MA-414 (Galois Theory)
Tutorial-3

March 15, 2023

Notation: If F is a subfield of a field E, then the field extension $F \subseteq E$ is usually denoted by E/F.

1. Let E be a field. Show that $E(X)(Y)$ is isomorphic to $E(S,T)$.
2. Find the degree of the field extension $\mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q}$.
3. If α is transcendental over $\mathbb{Q}$, show that $\alpha + \sqrt{2} \in \mathbb{C}$ is transcendental over $\mathbb{Q}$.
4. Let E/F be a field extension such that $[E : F] = 5$. Show that there is no proper intermediate field for the extension $F \subset E$.
5. Let F be any field and let α be a root of some polynomial $f(t) \in F[t]$. If $[F(\alpha) : F]$ is odd, then show that $F(\alpha) = F(\alpha^2)$. (Hint: Consider the tower of field extensions $F \subseteq F(\alpha^2) \subseteq F(\alpha)$.)
6. Let E/F be a field extension of degree 3. If $\alpha \in E$ such that $\alpha \notin F$, show that $E = F(\alpha)$.
7. Prove that any algebraically closed field is infinite. (Hint: If F is a finite field, then look at the **polynomial** $f(t) = 1 + \prod_{\alpha \in F} (t - \alpha) \in F[t]$.)
8. Let F be a field and $f(t) \in F[t]$ a polynomial such that $f(\alpha) = 0$ for all $\alpha \in F$. Is it true that $f(t) = 0$ in $F[t]$? (Hint: Take $F = \mathbb{Z}/(2)$ and $f(t) = t^2 + t \in F[t]$.)
9. Determine the monic irreducible polynomial of $1 + i$, $2 + \sqrt{3}$, $1 + \sqrt[3]{2} + \sqrt[3]{4}$.
10. Find the degrees of the field extensions $\mathbb{Q}(\sqrt{2 + \sqrt{3}})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})/\mathbb{Q}$.
11. Let F be a field of characteristic $\neq 2$. Let $\alpha, \beta \in F$ such that both $X^2 - \alpha$ and $X^2 - \beta$ has no roots in F. Show that
    (a) $[F(\sqrt{\alpha}, \sqrt{\beta}) : F] = 4$ if $X^2 - \alpha\beta$ has no root in F, and
    (b) $[F(\sqrt{\alpha}, \sqrt{\beta}) : F] = 2$, otherwise.