MA-414 (Galois Theory)
Tutorial-5

March 15, 2023

Notation and definitions: If F is a subfield of a field E, then the field extension $F \subseteq E$ is usually denoted by E/F. For any polynomial $f(X) \in F[X]$, we denote by $f'(X)$ its formal derivative $D_F(f(X)) \in F[X]$. Two non-zero non-unit polynomials $f(X), g(X) \in F[X]$ are said to be **relatively prime** if they have no common divisor in $F[X]$ except for units of $F[X]$. A field F is said to be **perfect** if all algebraic field extensions of F are separable over F.

1. Let F be a field of characteristic $p > 0$. Show that for any integer $n \geq 0$, $F^{p^n} := \{a^{p^n} \mid a \in F\}$ is a subfield of F.

2. Let $E = \mathbb{F}_p(X)$. Show that the polynomial $T^p - X \in E[T]$ is irreducible. Let E_1 denote the field $E[T]/(T^p - X)$. Let $\bar{E}$ denote an algebraic closure of E and fix an embedding $\phi : E \subset \bar{E}$. Compute cardinality of the set $\mathrm{Hom}_\phi(E_1, \bar{E})$.

3. Let F be a field of characteristic $p > 0$. Let $\alpha \in F$. Show that the polynomial $T^p - \alpha \in F[T]$ is irreducible iff it does not have a root in F.

4. Let F be a field of characteristic $p > 0$. Let α be an element in F which does not have a p^{th} root in F.

   (a) Show that the polynomial $T^p - \alpha \in F[T]$ is irreducible.

   (b) Let α_1 be a root of this polynomial in $\bar{F}$. Consider the field $F_1 := F[\alpha_1] \subset \bar{F}$. Show that $(F_1)^p \subset F$.

   (c) Show that the polynomial $T^p - \alpha_1 \in F_1[T]$ is irreducible.

5. Use exercises 3 and 4 to show that there is an algebraic extension L/E with $[L : E] = \infty$ and $[L : E]_s = 1$.

6. Let F be a finite field of characteristic $p > 0$. Show that $F = F^p$.

7. Let $F \subset K \subset E$ be tower of field extensions. Let $\alpha \in E$ be separable over F. Show that α is separable over K.

8. Let F be a perfect field. Let $F \subset E$ be an algebraic extension. Show E is perfect.

9. Let $F := \mathbb{Z}/(2)$, and let $\alpha \in \overline{F}$ be a root of the polynomial $f(t) = t^4 + t^2 + 1 \in F[t]$. Determine if α is separable over F or not.

10. Let E/F be a field extension. Let $f(X) \in F[X]$ and $\alpha \in E$. Show that $(X-\alpha)^2 \mid f(X)$ if and only if $(X-\alpha) \mid f(X)$ and $(X-\alpha) \mid D_F(f(X))$.

11. Show that two polynomials in $F[X]$ are relatively prime iff they are relatively prime in $\bar{F}[X]$.

12. Let F be a field and $f(X) \in F[X]$. Show that $f(X)$ has no repeated roots if and only if $f(X)$ and $D_F(f(X))$ are relatively prime in $\bar{F}[X]$. Shows that $f(X)$ has repeated roots iff $f(X)$ and $D_F(f(X))$ are not relatively prime.

13. Let F be a field of characteristic $p > 0$. Show that $X \in F(X)$ is inseparable over $F(X^p)$.

14. Let F be a field of characteristic $p > 0$. Show that F is perfect if and only if $F = F^p$.

15. Let F be a field of characteristic $p > 0$. Show that $\bigcap_{n=0}^{\infty} F^{p^n}$ is the largest perfect subfield of F.

16. Let E/F be a finite degree field extension and $\text{char}(F) = p > 0$. If p does not divide $[E : F]$, show that E/F is separable.

17. Let F be a field of characteristic $p > 0$, and let $f(X) = X^p - a \in F[X]$. Show that either $f(X)$ is irreducible in $F[X]$ or $f(X) = (X-\alpha)^p$, for some $\alpha \in F$.

18. Let F be a field of characteristic $p > 0$. Let $f(X) \in F[X]$ be an irreducible polynomial. Show that there exists an irreducible separable polynomial $g(X) \in F[X]$ and an integer $n \geq 0$ such that $f(X) = g(X^{p^n})$. Show that all roots of $f(X)$ have the same multiplicity p^n.

19. Let F be a field of characteristic $p > 0$. Let $\alpha \in \overline{F(X)}$ be a root of $p(Y) := Y^{2p} + XY^p + X \in F(X)[Y]$. Show that α is inseparable over $F(X)$. (Hint: Use Eisenstein's irreducibility criterion to conclude that $p(Y)$ is irreducible.)

20. Let F be a field of characteristic $p > 0$. Consider the field $E := F(X, Y)$ and its extensions $E_1 := E[T]/(T^p - X)$ and $E_2 := E_1[S]/(S^p - Y)$. In particular, show that both E_1 and E_2 are fields. Show that $[E_2 : E] = p^2$. Show that for any $\alpha \in E_2$, one has $\alpha^p \in E$. Use this to conclude that there is no $\alpha \in E_2$ such that $E_2 = E[\alpha]$.

21. Let F be a field of characteristic 0, and $f(X) \in F[X]$. Let
$$h(X) := \gcd(f(X), f'(X)) .$$
Show that both $g(X) := f(X)/h(X)$ and $f(X)$ have the same roots and $g(X)$ has no repeated root.

22. Let F be a field of characteristic $p > 0$. Let α be a root of $X^p - a \in F[X]$. Let $E = F(\alpha)$. Find the separable degree and inseparable degree of E over F.

23. Let $p > 0$ be a prime number. Let $F = \mathbb{F}_p(X, Y)$ be the field of fractions of the polynomial ring $\mathbb{F}_p[X, Y]$. Let $E = F(\sqrt[p]{X}, \sqrt[p]{Y})$. Find the separable degree and inseparable degree of E over F.