Ordinary Differential Equations

Homework 1

Important

- *Write your solutions neatly and submit it on* **23 August** *(tutorials). Late submission will not be allowed.*
- *Simplify all your answers as much as possible and express answers in terms of fractions or constants such as* $\sqrt{e}$ *or* $\ln(4)$ *rather than decimals.*
- *Show all your work and explain your reasonings clearly! Copying will not be tolerated.*

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1. Find the general solution of the ODEs:
   (a) $\dfrac{dx}{dt} = te^t$.
   (b) $\dfrac{dx}{dt} = \dfrac{2x^4 + t^4}{tx^3}$.
   (c) $\dfrac{dx}{dt} = \begin{cases} -t\sqrt{x} & \text{for } x \geq 0, \\ t\sqrt{-x} & \text{for } x \leq 0. \end{cases}$

2. Solve the following initial value problems:
$$\frac{dx}{dt} + a(t)x = t, \text{ with } x(0) = 1,$$
   where $a(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 2, \\ 3 & \text{for } t > 2. \end{cases}$

   Is the solution differentiable in $(0, +\infty)$.

3. Suppose $x(t)$ satisfies the ODE:
$$\dot{x}(t) = \alpha(t)x(t) + \beta(t),$$
   where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $\alpha(t) \leq -c < 0$ for some $c > 0$, and $\lim_{|t| \to +\infty} \beta(t) = 0$. Find $\lim_{t \to \infty} x(t)$.

4. Suppose $x(t)$ satisfies the ODE:
$$\frac{dx}{dt} - \frac{x}{2} = 2\cos t.$$
   Find the initial values $x(0) = x_0$ such that:
   (a) $x(t) > 0$ as $t \to +\infty$.
   (b) $x(t) < 0$ as $t \to +\infty$.

5. Solve the Bernoulli type equation
$$tx^2 \frac{dx}{dt} + x^3 = t \cos t.$$

6. Determine the constant a so that the following equations are exact, and then solve the resulting ODEs.

   (a) $(2xt^3 + e^{at+x})\dfrac{dx}{dt} + e^{at+x} + 3t^2x^2 = 0.$

   (b) $\left(\dfrac{at+1}{x^3}\right)\dfrac{dx}{dt} + \dfrac{1}{x^2} + \dfrac{1}{t^2} = 0.$

7. Finding an integrating factor, or otherwise, solve the following ODEs:

   (a) $(x^2 + t^2)\dfrac{dx}{dt} + 3t^2x + 2tx + x^3 = 0.$

   (b) $(e^t \cos x + 2\cos t)\dfrac{dx}{dt} + e^t \sin x - 2x \sin t = 0$, with $x(0) = \dfrac{\pi}{2}$.

   (c) $(te^{tx}\cos(2t) - 3)\dfrac{dx}{dt} + xe^{tx}\cos(2t) - 2e^{tx}\sin(2t) + 2t = 0$, with $x(0) = 0$.
