MA-414 (Galois Theory)

Tutorial-2

January 31, 2023

Notation: If F is a subfield of a field E, then the field extension $F \subseteq E$ is usually denoted by E/F.

1. If F is a subfield of a field E, show that E is a vector space over F. Conclude that $[E : F] = 1$ if and only if $E = F$.
2. Let F be a finite field. Show that the number of elements of F is of the form p^n, for some prime number p and some integer $n \geq 1$. (Hint: Use the above exercise and the fact that F has a subfield isomorphic to $\mathbb{Z}/(p)$, for some prime number p. (recall Ex.1 from Tutorial 1))
3. Find all roots of the polynomial $f(t) = x^p - 1 \in F[t]$, where $F = \mathbb{Z}/(p)$, for some prime number p.
4. Show that $[\mathbb{C} : \mathbb{R}] = 2$.
5. Let F be a field of characteristic 0. Let $p > 0$ be a prime number. Show that any ring homomorphism $f : \mathbb{Z}/p\mathbb{Z} \to F$ must be a zero map.
6. Let E/F be a field extension and $\alpha \in E$ is algebraic over F. Show that for any polynomial $f(t) \in F[t]$, $f(\alpha) \in E$ is algebraic over F.
7. Consider the subfield $F(X^2) \subset F(X)$. Is X algebraic over $F(X^2)$?
8. Find the degree of the field extension $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$.
9. Let p, q be distinct positive primes. Show that the polynomial $X^2 - p$ does not have a root in $\mathbb{Q}[\sqrt{q}]$.
10. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Find a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$. (Hint: Compute the inverse of $\sqrt{2} + \sqrt{3}$.)
11. Show that the degree of the field extension $\mathbb{C}/\mathbb{Q}(\sqrt{-1})$ is infinite.