MA-414 (Galois Theory)
Tutorial-4

March 15, 2023

Notation: If F is a subfield of a field E, then the field extension $F \subseteq E$ is usually denoted by E/F.

1. Let E/F be a field extension. Let K, L be subfields of E containing F. If $[K : F] = p$, for some prime number $p\,(> 0\,)$, show that either $K \cap L = F$ or $K \subset L$.

2. Let F be an algebraically closed field. Let R be an integral domain containing F. If R is a finite dimensional vector space over F, show that $R = F$.

3. Let E/F be an algebraic extension. If F is algebraically closed, show that $E = F$.

4. Let $\omega = e^{2\pi i/3}$. Find all field homomorphisms from $\mathbb{Q}[\sqrt[3]{2}, \omega\sqrt[3]{2}] \to \bar{\mathbb{Q}}$.

5. Let $f(X) = X^2 + 2X - 1$ and $g(X) = X^2 - 2$. Show that $\mathbb{Q}[X]/(f(X))$ is isomorphic to $\mathbb{Q}[X]/(g(X))$.

6. Let $p(X) \in \mathbb{Q}[X]$ be a non-zero irreducible polynomial. Let $\alpha, \beta \in \mathbb{C}$ be two distinct numbers such that $p(\alpha) = 0 = p(\beta)$. Show that $\mathbb{Q}(\alpha) \cong \mathbb{Q}(\beta)$.

7. Let F be a field and $F(X)$ the field of fractions of the polynomial ring $F[X]$. Show that $F(X)$ is not algebraically closed.

8. Let $f(X) = X^4 - X^2 - 6 \in \mathbb{Q}[X]$. Let $E \subseteq \mathbb{C}$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and all roots of $f(X)$. Find the degree of the field extension $E/\mathbb{Q}$.

9. Let $f(X) = X^p - 1 \in \mathbb{Q}[X]$, where $p\,(>\ 0)$ is a prime number. Let $E \subseteq \mathbb{C}$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and all roots of $f(X)$. Show that $[E : \mathbb{Q}] = p - 1$.

10. Let F be a field, and $f(X) \in F[X]$ a polynomial of degree n. Fix an algebraic closure $\overline{F}$ of F. Let $E \subseteq \overline{F}$ be the smallest subfield containing F and all roots of $f(X)$. Show that $[E : F] \leq n!$.

11. Fix an algebraic closure $\overline{F}$ of a field F. Let $f(X) \in F[X]$. For $a, b \in F \setminus \{0\}$, let $g(X) = f(aX + b) \in F[X]$. Let E_f (respectively, E_g) be the smallest subfield of $\overline{F}$ containing F and all roots of $f(X)$ (respectively, $g(X)$). Show that $E_f = E_g$.

12. Let $E \subset K$ be an algebraic extension. Let R be a ring such that $E \subset R \subset K$. Show that R is a field.

13. Show that $[\bar{\mathbb{Q}} : \mathbb{Q}] = \infty$.

14. Show that $[L : E]_s \leqslant [L : E]$.

15. Let $\omega = e^{2\pi i/3}$ and let $\beta = \omega \sqrt[3]{2}$. Let $K = \mathbb{Q}(\beta)$. Show that the equation $x_1^2 + \ldots + x_7^2 = -1$ has no solution with $x_i \in K$.

16. Let a be a positive rational number that is not a square in $\mathbb{Q}$. Prove that $\sqrt[4]{a}$ has degree 4 over $\mathbb{Q}$.

17. Let $\alpha \in \bar{\mathbb{Q}}$ be such that for all automorphisms $\sigma : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$ we have $\sigma(\alpha) = \alpha$. Show that $\alpha \in \mathbb{Q}$.

18. Let $f(X) \in F[X]$ be an irreducible polynomial of degree n, and let E be a field extension of F with $[E : F] = m$. If $\gcd(m, n) = 1$, show that $f(X)$ is irreducible over E.

19. Let E be a field. Let $\alpha, \beta \in \bar{E}$ be such that $[E[\alpha] : E] = n$ and $[E[\beta] : E] = m$ are coprime. Show that $[E[\alpha, \beta] : E] = nm$.

20. Let $\zeta := e^{2\pi i/5}$. Let $\phi : \mathbb{Q}(\sqrt[5]{2}, \zeta) \to \bar{\mathbb{Q}}$ be a homomorphism. Show that the image is contained in $\mathbb{Q}(\sqrt[5]{2}, \zeta)$.

21. Let p be a primes. Let $\Phi_p(X) = 1 + X + \ldots + X^{p-1} \in \mathbb{Q}[X]$. Let $0 < i < p$. Show that $\Phi_p(X)$ divides $\Phi_p(X^i)$.