MA-414 (Galois Theory)
Tutorial-1

January 31, 2023

1. Show that any field contains either $\mathbb{Q}$ or $\mathbb{Z}/(p)$, for some prime number p, as the smallest subfield. Show that the characteristic of a finite field is a prime number.

2. For any integer n, show that $\sqrt{n}$ is algebraic over $\mathbb{Q}$.

3. For any prime number p, show that the polynomial $x^2 - p$ has no solution in $\mathbb{Q}$.

4. For any two prime numbers p and q, show that $\sqrt{p} + \sqrt{q}$ is algebraic over $\mathbb{Q}$.

5. If $\alpha \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, show that for any integer $n \geq 1$, the number α^n is transcendental over $\mathbb{Q}$.

6. Let $f : [0, 1] \to \mathbb{R}$ be a non-constant continuous function. Show that the set $f([0, 1]) := \{f(a) : a \in [0, 1]\}$ contains a number transcendental over $\mathbb{Q}$.

7. Let $\alpha \in \mathbb{C}$ be a root of the equation $X^2 + 2 = 0$. Show that
$$\mathbb{Q}[\alpha] := \{a + \alpha b : a, b \in \mathbb{Q}\}$$
is a field.

8. Show that the field described in Section 1.2 of the notes is the smallest subfield of K containing all the α_i.

For the next three questions, let F be a field and $f(t) \in F[t]$ be a nonzero polynomial. Show the following:

9. For any $a \in F$, there is a polynomial $g(t) \in F[t]$ such that
$$f(t) = g(t)(t - a) + f(a).$$

10. For $a \in F$, we have $f(a) = 0$ if and only if $t - a$ divides $f(t)$ in $F[t]$.

11. The polynomial $f(t)$ can have at most $\deg(f(t))$ number of roots.

12. Let E, K be fields and let $\phi : E \to K$ be a ring homomorphism. Show that ϕ has to injective. Recall that our rings will always contain 0,1 and $1 \neq 0$ and a ring homomorphism always sends 1 to 1.

13. Let R be an integral domain and let K be its field of fractions. Let F be a field. Show that there is a bijective correspondence between the following two sets: $\{$Injective ring homomorphisms $\phi : R \to F\}$ and $\{$Field homomorphisms $\psi : K \to F\}$.