mathrm{Hom}(-, M)$ to the exact sequence $0 \to B_n \xrightarrow{i_n} Z_n \to H_n(A_\bullet) \to 0$ yields the exact sequence

$$0 \to \mathrm{Hom}(H_n(A_\bullet), M) \to \mathrm{Hom}(Z_n, M) \xrightarrow{\tilde{i}_n} \mathrm{Hom}(B_n, M)\,.$$

We will define a map $Z^n(\mathrm{Hom}(A_\bullet, M)) \xrightarrow{a} \mathrm{Hom}(Z_n, M)$ and check that the image lands inside $\mathrm{Hom}(H_n(A_\bullet), M)$. By exactness of the preceding sequence, it suffices to check that $\tilde{i}_n \circ a = 0$. Let $f \in Z^n(\mathrm{Hom}(A_\bullet, M))$. Then $f : A_n \to M$ is such that $f \circ d_{n+1} = 0$. We may restrict f to Z_n to get a map $Z_n \to M$. Clearly, $f(B_n) = f \circ d_{n+1}(A_{n+1}) = 0$, that is, $\tilde{i}_n(f) = 0$. This defines a map $Z^n(\mathrm{Hom}(A_\bullet, M)) \to \mathrm{Hom}(H_n(A_\bullet), M)$. If we take an element of the type $g \circ d_n \in Z^n(\mathrm{Hom}(A_\bullet, M))$ then it is easily checked that its restriction to Z_n is 0. This shows that elements in $B^n(\mathrm{Hom}(A_\bullet, M))$ get mapped to 0. Thus, we get a map $H^n(\mathrm{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \mathrm{Hom}(H_n(A_\bullet), M)$.

We have a short exact sequence $0 \to Z_n \to A_n \xrightarrow{d_n} B_{n-1} \to 0$. Since B_{n-1} is free, we have $\mathrm{Ext}^1(B_{n-1}, M) = 0$. Thus, applying $\mathrm{Hom}(-, M)$ to the above sequence we get an exact sequence

$$0 \to \mathrm{Hom}(B_{n-1}, M) \to \mathrm{Hom}(A_n, M) \to \mathrm{Hom}(Z_n, M) \to 0\,.$$

These sit in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, M) \arrow[r, "\tilde{d}_n"] \arrow[d, "0"'] & \mathrm{Hom}(A_n, M) \arrow[r] \arrow[d, "d'_n"] & \mathrm{Hom}(Z_n, M) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_n, M) \arrow[r, "\tilde{d}_{n+1}"] & \mathrm{Hom}(A_{n+1}, M) \arrow[r] & \mathrm{Hom}(Z_{n+1}, M) \arrow[r] & 0
\end{tikzcd}$$

Define a cochain complex $Z'_\bullet$ by taking the n^{th} degree term to be $\mathrm{Hom}(Z_n, M)$ and all differentials to be 0. Note that n^{th} differential is a map from Z'_n to Z'_{n+1}. Similarly, we define a cochain complex $B'_\bullet$. Putting together the preceding commutative diagrams we get a short exact sequence of cochain complexes

$$0 \to B'_\bullet[-1] \to \mathrm{Hom}(A_\bullet, M) \to Z'_\bullet \to 0\,.$$

Taking cohomology we get a long exact sequence (note that the indices will increase as this is a short exact sequence of cochain complexes). One checks that the maps in the long exact sequence have the following description. This is a simple diagram chase which is left to the reader.

$$\operatorname{Hom}(Z_{n-1}, M) \xrightarrow{\tilde{i}_{n-1}} \operatorname{Hom}(B_{n-1}, M) \to H^n(\operatorname{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \operatorname{Hom}(Z_n, M) \xrightarrow{\tilde{i}_n} \operatorname{Hom}(B_n, M)\,. \tag{17.2.9}$$

Thus, we have a short exact sequence

$$0 \to \operatorname{Coker}(\tilde{i}_{n-1}) \to H^n(\operatorname{Hom}(A_\bullet, M)) \to \operatorname{Ker}(\tilde{i}_n) \to 0\,.$$

To compute the cokernel and kernel apply $\operatorname{Hom}(-, M)$ to the exact sequence $0 \to B_n \xrightarrow{i_n} Z_n \to H_n(A_\bullet) \to 0$ and use Z_n is free to get the exact sequence

$$0 \to \operatorname{Hom}(H_n(A_\bullet), M) \to \operatorname{Hom}(Z_n, M) \xrightarrow{\tilde{i}_n} \operatorname{Hom}(B_n, M) \to \operatorname{Ext}^1(H_n(A_\bullet), M) \to 0\,. \tag{17.2.10}$$

This shows $\operatorname{Coker}(\tilde{i}_{n-1}) = \operatorname{Ext}^1(H_{n-1}(A_\bullet), M)$ and $\operatorname{Ker}(\tilde{i}_n) = \operatorname{Hom}(H_n(A_\bullet), M)$. This proves the existence of the exact sequence

$$0 \to \operatorname{Ext}^1(H_{n-1}(A_\bullet), M) \to H^n(\operatorname{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \operatorname{Hom}(H_n(A_\bullet), M) \to 0\,.$$

Next we show that this sequence is split. We have the short exact sequence $0 \to Z_n \xrightarrow{j_n} A_n \to B_{n-1} \to 0$. Since B_{n-1} is free, we can choose a splitting $s : A_n \to Z_n$ of the inclusion j_n. This induces a splitting $\tilde{s} : \operatorname{Hom}(Z_n, M) \to \operatorname{Hom}(A_n, M)$ of the short exact sequence

$$0 \to \operatorname{Hom}(B_{n-1}, M) \to \operatorname{Hom}(A_n, M) \to \operatorname{Hom}(Z_n, M) \to 0\,.$$

It is easily checked that there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(Z_n, M) \arrow[r, "\tilde{s}"] & \operatorname{Hom}(A_n, M) \arrow[rr] & & \operatorname{Hom}(Z_n, M) \arrow[d, equal] \\
\operatorname{Hom}(H_n(A_\bullet), M) \arrow[r] \arrow[u, hook] & Z^n(\operatorname{Hom}(A_\bullet, M)) \arrow[r] \arrow[u, hook] & H^n(\operatorname{Hom}(A_\bullet, M)) \arrow[r, "\bar{a}"] & \operatorname{Hom}(Z_n, M)
\end{tikzcd}$$

We have already seen that the image of $\bar{a}$ lands inside $\operatorname{Hom}(H_n(A_\bullet), M)$. Thus, the bottom row factors as

$$\operatorname{Hom}(H_n(A_\bullet), M) \to H^n(\operatorname{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \operatorname{Hom}(H_n(A_\bullet), M) \subset \operatorname{Hom}(Z_n, M)\,.$$

The composite in the top row is identity as $\tilde{s}$ is a splitting. It follows that the composite map

$$\operatorname{Hom}(H_n(A_\bullet), M) \to H^n(\operatorname{Hom}(A_\bullet, M)) \xrightarrow{\bar{a}} \operatorname{Hom}(H_n(A_\bullet), M)$$

is also the identity. This is a splitting of the short exact sequence. This completes the proof of the Theorem. ∎

Corollary 17.2.11.

$$H^n(\mathrm{Hom}(A_\bullet, M)) \cong \mathrm{Hom}(H_n(A_\bullet), M) \bigoplus \mathrm{Ext}^1(H_{n-1}(A_\bullet), M)\,.$$

17.2.12 **Cohomology of** $\mathrm{Hom}(A_\bullet, \mathbb{Z}) \otimes M$**.** Let $A_\bullet$ be a complex of free abelian groups. In section 17.2.1 we expressed the homologies of the complex $A_\bullet \otimes M$ in terms of the homologies of the complex $A_\bullet$. In a similar spirit, in this section we shall express the homologies of the complex $\mathrm{Hom}(A_\bullet, M)$ in terms of the homologies of the complex $\mathrm{Hom}(A_\bullet, \mathbb{Z})$ when $A_\bullet$ is a complex of free abelian groups for which all the $H_n(A_\bullet)$ are finitely generated abelian groups.

Remark 17.2.13. First consider the case when each A_n is a **finitely generated** free abelian group. In this case the cochain complex $\mathrm{Hom}(A_\bullet, \mathbb{Z})$ is a complex of free abelian groups and $\mathrm{Hom}(A_n, M) \cong \mathrm{Hom}(A_n, \mathbb{Z}) \otimes M$. Thus, by Theorem 17.2.2 (use Remark 17.1.3 to apply this Theorem) we get that there is a split exact sequence

$$\begin{aligned} 0 \to H^n(\mathrm{Hom}(A_\bullet, \mathbb{Z})) \otimes M \to & H^n(\mathrm{Hom}(A_\bullet, M)) \\ & \to H^{n+1}(\mathrm{Hom}(A_\bullet, \mathbb{Z})) * M \to 0\,. \end{aligned}$$

□

For the general case we need the following preliminary lemmas.

Definition 17.2.14. *Let $A_\bullet$ and $B_\bullet$ be chain complexes and let $f : A_\bullet \to B_\bullet$ be a map of complexes. If the induced map on homologies is an isomorphism, then we say that f is a quasi-isomorphism.*

Lemma 17.2.15. *Let $D_\bullet$ and $C_\bullet$ be complexes of free abelian groups. Let $f : D_\bullet \to C_\bullet$ be a quasi-isomorphism. Then for any abelian group M, the induced map of complexes $\tilde{f} : \mathrm{Hom}(C_\bullet, M) \to \mathrm{Hom}(D_\bullet, M)$ is a quasi-isomorphism.*

Proof. This follows easily using Lemma 16.1.3, Theorem 17.2.8 and the following commutative diagram of short exact sequences in which the extreme vertical arrows are isomorphisms.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(C_\bullet), M) \arrow[r] \arrow[d] & H^n(\mathrm{Hom}(C_\bullet, M)) \arrow[r] \arrow[d] & \mathrm{Hom}(H_n(C_\bullet), M) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(D_\bullet), M) \arrow[r] & H^n(\mathrm{Hom}(D_\bullet, M)) \arrow[r] & \mathrm{Hom}(H_n(D_\bullet), M) \arrow[r] & 0 \end{tikzcd}$$

We leave it to the reader to check the details. □

Lemma 17.2.16. *Let $C_\bullet$ be a complex of free abelian groups such that the homology groups $H_n(C_\bullet)$ are finitely generated. Then we can find a complex $D_\bullet$ such that each D_n is a finitely generated free abelian group and a quasi-isomorphism $f : D_\bullet \to C_\bullet$.*

Proof. We have a surjective map $Z_n(C_\bullet) \to H_n(C_\bullet)$. As $H_n(C_\bullet)$ is finitely generated, we can find a finitely generated submodule $F_n \subset Z_n$ such that the composite $F_n \to Z_n \to H_n(C_\bullet)$ is surjective. Let G_n denote the kernel of $F_n \to H_n(C_\bullet)$. It is clear that $G_n = F_n \cap B_n(C_\bullet)$. Both G_n and F_n are free, being submodules of the free abelian group C_n. Define $D_n := F_n \oplus G_{n-1}$ and the differential $d_n(x,y) := (y,0)$. It is clear that this defines a complex $\{D_\bullet, d_\bullet\}$.

We define a map $f_n : D_n \to C_n$. It suffices to define f_n on F_n and G_{n-1}, and then we can extend it linearly to D_n. Let $f_n : F_n \to C_n$ be the inclusion. As $G_{n-1} \subset B_{n-1}(C_\bullet)$ and G_{n-1} is free, we can find a map $f_n : G_{n-1} \to C_n$ so that $d_n \circ f_n$ is the identity. It is easily checked that the f_n define a map of complexes $D_\bullet \to C_\bullet$ which is a quasi-isomorphism. □

Theorem 17.2.17 (Universal coefficients)**.** *Let $A_\bullet$ be a complex of free abelian groups for which all the $H_n(A_\bullet)$ are finitely generated abelian groups. Then we have a split exact sequence*

$$\begin{aligned} 0 \to H^n(\mathrm{Hom}(A_\bullet, \mathbb{Z})) \otimes M \to & H^n(\mathrm{Hom}(A_\bullet, M)) \\ & \to H^{n+1}(\mathrm{Hom}(A_\bullet, \mathbb{Z})) * M \to 0\,. \end{aligned}$$

Proof. In view of Lemma 17.2.16 we can find a quasi-isomorphism $f : D_\bullet \to A_\bullet$ such that each D_n is a finitely generated free abelian group. In view of Lemma 17.2.15 the induced maps of complexes $\tilde{f}_{\mathbb{Z}} : \mathrm{Hom}(A_\bullet, \mathbb{Z}) \to \mathrm{Hom}(D_\bullet, \mathbb{Z})$ and $\tilde{f}_M : \mathrm{Hom}(A_\bullet, M) \to \mathrm{Hom}(D_\bullet, M)$ are quasi-isomorphisms. Let us denote $H^n(\mathrm{Hom}(A_\bullet, M))$ by $H^n(A_\bullet, M)$. It is easily checked that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^n(A_\bullet, \mathbb{Z}) \otimes M \arrow[r] \arrow[d] & H^n(A_\bullet, M) \arrow[r] \arrow[d] & H^{n+1}(A_\bullet, \mathbb{Z}) * M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^n(D_\bullet, \mathbb{Z}) \otimes M \arrow[r] & H^n(D_\bullet, M) \arrow[r] & H^{n+1}(D_\bullet, \mathbb{Z}) * M \arrow[r] & 0 \end{tikzcd}$$

in which the vertical arrows are isomorphisms. The Theorem now follows using Remark 17.2.13. □

17.3 Kunneth Formula

17.3.1 **Tensor product of complexes.** Let $i : M' \subset M$ be abelian groups. In general, the map $i \otimes 1_N : M' \otimes N \to M \otimes N$ will not be an inclusion. However, sometimes we will abuse notation and write $M \otimes N/(M' \otimes N)$ instead of writing $M \otimes N/i \otimes 1_N(M' \otimes N)$. Similarly, if $h : M \otimes N \to G$ is a map, then by the restriction of h to $M' \otimes N$ we shall mean the composite $h \circ (i \otimes 1_N)$.

Let $\mathcal{D}_\bullet$ and $\mathcal{E}_\bullet$ be complexes. Define a complex $\mathcal{C}_\bullet$ as follows. Let $\mathcal{C}_n := \bigoplus_{j \in \mathbb{Z}} \mathcal{D}_j \otimes \mathcal{E}_{n-j}$. Define $d_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ by defining it on pure tensors $a \otimes b \in \mathcal{D}_l \otimes \mathcal{E}_k$ (where $l + k = n$) by

$$d_n(a \otimes b) := d_l(a) \otimes b + (-1)^l a \otimes d_k(b)\,.$$

One easily checks that $d_{n+1} \circ d_n = 0$. It is clear that there is a map $Z_p(\mathcal{D}_\bullet) \otimes Z_q(\mathcal{E}_\bullet) \to Z_{p+q}(\mathcal{C}_\bullet)$. We claim that under this map the image of $B_p(\mathcal{D}_\bullet) \otimes Z_q(\mathcal{E}_\bullet)$ lands inside $B_{p+q}(\mathcal{C}_\bullet)$. This is clear as (using $d_q(z) = 0$)

$$d_{p+1}(a) \otimes z = d_{p+q+1}(a \otimes z)\,.$$

Similarly, the image of $Z_p(\mathcal{D}_\bullet) \otimes B_q(\mathcal{E}_\bullet)$ lands inside $B_{p+q}(\mathcal{C}_\bullet)$. Thus, we have a map

$$\frac{Z_p(\mathcal{D}_\bullet) \otimes Z_q(\mathcal{E}_\bullet)}{B_p(\mathcal{D}_\bullet) \otimes Z_q(\mathcal{E}_\bullet) + Z_p(\mathcal{D}_\bullet) \otimes B_q(\mathcal{E}_\bullet)} \to \frac{Z_{p+q}(\mathcal{C}_\bullet)}{B_{p+q}(\mathcal{C}_\bullet)}\,.$$

Using tensor product is right exact we easily check that the LHS in the above is isomorphic to $H_p(\mathcal{D}_\bullet) \otimes H_q(\mathcal{E}_\bullet)$. Thus, there is a map

$$H_p(\mathcal{D}_\bullet) \otimes H_q(\mathcal{E}_\bullet) \to H_{p+q}(\mathcal{C}_\bullet)\,.$$

Theorem 17.3.2. *Let $\mathcal{D}_\bullet$ be a complex of free abelian groups. Let $\mathcal{E}_\bullet$ be a complex of abelian groups. Then we have a short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(\mathcal{D}_\bullet) \otimes H_j(\mathcal{E}_\bullet) \to H_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet) \to \bigoplus_{i+j=n-1} H_i(\mathcal{D}_\bullet) * H_j(\mathcal{E}_\bullet) \to 0\,. \tag{17.3.3}$$

If $\mathcal{E}_\bullet$ is a complex of free abelian groups then the above sequence splits.

Proof. The proof is similar to that of Theorem 17.2.2 and so we only sketch it. We consider the complexes

$$Z_\bullet = Z_\bullet(\mathcal{D}_\bullet) : \ldots \xrightarrow{0} Z_{n+1}(\mathcal{D}_\bullet) \xrightarrow{0} Z_n(\mathcal{D}_\bullet) \xrightarrow{0} Z_{n-1}(\mathcal{D}_\bullet) \xrightarrow{0} \ldots$$
$$B_\bullet = B_\bullet(\mathcal{D}_\bullet) : \ldots \xrightarrow{0} B_{n+1}(\mathcal{D}_\bullet) \xrightarrow{0} B_n(\mathcal{D}_\bullet) \xrightarrow{0} B_{n-1}(\mathcal{D}_\bullet) \xrightarrow{0} \ldots$$

These are complexes of free abelian groups as $B_n \subset Z_n \subset \mathcal{D}_n$ and $\mathcal{D}_n$ is free. As before, the short exact sequences $0 \to Z_n \to \mathcal{D}_n \to B_{n-1} \to 0$ can be put together to get a short exact sequence of complexes:

$$0 \to Z_\bullet \to \mathcal{D}_\bullet \to B_\bullet[-1] \to 0\,.$$

As B_{n-1} is free, the above short exact sequence remains exact after tensoring with $\mathcal{E}_j$. Thus, tensoring and taking direct sums we get a short exact sequence of complexes

$$0 \to Z_\bullet \otimes \mathcal{E}_\bullet \to \mathcal{D}_\bullet \otimes \mathcal{E}_\bullet \to B_\bullet \otimes \mathcal{E}_\bullet[-1] \to 0\,. \tag{17.3.4}$$

Note that the differential of $Z_\bullet \otimes \mathcal{E}_\bullet$ has the following description. If $z \otimes b \in Z_j \otimes \mathcal{E}_k$ then $d(z \otimes b) = (-1)^j z \otimes d(b)$. Consider the free abelian group Z_j and the complex $\mathcal{E}_\bullet$. We obtain the complex $Z_j \otimes \mathcal{E}_\bullet$. The abelian group which appears in degree l in this complex is $Z_j \otimes \mathcal{E}_l$. Notice that the complex $Z_\bullet \otimes \mathcal{E}_\bullet$ is simply the direct sum of complexes

$$Z_\bullet \otimes \mathcal{E}_\bullet = \bigoplus_{j\in\mathbb{Z}} (Z_j \otimes \mathcal{E}_\bullet[j])\,.$$

Similarly, the complex $B_\bullet \otimes \mathcal{E}_\bullet$ is simply the direct sum of complexes

$$B_\bullet \otimes \mathcal{E}_\bullet = \bigoplus_{j\in\mathbb{Z}} (B_j \otimes \mathcal{E}_\bullet[j])\,.$$

Observe that $H_n(Z_i \otimes \mathcal{E}_\bullet) = Z_i \otimes H_n(\mathcal{E}_\bullet)$. It follows easily that

$$H_n(Z_\bullet \otimes \mathcal{E}_\bullet) = \bigoplus_{k+j=n} Z_k \otimes H_j(\mathcal{E}_\bullet)\,.$$

Similarly,

$$H_n(B_\bullet \otimes \mathcal{E}_\bullet) = \bigoplus_{k+j=n} B_k \otimes H_j(\mathcal{E}_\bullet)\,.$$

Taking the long exact homology sequence of (17.3.4) we get

$$\begin{aligned}\bigoplus_{k+j=n} B_k \otimes H_j(\mathcal{E}_\bullet) \xrightarrow{a} \bigoplus_{k+j=n} Z_k \otimes H_j(\mathcal{E}_\bullet) \to H_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet) \to \\ \bigoplus_{k+j=n-1} B_k \otimes H_j(\mathcal{E}_\bullet) \to \bigoplus_{k+j=n-1} Z_k \otimes H_j(\mathcal{E}_\bullet)\end{aligned}$$

Using the definition of the connecting homomorphism it is easily checked that $a = i_k \otimes \mathrm{Id}$, where $i_k : B_k \to Z_k$ is the inclusion. From this it follows easily that we have a short exact sequence

$$0 \to \bigoplus_{k+j=n} H_k(\mathcal{D}_\bullet) \otimes H_j(\mathcal{E}_\bullet) \to H_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet) \to \bigoplus_{k+j=n-1} H_k(\mathcal{D}_\bullet) * H_j(\mathcal{E}_\bullet) \to 0\,.$$

Now assume that $\mathcal{E}_\bullet$ is a complex of free abelian groups. As $B_{n-1}(\mathcal{E}_\bullet)$ is free, we may choose a splitting $t_n : \mathcal{E}_n \to Z_n(\mathcal{E}_\bullet)$ of the inclusion $Z_n(\mathcal{E}_\bullet) \subset \mathcal{E}_n$. Let $s_n : \mathcal{D}_n \to Z_n(\mathcal{D}_\bullet)$ of the inclusion $Z_n(\mathcal{D}_\bullet) \subset \mathcal{D}_n$. Define a map

$$u_n : (\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet)_n = \bigoplus_{i+j=n} \mathcal{D}_i \otimes \mathcal{E}_j \to Z_i(\mathcal{D}_\bullet) \otimes Z_j(\mathcal{E}_\bullet)$$

by defining it on pure tensors $a \otimes b \in \mathcal{D}_i \otimes \mathcal{E}_j$ by $s_i(a) \otimes t_j(b) \in Z_i(\mathcal{D}_\bullet) \otimes Z_j$. Obviously the composite map

$$\bigoplus_{i+j=n} Z_i(\mathcal{D}_\bullet) \otimes Z_j(\mathcal{E}_\bullet) \to Z_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet) \subset (\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet)_n \to \bigoplus_{i+j=n} Z_i(\mathcal{D}_\bullet) \otimes Z_j(\mathcal{E}_\bullet)$$

is the identity. We claim that the image of $B_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet) \subset Z_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet)$ lands inside

$$\bigoplus_{i+j=n} \left(B_i(\mathcal{D}_\bullet) \otimes Z_j(\mathcal{E}_\bullet) + Z_i(\mathcal{D}_\bullet) \otimes B_j(\mathcal{E}_\bullet)\right).$$

It is enough to check this for pure tensors $a \otimes b \in \mathcal{D}_{i+1} \otimes \mathcal{E}_j$. This is clear as

$$\begin{aligned} u_n(d(a \otimes b)) &= u_n(d(a) \otimes b) + (-1)^{i+1} u_n(a \otimes d(b)) \\ &= d(a) \otimes t_j(b) + (-1)^{i+1} s_{i+1}(a) \otimes d(b), \end{aligned}$$

$d(a) \otimes t_j(b) \in B_i(\mathcal{D}_\bullet) \otimes Z_j(\mathcal{E}_\bullet)$ and $s_{i+1}(a) \otimes d(b) \in Z_{i+1}(\mathcal{D}_\bullet) \otimes B_{j-1}(\mathcal{E}_\bullet)$. Thus, it follows that we get a map

$$H_n(\mathcal{D}_\bullet \otimes \mathcal{E}_\bullet) \to \bigoplus_{i+j=n} H_i(\mathcal{D}_\bullet) \otimes H_j(\mathcal{E}_\bullet)$$

which is a splitting for the short exact sequence (17.3.3). This completes the proof of the Theorem. □

Chapter 18

CW complexes

In this chapter we introduce a class of spaces central to algebraic topology, called CW complexes. For CW complexes we shall define cellular homology and show that this coincides with singular homology. Thus, this will give a different way to compute the homology groups of spaces homotopy equivalent to a CW complex.

18.1 CW complexes

A CW complex is a space X which is inductively built out of cells as follows. For $k \geqslant 1$, a k-cell is defined to be the space $D^k := \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$. By ∂D^k we shall mean the boundary of D^k, that is, $D^k \setminus \mathrm{Int}(D^k)$. Thus, for $k \geqslant 2$ we have $\partial D^k = S^{k-1}$ and ∂D^1 is a discrete set consisting of two points. A 0-cell is just a point and $\partial D^0 = \emptyset$. With these definitions we build X as follows.

(1) Start with a non-empty discrete set of points X^0. This is called the 0-skeleton of X

(2) Let $n \geqslant 1$ and assume that we have constructed X^{n-1}.

(3) Suppose we have a collection of n-cells D^n_α indexed by α and maps $f_\alpha : \partial D^n_\alpha \to X^{n-1}$. Define $X^n := (X^{n-1} \coprod_\alpha D^n_\alpha)/\sim$, where $x \in \partial D^n_\alpha$ is identified with $f_\alpha(x) \in X^{n-1}$. Note that the map $X^{n-1} \to X^n$ is an inclusion. The maps f_α are called the ***attaching maps***. We get a chain of inclusions $X^0 \subset X^1 \subset X^2 \subset \ldots$ where X^i is a closed subset of X^{i+1}.

(4) Define $X = \cup_{n \geqslant 0} X^n$ and give it the weak topology, that is, a set A is defined to be closed iff $A \cap X^n$ is closed for each n.

The subspace X^n is called the n-skeleton of X. Thus, it is a closed subspace of X. If X is connected then it is clear that X^n/X^{n-1} is a wedge of n-spheres where each sphere corresponds to an n-cell. By an open n-cell in X^n we shall mean the following. Note that

$\mathrm{Int}(D^n_\alpha) \to X^n$ is an inclusion and is an open subset. We shall refer to this as an open n-cell in X^n and denote it by e_α.

18.2 Examples of CW complexes

(1) It is clear that a point is a CW complex.

(2) The sphere S^n is a CW complex as follows. Take X^0 to be a single point. There are no k-cells for $1 \leqslant k \leqslant n-1$. Thus, $X^0 = X^1 = \ldots = X^{n-1}$. Take one n-cell and take the attaching map $f : S^{n-1} = \partial D^n \to X^0$ to be the constant map.

(3) $\mathbb{P}^1_{\mathbb{R}}$ is S^1 and this has a CW structure by the previous example.

(4) Now consider the case $n \geqslant 2$. We view $\mathbb{P}^n_{\mathbb{R}}$ as the open upper hemisphere of S^n along with the identification by the antipodal map on the equatorial S^{n-1}. From this it is clear that we take $X^{n-1} = \mathbb{P}^{n-1}_{\mathbb{R}}$, take one n-cell and attach the boundary to X^{n-1} using the usual quotient map $S^{n-1} \to \mathbb{P}^{n-1}_{\mathbb{R}}$. Thus, for each $0 \leqslant k \leqslant n$ there is exactly one k-cell.

(5) $\mathbb{P}^1_{\mathbb{C}}$ is S^2 and this has a CW structure as shown above.

(6) There is a surjective map $D^{2k} \to \mathbb{P}^k_{\mathbb{C}}$ given as follows,

$$f(y_1, \ldots, y_{2k}) = [y_1 + iy_2 : \ldots : y_{2k-1} + iy_{2k} : 1 - \|y\|] \, .$$

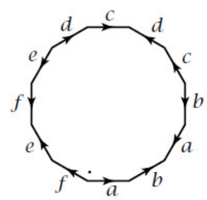
It is clear that the sphere $\partial D^{2k} = S^{2k-1}$ maps to $\mathbb{P}^{k-1}_{\mathbb{C}}$ sitting as the hyperplane $x_k = 0$. One checks easily that f is a bijective homeomorphism from $D^{2k} \setminus S^{2k-1} \to \mathbb{P}^k_{\mathbb{C}} \setminus \mathbb{P}^{k-1}_{\mathbb{C}}$. In view of this, we get a bijective continuous map

$$X := (D^{2k} \coprod \mathbb{P}^{k-1}_{\mathbb{C}})/\sim \,\to \mathbb{P}^k_{\mathbb{C}} \, .$$

The equivalence on the left is given by, for $x \in S^{2k-1}$ identify $x \sim f(x) \in \mathbb{P}^{k-1}_{\mathbb{C}}$. Thus, if we can show that X is Hausdorff, then it will also be compact and this map will be a homeomorphism. That X is Hausdorff is left as an exercise.

This description shows that we can inductively give $\mathbb{P}^k_{\mathbb{C}}$ a CW structure as follows. Assume that we have given $\mathbb{P}^{k-1}_{\mathbb{C}}$ a CW structure. Take one $2k$-cell and use the attaching map $S^{2k-1} \to X^{k-1}$. Then we get that $X^k = \mathbb{P}^k_{\mathbb{C}}$. Thus, cells exist only in even dimension and for each $0 \leqslant k \leqslant n$ there is exactly one cell of dimension $2k$.

(7) For compact orientable surfaces of genus g, we have the following well known representation.



We leave it to the reader to use this representation and give this surface the structure of a CW complex.

In all the above examples, for each n there are only finitely many n-cells. However, in general the number of n-cells is allowed to be infinite.

18.3 Cellular Homology

Throughout we will assume that X is connected. It is easy to check that this happens iff the 1-skeleton X^1 is connected. The reader will also easily check that connected components and path components for a CW complex are the same. Moreover, the reader will correctly guess and prove that when X has more than one path component then the algebraic objects we associate to X are simply direct sums of the algebraic objects we associate to each connected component of X.

For a CW-complex X we define a chain complex as follows. For $n \geqslant 1$ define

$$CW_n(X) := H_n(X^n/X^{n-1}),$$

and define $CW_0(X) = H_0(X^0)$. If $n \geqslant 1$ then since X^n/X^{n-1} is a wedge of S^n's, it follows that $CW_n(X)$ is free abelian of rank equal to the number of n-cells. It is clear that the same is true when $n = 0$.

Next we define maps $d_{n+1} : CW_{n+1}(X) \to CW_n(X)$. First consider the case $n \geqslant 1$. The long exact sequence for the pair $(X^{n+1}/X^{n-1}, X^n/X^{n-1})$ gives

$$\begin{array}{c} CW_{n+1}(X) = H_{n+1}(X^{n+1}/X^n) \\ \downarrow \sim \\ H_{n+1}(X^{n+1}/X^{n-1}, X^n/X^{n-1}) \xrightarrow{\delta_{n+1}} H_n(X^n/X^{n-1}) = CW_n(X) \end{array}$$

In the above we have used Corollary 16.7.6. Define $d_{n+1} : CW_{n+1}(X) \to CW_n(X)$ by

$$d_{n+1} := \delta_{n+1}\,.$$

Next we define $d_1 : CW_1(X) \to CW_0(X)$. In the same way as above, using Corollary 16.7.6, the long exact sequence for the pair (X^1, X^0) gives

$$CW_1(X) = H_1(X^1/X^0) \xrightarrow{\delta_1} H_0(X^0) = CW_0(X)\,.$$

Define $d_1 = \delta_1$. Define $d_0 = 0$.

Next let us show that the d_i's define a chain complex. We first describe a way to compute the differential $d_{n+1}(e)$ in terms of the n-cells. Let $n \geqslant 1$. Let $e \in CW_{n+1}(X)$ be a generator corresponding to an $(n+1)$-cell. We have a map of pairs $(D^{n+1}, S^n) \to (X^{n+1}/X^{n-1}, X^n/X^{n-1})$, where D^{n+1} is the $(n+1)$-cell corresponding to e. Let q denote the map $S^n \to X^n \to X^n/X^{n-1}$ which is the attaching map followed by the quotient map. We get the following commutative diagram from the long exact sequences attached to the above map of pairs.

(18.3.1)
$$\begin{tikzcd} H_{n+1}(D^{n+1}, S^n) \arrow[r, "\sim"] \arrow[d] & H_n(S^n) \arrow[d, "q_*"] \\ H_{n+1}(X^{n+1}/X^n) \arrow[r, "\delta_{n+1}"] & H_n(X^n/X^{n-1}) \end{tikzcd}$$

If $X^n/X^{n-1} = \vee_\alpha S^n_\alpha$ then for the map $q_* : \mathbb{Z} \to \bigoplus_\alpha \mathbb{Z}_\alpha$, each component can be found by considering the composition $S^n \to \vee_\alpha S^n_\alpha \to S^n_\alpha$. In the second map all the spheres other than the one corresponding to α are smashed to a point. This induces a map $\mathbb{Z} \to \mathbb{Z}_\alpha$ and that map is precisely the projection of q_* to the α'th coordinate. This gives a topological description of the differentials.

Lemma 18.3.2. $d_n \circ d_{n+1} = 0$

Proof. This is clear when $n = 0$ since $d_0 = 0$. First assume that $n \geqslant 2$. It suffices to show that $d_n \circ d_{n+1} = 0$ on the generators. Thus, from the preceding discussion, it suffices to show that $d_n(q_*(1)) = 0$. Consider the map j which is the composite $S^n \to X^n \to X^n/X^{n-2}$. Then q is the composite $S^n \xrightarrow{j} X^n/X^{n-2} \xrightarrow{\pi} X^n/X^{n-1}$. Thus, we have a commutative diagram where the bottom row is from the long exact sequence of pairs $(X^n/X^{n-2}, X^{n-1}/X^{n-2})$ and using Corollary 16.7.6

$$\begin{tikzcd} & H_n(S^n) \arrow[dl, "j_*"'] \arrow[d, "q_*"] & \\ H_n(X^n/X^{n-2}) \arrow[r, "\pi_*"] & H_n(X^n/X^{n-1}) \arrow[r, "\delta_n"] & H_{n-1}(X^{n-1}/X^{n-2}) \end{tikzcd}$$

From the long exact sequence of the pair $(X^n/X^{n-2}, X^{n-1}/X^{n-2})$ we have $\delta_n \circ \pi_* = 0$. The lemma follows when $n \geqslant 2$.

Next consider the case $n = 1$. It suffices to show that $d_1 \circ d_2 = 0$ on the generators. Thus, from the preceding discussion, it suffices to show that $d_1(q_*(1)) = 0$. Then q is the composite $S^1 \xrightarrow{j} X^1 \xrightarrow{\pi} X^1/X^0$. Thus, we have a commutative diagram where the bottom row is from the long exact sequence of pairs (X^1, X^0) and using Corollary 16.7.6

$$\begin{tikzcd}
 & H_1(S^1) \arrow[dl, "j_*"'] \arrow[d, "q_*"] & \\
H_1(X^1) \arrow[r, "\pi_*"] & H_1(X^1/X^0) \arrow[r, "\delta_1"] & H_0(X^0)
\end{tikzcd}$$

From the long exact sequence of the pair (X^1, X^0) we have $\delta_1 \circ \pi_* = 0$. The lemma follows when $n = 1$. □

This constructs a chain complex and the homology groups of this complex are denoted $H_n^{CW}(X)$.

Definition 18.3.3. *Let X and Y be CW complexes. A map $f : X \to Y$ is said to be cellular if $f(X^n) \subset Y^n$.*

Using the idea in equation (18.3.1) we easily check that a cellular map f induces a map of complexes $f_* : CW_\bullet(X) \to CW_\bullet(Y)$ and so induces maps $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$. This is left as an exercise to the reader, see Exercise 18.5.6.

18.4 Comparing cellular and singular homology

We begin with some preliminaries. We first want to show that a compact set $C \subset X$ in a CW complex is contained in a union of finitely many cells. We first prove the following easy lemma.

Lemma 18.4.1. *Let X be a CW complex. Let $C \subset X^n$ be a compact set. Then C meets only finitely many open n-cells in X^n.*

Proof. On the contrary, let us assume that C meets infinitely many open n-cells. If $e_\alpha \subset X^n$ is one such open n-cell, let x_α be a point in $C \cap e_\alpha$. Then the set $\{x_\alpha\}$ is a subset of C which has the discrete topology. Since C is compact, it follows that this set has finite cardinality, a contradiction. □

Corollary 18.4.2. *Let $C \subset X$ be a compact set. Then for all n the set $C \cap X^n$ is contained in a finite union of cells.*

Proof. Let us assume that this is not the case. Choose the smallest n for which $C \cap X^n$ is not contained in a finite union of cells. This means that $C \cap X^n$ meets infinitely many open n-cells in X^n, which contradicts the previous lemma. □

Proposition 18.4.3. *Let X be a CW complex. Let $C \subset X$ be a compact set. Then C is contained in a finite union of cells.*

Proof. Let us assume that this is not the case. This would mean that for infinitely many n the set $C \cap X^n$ meets some open n-cells in X^n. For each n, and each open n-cell e_α in X^n choose a point $x_\alpha \in C \cap e_\alpha$, whenever this intersection is non-empty. Let S denote the collection of these x_α. For each n, the set $S \cap X^n$ has finite cardinality, from the previous result. Thus, the same is true for every subset $T \subset S$. This shows that every such T is closed in X and so closed in S. That is, every subspace of S is closed and so S has the discrete topology. Since $S \subset C$ which is compact, this forces that S is finite, a contradiction. $\square$

Lemma 18.4.4. *Let X be a CW complex. Then*

(1) $H_r(X^n) = 0$ *for* $r > n$.

(2) $H_r(X^n) \xrightarrow{\sim} H_r(X^{n+1})$ *is an isomorphism if* $1 \leqslant r < n$.

(3) $H_n(X^{n+1}) \xrightarrow{\sim} H_n(X)$ *is an isomorphism for all* n.

Proof. We shall prove (1) by induction on n. This is clearly true when $n = 0$. Assume this is true for $n \geqslant 0$ and consider the long exact sequence for the pair (X^{n+1}, X^n). We get

$$H_r(X^n) \to H_r(X^{n+1}) \to H_r(X^{n+1}, X^n)\,.$$

Since $r > n+1$, it follows that $r \geqslant 1$. It is clear that there is a neighborhood of X^n in X^{n+1} which deformation retracts onto X^n. Thus, applying Corollary 16.7.6 we get that $H_r(X^{n+1}, X^n) \cong H_r(X^{n+1}/X^n)$. The space X^{n+1}/X^n is either a point of a wedge of spheres S^{n+1}. Since $r > n+1$ it follows thiat this homology group is 0. Thus, it follows that $H_r(X^{n+1}) = 0$. This proves (1).

Let us prove (2). If $1 \leqslant r < n$ then, as above, we have

$$\begin{aligned} H_{r+1}(X^{n+1}/X^n) \cong H_{r+1}(X^{n+1}, X^n) \to H_r(X^n) \to H_r(X^{n+1}) \\ \to H_r(X^{n+1}, X^n) \cong H_r(X^{n+1}/X^n)\,. \end{aligned}$$

This proves (2) since the ends are 0.

Let us prove (3). We leave it as an exercise to the reader to check that the path components of X are in bijective correspondence with the path components in X^1. From this the assertion in (3) follows for $n = 0$. So let us assume that $n \geqslant 1$. In view of Proposition 18.4.3 it follows that there is an isomorphism

$$\varinjlim_m H_n(X^m) \xrightarrow{\sim} H_n(X)\,.$$

By part (2) since $H_n(X^m) \xrightarrow{\sim} H_n(X^{m+1})$ is an isomorphism for $1 \leqslant n < m$, it follows that $H_n(X^{n+1}) \xrightarrow{\sim} \varinjlim_m H_n(X^m) \xrightarrow{\sim} H_n(X)$. Thus, (3) is proved. $\square$

Theorem 18.4.5. *Let X be a CW complex. There are isomorphisms $H_n^{CW}(X) \cong H_n(X)$. If $f : X \to Y$ is a cellular map then we have a commutative diagram*

$$\begin{tikzcd} H_n^{CW}(X) \arrow[r, "\sim"] \arrow[d, "f_*"'] & H_n(X) \arrow[d, "f_*"] \\ H_n^{CW}(Y) \arrow[r, "\sim"] & H_n(Y) \end{tikzcd}$$

The left vertical arrow is defined using Exercise 18.5.6.

Proof. It suffices to prove the theorem when X is connected.

For $n = 0$ the CW homology $H_0^{CW}(X)$ is the cokernel of the map $H_1(X^1, X^0) \to H_0(X^0)$. This fits in the exact sequence

$$H_1(X^1, X^0) \to H_0(X^0) \to H_0(X^1)$$

Since X is connected it follows that X^1 is connected and the map $H_0(X^0) \to H_0(X^1)$ is surjective. Thus, in this case we get that $H_0^{CW}(X) \xrightarrow{\sim} H_0(X)$.

Now consider the case when $n = 1$. Consider the following diagram

$$\begin{tikzcd} & H_1(X^1/X^0) = CW_1(X) \arrow[d, "a"', "\sim"] & \\ H_1(X^1) \arrow[r, "b"] & H_1(X^1, X^0) \arrow[r, "c"] & H_0(X^0) = CW_0(X) \end{tikzcd}$$

The bottom row is from the long exact sequence of the pair (X^1, X^0). The arrow a is an isomorphism using Corollary 16.7.6. By definition of $d_1 : CW_1(X) \to CW_0(X)$ we have that $d_1 = c \circ a$. Thus, $\mathrm{Ker}(d_1) = \mathrm{Im}(b)$. Consider the following commutative diagram for $n \geqslant 2$. The middle row is from the long exact sequence of the pair (X^n, X^{n-1}). The bottom row is from the long exact sequence of the pair $(X^n/X^{n-2}, X^{n-1}/X^{n-2})$.

$$\begin{tikzcd} & H_n(X^n/X^{n-1}) = CW_n(X) \arrow[d, "a"', "\sim"] & \\ H_n(X^n) \arrow[r, "b"] & H_n(X^n, X^{n-1}) \arrow[r, "c"] \arrow[d, "d"', "\sim"] & H_{n-1}(X^{n-1}) \arrow[d, hook, "e"'] \\ & H_n(X^n/X^{n-2}, X^{n-1}/X^{n-2}) \arrow[r, "f"] & H_{n-1}(X^{n-1}/X^{n-2}) = CW_{n-1}(X) \end{tikzcd}$$

The arrows a and d are isomorphisms using Corollary 16.7.6. The arrow e is an inclusion follows from the long exact of the pair (X^{n-1}, X^{n-2}) and Lemma 18.4.4 (1). By definition the map $d_n = f \circ d \circ a$. If we put $n = 2$ then it is easy to see that $H_1^{CW}(X)$ is the cokernel of the map c. But from the long exact sequence of the pair (X^2, X^1),

$$H_2(X^2) \to H_2(X^2, X^1) \to H_1(X^1) \to H_1(X^2) \to H_1(X^2/X^1) = 0,$$

we get that $H_1^{CW}(X) \cong H_1(X^2)$. Now applying Lemma 18.4.4 (3) we see that the theorem is true for $n = 1$.

The general case proceeds in the same way and we get that $H_n^{CW}(X)$ is isomorphic to cokernel of the map

$$H_{n+1}(X^{n+1}, X^n) \to H_n(X^n),$$

which by same considerations as above is isomorphic to $H_n(X^{n+1})$. Again applying Lemma 18.4.4 (3) we see that the theorem is true for all n.

Let $f : X \to Y$ be a map of CW complexes. On the one hand we saw that f induces a map $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$. On the other hand it also induces a map on singular homology groups. These induced maps are the same modulo the isomorphism of cellular homology with singular homology. In other words the above isomorphisms and maps f_* fit into a commutative diagram

(18.4.6)
$$\begin{tikzcd} H_n^{CW}(X) \arrow[r, "\sim"] \arrow[d, "f_*"'] & H_n(X) \arrow[d, "f_*"] \\ H_n^{CW}(Y) \arrow[r, "\sim"] & H_n(Y) \end{tikzcd}$$

Again this check follows by combining the above proof with equation (18.3.1). We leave this check to the reader. □

18.5 Exercises

18.5.1. Let X be a CW complex. Show that the path components of X are in bijective correspondence with the path components of X^1. Show that the path components and the components of X are the same.

18.5.2. Let X be a CW complex and let $Z \subset X^n$ be a subspace. Show that the topology on Z induced from X^n is that the same as the topology on Z induced from X.

18.5.3. In Example 6 show that X is Hausdorff.

18.5.4. Use CW homology to compute the homology groups of all examples in section 18.2.

18.5.5. Let X and Y be CW complexes and let $Z = X \bigsqcup Y$. Show that $H_n^{CW}(Z) \cong H_n^{CW}(X) \oplus H_n^{CW}(Y)$.

18.5.6. Let X and Y be CW complexes and let $f : X \to Y$ be a cellular map. Using the idea in equation (18.3.1) check that a cellular map f induces a map of complexes $f_* : CW_\bullet(X) \to CW_\bullet(Y)$ and so induces maps $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$.

18.5.7. Let the setup be as in Theorem 18.4.5. Prove the commutativity of (18.4.6).

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