Ordinary Differential Equations

Homework 2

Important

- *Write your solutions neatly and submit it on* **13 September** *(tutorials). Late submission will not be allowed.*
- *Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\sqrt{e}$ or $\ln(4)$ rather than decimals.*
- *Show all your work and explain your reasonings clearly! Copying will not be tolerated.*

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1. (a) Show that the initial value problem: $\begin{cases} \dot{x} = x^2 + \cos t^2 \\ x(0) = 0 \end{cases}$, has a unique solution in some interval $-\delta \leq t \leq \delta$.

   (b) Show that the initial value problem: $\begin{cases} \dot{x} = \left(4x + e^{-t^2}\right) e^{2x} \\ x(0) = 0 \end{cases}$, has a unique solution in some interval $-\delta \leq t \leq \delta$.

2. Suppose $f \in \mathbb{C}^1(\mathbb{R})$, that is, $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. Show that the initial value problem: $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$, has at most one solution in some interval $-\delta \leq t \leq \delta$.

3. (a) Let α, β be real valued continuous function on $\mathbb{R}$. Show that the initial value problem:

$$\begin{cases} \dot{x} = \alpha(t) \sin x + \beta(t) \cos x \\ x(0) = x_0 \in \mathbb{R}, \end{cases}$$

   has a unique solution for all $t \in \mathbb{R}$.

   (b) Show that the initial value problem:

$$\begin{cases} \dot{x} = \dfrac{\sin^2 t}{1 + t^2} e^{-x^2} \\ x(0) = 1, \end{cases}$$

   has a unique solution for all $t \in \mathbb{R}$.

4. Consider the initial value problem:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0, \end{cases}$$

   (a) For $f(x) = \begin{cases} x \log\left(\dfrac{1}{x}\right) & \text{if } 0 < x < 1 \\ 0 & \text{for } x = 0. \end{cases}$ and $x_0 = 1/2$, show that the above initial value problem has at most one solution.

   (b) For $f(x) = 3x^{2/3}$ and $x_0 = 0$, show that above initial value problem has a solution. (This shows that the Lipschitz condition is not necessary for existence of a solution).

(c) For $f(x) = 1 + x^{2/3}$ and $x_0 = 0$, show that above initial value problem has at most one solution. (This shows that the Lipschitz condition is not necessary for uniqueness of solutions).

5. Consider the ordinary differential equation: $\dot{x} = -x^2$.

   (a) For the initial condition $x(1) = 1$, show that the above ODE has a unique solution for t in $(0, +\infty)$.

   (b) For the initial condition $x(0) = 0$, show that the above ODE a unique solution for $-\infty < t < +\infty$.
