Ordinary Differential Equations

Mid-Semester Exam

Total Points: 30 **Time: 90 minutes**

Important

1. *Simplify your answers as much as possible and write them in terms of fractions or constants such as $\sqrt{e}$ or $\log 4$ rather than decimals.*
2. *Show all your work and explain your reasonings clearly.*
3. *No books, notes, calculators or any papers are allowed. Additional paper is available upon request.*
4. *You are not allowed to communicate with other students during the exam. Copying will not be tolerated.*

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1. (6 points) Find the *general solution* of the following ODEs.

   (a) $(e^{2t} - t^2 \sin x)\dfrac{dx}{dt} + 2xe^{2t} + 2t\cos x = 0.$ (3 points)

   (b) $\dfrac{dx}{dt} = x \log\left(\dfrac{1}{x}\right).$ (3 points)

2. (6 points) Solve the following *initial value problems*.

   (a) $\dfrac{dx}{dt} = \dfrac{t}{1+t}x - 1$, with $x(0) = 0$. (3 points)

   (b) $\dfrac{dx}{dt} = (x - \sin t)^8 + \cos t$, with $x(0) = 0$. (3 points)

3. (3 points) *Evaluate* $\lim\limits_{t\to\infty} x(t)$, where $x(t)$ satisfies: $\dot{x}(t) = (\cos t - 2)x(t) - (1+t^6)^{-1}$.

4. (5 points) Construct *three distinct solutions* of the the initial value problem: $\begin{cases} \dot{x} = |x|^{3/4}, \\ x(0) = 0. \end{cases}$

5. (3 points) Show that the initial value problem: $\begin{cases} \dot{x} = \dfrac{3e^{-t^2}}{1+x^4} - \dfrac{x \sin x}{2+\text{[illegible]}}, \\ x(0) = 0. \end{cases}$ has a *unique solution defined for all* $t \in \mathbb{R}$.

6. (4 points) Suppose $o(t)$ satisfies the initial value problem: $\begin{cases} \dot{x} = (1 - \text{[illegible]})\sin(1 - x^2), \\ x(0) = 0 \end{cases}$

   Show that $o(t)$ is *defined for all* $t \in \mathbb{R}$ and $|o(t)| \leq 1$ for all $t \in \mathbb{R}$.

7. (3 points) Suppose $x(t)$ satisfies the differential equation: $2\dot{x} = 2\cos\text{[illegible]} - 3x$.
   Show that

$$x^2(t) \leq x^2(0)e^{5t} \quad \text{for all } t \geq 0.$$

1.
(a) $$\left(e^{2t} - t^2 \sin x\right)\frac{dx}{dt} + 2xe^{2t} + 2t\cos x = 0$$

Let $P = e^{2t} - t^2 \sin x$

$Q = 2xe^{2t} + 2t\cos x$

1

$$\frac{\partial P}{\partial t} = 2e^{2t} - 2t\sin x$$

and $\dfrac{\partial Q}{\partial x} = 2e^{2t} - 2t\sin x$

since $\dfrac{\partial P}{\partial t} = \dfrac{\partial Q}{\partial x}$ in $\mathbb{R} \times \mathbb{R}$, the given ODE is an exact one in $\mathbb{R} \times \mathbb{R}$.

So ~~Suppose~~ $\exists$ a function $F(x,t) \in C^2(\mathbb{R} \times \mathbb{R})$ such that

$$\frac{\partial F}{\partial x} = P \quad \text{and} \quad \frac{\partial F}{\partial t} = Q.$$

Now $\dfrac{\partial F}{\partial x} = P$ gives $F(x,t) = xe^{2t} + t^2\cos x + \varphi_1(t)$; φ_1 being

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a f^n of t on $\mathbb{R}$ and $\varphi_1 \in C^1(\mathbb{R})$

and $\dfrac{\partial F}{\partial t}$ gives, $\dfrac{\partial F}{\partial t} = 2xe^{2t} + 2t\cos x + \varphi_1'(t)$

$= Q$

$= 2xe^{2t} + 2t\cos x$

This implies that $\varphi_1'(t) = 0$

i.e. $\varphi_1(t) = \text{constant} = C$, say

Hence the required sol^n to the given ODE is,

$$\boxed{F(x,t) = xe^{2t} + t^2\cos x = \text{constant}.}$$

1.
(b) $\dfrac{dx}{dt} = x\log\left(\dfrac{1}{x}\right) = -x\log x = f(x)\,g(t)$; where $f(x) = -x\log x$
$g(t) = 1$.

The given ODE is of separable form.

Let f, g be continuous f^{n}s defined on intervals $\tilde{I}$ and I
1 respectively such that $|f(y)| > 0 \;\forall\, y \in \tilde{I}$.

Let φ be a solution of the given ODE.

Then
$$\varphi(t) = \Psi^{-1}\left(\int_{t_0}^{t} g(s)\,ds + c\right) \text{ with } \Psi(u) = \int_{u_0}^{u} \frac{1}{f(y)}\,dy$$

for some constant c
and $t_0 \in I$, $u_0 \in \tilde{I}$.
Also Ψ^{-1} exists !!

Now $\Psi(u) = -\displaystyle\int_{u_0}^{u} \frac{1}{y\log y}\,dy = -\log\log u + \log\log u_0 = \log\frac{\log u_0}{\log u}$

Hence $\varphi(t) = \Psi^{-1}\left(\displaystyle\int_{t_0}^{t} ds + c\right)$

or, $\Psi(\varphi(t)) = (t - t_0) + c$

2 or, $\log\dfrac{\log u_0}{\log \varphi(t)} = (t - t_0) + c$

or, $\dfrac{k}{\log\varphi(t)} = c' e^{(t - t_0)}$

or, $\log\varphi(t) = c'' e^{-(t - t_0)}$

or, $\varphi(t) = e^{c'' e^{-(t - t_0)}}$.

2.(a)

Given IVP
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{t}{1+t}x + 1 = \alpha(t)x(t) + \beta(t) \\ x(0) = 0 \end{cases}$$

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$$\varphi(t) = e^{\Psi(t)}\left(\int_{t_0}^{t} e^{-\Psi(s)}\beta(s)\,ds + c\right)$$

where, $\Psi(t) = \int_{t_0}^{t} \alpha(y)\,dy$

$$\Psi(t) = \int_{t_0}^{t}\left(\frac{y}{y+1}\right)dy = \Big[y - \ln(y+1)\Big]_{y=t_0}^{t}$$

$$= (t - t_0) - \ln(t+1) + \ln(t_0+1)$$

$$e^{-\Psi(s)} = e^{-(s-t_0)+\ln(s+1) - \ln(t_0+1)}$$

$$= e^{-s}(s+1) \quad ; \text{as } t_0 = 0 \text{ in this case}$$

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and $e^{\Psi(t)} = e^{t}(t+1)^{-1}$

Hence
$$\varphi(t) = e^{t}(1+t)^{-1}\left(\int_{0}^{t} e^{-s}(s+1)\,ds + c\right)$$

$$= e^{t}(1+t)^{-1}\left[\Big\{-(s+1)e^{-s} - e^{-s}\Big\}_{s=0}^{t} + c\right]$$

$$= e^{t}(1+t)^{-1}\left[-e^{-t}(1+t) - e^{-t} + 2 + c\right]$$

$$= \frac{e^{t}}{(1+t)}\left(e^{-t}(-2-t)+2\right) \quad \left(c=0 \text{ using } \varphi(0)=0\right)$$

$$= -\frac{(2+t)}{(1+t)} + \frac{2e^{t}}{1+t}$$

2. (b) Given IVP $\begin{cases} \frac{dx}{dt} = (x - \sin t)^8 + \cos t \\ x(0) = 0 \end{cases}$

put $y = x - \sin t$

$$\frac{dy}{dt} = \frac{dx}{dt} - \cos t$$

substituting these things, we've our IVP,

$$\frac{1}{2} \quad \begin{cases} \frac{dy}{dt} = y^8 \\ y(0) = 0 \end{cases} \quad \cdots \circledast$$

$\circledast$ is of separable form.

$$\varphi(t) = \Psi^{-1}\left(\int_{t_0}^{t} 1\, ds + C\right) \quad ; \quad \Psi(u) = \int_{u_0}^{u} \frac{1}{y^8}\, dy$$

1

$$= \frac{1}{7}\left(u_0^{-7} - u^{-7}\right)$$

$$\Psi(\varphi(t)) = t - t_0 + C$$

$$\Rightarrow -\frac{1}{7}\left(\varphi(t)\right)^{-7} = t + \tilde{C}$$

$$\Rightarrow \varphi(t)^{-7} = -7t - 7\tilde{C}$$

$$\Rightarrow \varphi(t) = \left(-7(t + \tilde{C})\right)^{-\frac{1}{7}}$$

$\varphi(0) = 0$ gives $\left.(\tilde{C} + t)^{-1/7}\right|_{t=0} = 0$

1.5

$$\Rightarrow (\tilde{C})^{-\frac{1}{7}} = 0$$

which is impossible.

Hence $\varphi(t) \equiv 0$ is the only possible solution of $\circledast$.
That is $x(t) = \sin t$, $t \in \mathbb{R}$ is the only solution of the given IVP.

3.

$x(t)$ satisfy

$$\dot{x}(t) = (\cos t - 2)x(t) + (1+t^6)^{-1} \quad \dots \circledast$$
$$= \alpha(t)\, x(t) + \beta(t)$$

; where $\alpha(t) = \cos t - 2$, $t \in \mathbb{R}$

$\beta(t) = \dfrac{1}{(1+t^6)}$, $t \in \mathbb{R}$

let $\varphi(t)$ be a sol^n of $\circledast$.

Then
$$\varphi(t) = e^{\psi(t)} \left(\int_{t_0}^{t} e^{-\psi(s)} \beta(s)\, ds + c \right)$$

, where $\psi(t) = \int_{t_0}^{t} \alpha(s)\, ds$

$$= \int_{t_0}^{t} (\cos s - 2)\, ds$$

$$= \sin t - 2t - \sin t_0 + 2t_0$$

$$\cancel{\psi(\varphi(t))} = \varphi(t) = e^{(\sin t - 2t - c_1)} \left(\int_{t_0}^{t} e^{2s - \sin s} \frac{\tilde{c}}{(1+s^6)} + c \right)$$

$$= \frac{e^{\sin t}}{e^{c_1 + 2t}} \left(\int_{t_0}^{t} e^{2s - \sin s} \frac{\tilde{c}\, ds}{1+s^6} \right) + \frac{c\, e^{\sin t}}{e^{2t} \cdot e^{c_1}}$$

$$= \frac{\tilde{c} \int_{t_0}^{t} \frac{e^{2s - \sin s}}{1+s^6} ds}{c_2\, e^{2t - \sin t}} + \frac{c_3}{e^{2t - \sin t}}$$

Now since $|\sin t| \leq 1 \ \forall\, t \in \mathbb{R}$, hence

$$\lim_{t \to \infty} \frac{c_3}{e^{2t - \sin t}} = 0$$

Now for evaluating

$$\lim_{t\to\infty} \frac{\tilde{c}\int_{t_0}^{t} \frac{e^{2s-\sin s}}{1+s^6}\,ds}{c_2\, e^{2t-\sin t}},$$

we'll consider two cases.

Case-1:

Let $\lim_{t\to\infty} \tilde{c}\int_{t_0}^{t} \frac{e^{2s-\sin s}}{1+s^6}\,ds = M < \infty$

Then clearly $\lim_{t\to\infty} \frac{\tilde{c}\int_{t_0}^{t} \frac{e^{2s-\sin s}}{1+s^6}\,ds}{c_2\, e^{2t-\sin t}} = 0$

Case-2:

Let $\lim_{t\to\infty} \tilde{c}\int_{t_0}^{t} \frac{e^{2s-\sin s}}{1+s^6}\,ds = \infty$

Then using L Hospital's rule ($\frac{\infty}{\infty}$ form here), we have

$$\lim_{t\to\infty} \frac{\tilde{c}\int_{t_0}^{t} \frac{e^{2s-\sin s}}{1+s^6}\,ds}{c_2\, e^{2t-\sin t}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t\to\infty} \frac{\tilde{c}\,\frac{e^{2t-\sin t}}{1+t^6}}{c_2\, e^{2t-\sin t}\cdot(2-\cos t)}$$

$$= \lim_{t\to\infty} \frac{\tilde{c}}{c_2\,(1+t^6)\,(2-\cos t)}$$

Since $1 \le (2-\cos t) \le 3 \quad \forall\, t \in \mathbb{R}$, hence

$$\lim_{t\to\infty} \frac{\tilde{c}}{c_2\,(1+t^6)(2-\cos t)} = 0.$$

Hence we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} x(t) = 0$.

Given IVP $\begin{cases} \dot{x} = |x|^{3/4} \\ x(0) = 0 \end{cases}$

$x > 0$ gives $\dfrac{dx}{dt} = x^{3/4}$ (separable form)

$= f(x)\, g(t)$; $f(x) = x^{3/4}$
$g(t) = 1$

$$\varphi(t) = \psi^{-1}\left(\int_{t_0}^{t} g(s)\, ds + C \right) \quad , \quad \psi(u) = \int_{u_0}^{u} \frac{1}{f(y)}\, dy$$

$$\psi(u) = \int_{u_0}^{u} y^{-3/4}\, dy = 4u^{1/4} - 4u_0^{1/4}$$

$$\psi(\varphi(t)) = t - t_0 + C$$

$$\Rightarrow 4\,\varphi(t)^{1/4} = t + \tilde{C} \qquad ; \quad \tilde{C} = 4u_0^{1/4} - t_0 + C$$

$$\Rightarrow \varphi(t) = \left(\frac{t + \tilde{C}}{4} \right)^{4}$$

For $x < 0$, $\dfrac{dx}{dt} = (x)^{3/4}$

put $y = -x$.

Then $-\dfrac{dy}{dt} = y^{3/4}$

or, $\dfrac{dy}{dt} = y^{3/4}(-1) = \tilde{f}(y)\, \tilde{g}(t)$

$$\tilde{\psi}(u) = \int_{u_0}^{u} y^{3/4}\, dy = 4u^{1/4} - 4u_0^{1/4}$$

$$\tilde{\psi}(\tilde{\varphi}(u)) = -t + t_0 + \tilde{C}' \Rightarrow +4\,\varphi(t)^{1/4} = -t - \tilde{C}' + t_0 + 4u_0^{1/4} = -t + \tilde{\tilde{C}}$$

$$\Rightarrow \tilde{\varphi}(t) = +\left(\frac{-t + \tilde{\tilde{C}}}{4} \right)^{4}$$

General solution:

$$\tilde{\tilde{\varphi}}(t) = \begin{cases} \left(\dfrac{-t + \tilde{\tilde{C}}}{4} \right)^{4} & ; \ t < \tilde{\tilde{C}} \\ 0 & ; \ \tilde{\tilde{C}} \le t \le -\tilde{C} \\ \left(\dfrac{t + \tilde{C}}{4} \right)^{4} & ; \ t > -\tilde{C} \end{cases}$$

for $-\tilde{C} > 0$ and $\tilde{\tilde{C}} < 0$

5. The given IVP $\begin{cases} \dot{x} = \dfrac{3e^{-t^2}}{1+x^4} + \dfrac{x\sin x}{2+t^2} \\ x(0) = 0 \end{cases}$

Let $f(t,x) = \dfrac{3e^{-t^2}}{1+x^4} + \dfrac{x\sin x}{2+t^2}$

Then $|f(t,x)| \le 3e^{-t^2} + \frac{1}{2}|x|$, which is of the form

$|f(t,x)| \le M_T + K_T |x| \quad \forall (t,x) \in [-T,T] \times \mathbb{R}^n$.

Again, to show $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is locally Lipschitz in the 2nd variable,

i.e. $|f(t,x_1) - f(t,x_2)| \le K|x_1 - x_2|$ for some constant K.

Now $\dfrac{\partial}{\partial x} f(t,x) = \dfrac{-3e^{-t^2} \cdot 4x^3}{(1+x^4)^2} + \dfrac{1}{2+t^2}\left(\sin x + x\cos x\right)$

$< \infty$ in a closed, bdd. rectangle around (t,x)

Hence $f(t,x)$ is locally Lipschitz in x.

So by the Global Existence theorem, the given IVP will have a global sol^n.

Uniqueness follows from the Gronwall's inequality.

6.

Let $\phi(t)$ satisfy the IVP

$$\begin{cases} \dot{x} = (1-x^2)\sin(1+x^2) & \cdots ① \\ x(0) = 0 \end{cases}$$

To show that $\phi(t)$ is defined for all $t \in \mathbb{R}$ and $|\phi(t)| \le 1 \ \forall t \in \mathbb{R}$

For that, we first prove the following result:

Considering the I.V.P. $\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \cdots \circledast$

, where f is locally Lipschitz w.r.to the 2nd variable, then any solution (if exists) on an interval $I \ni t_0$ will be unique.

Proof:

Let $\varphi(t), \psi(t)$ be two solutions of $\circledast$

Define the set $S := \{ t \in I : \varphi(t) = \psi(t) \} \subseteq I$.

As $t_0 \in S$, so S is a non-empty subset of I.

Our target is to show that $S = I$.

For that we'll show that S is both open and closed in I.

Then from the connectedness of I, it'll follow that $S = I$.

Since $S = (\varphi - \psi)^{-1}(\{0\})$, hence S is closed in I.

Now let $\tilde{t} \in S$.

Then $\varphi(\tilde{t}) = \psi(\tilde{t})$.

Considering the I.V.P. $\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(\tilde{t}) = \varphi(\tilde{t}) = \psi(\tilde{t}) \end{cases} \cdots (**)$

As $(**)$ satisfies all the hypotheses of Picard-Lindelöf theorem, hence $\exists \ \delta > 0$ such that $(**)$ has a unique solution in $(\tilde{t}-\delta, \tilde{t}+\delta) \subseteq I$.

~~But since~~ Since $\varphi(t), \psi(t)$ both solve $(**)$, so

$$\varphi(t) = \psi(t) \text{ on } (\tilde{t}-\delta, \tilde{t}+\delta).$$

$\therefore \ \tilde{t} \in (\tilde{t}-\delta, \tilde{t}+\delta) \subseteq S$

$\therefore$ S is open in I.

Consequently $S = I$.

Now let $\phi(t)$ be the solution of ① on the maximal open interval $I \ni 0$.

If possible, let $|\phi(t_0)| > 1$ for some $t_0 \in I$.

Then by Intermediate value theorem, $\exists\ t_1 \in (0, t_0) \subseteq I$ such that $\phi(t_1) \in \{-1, 1\}$

Without loss of generality, let $\phi(t_1) = 1$.

Now let's consider the I.V.P.

$$\begin{cases} \dot{x}(t) = (1-x^2)\sin(1+x^2) \\ x(t_1) = 1 \end{cases} \quad \cdots\ (***)$$

clearly $\psi(t) = 1$ is a solution of $(***)$ on I.

Again $\phi(t)$ is the unique solution of ① on I.

So $\phi(t) = \psi(t) = 1$ on I, which is a contradiction as $\phi(0) = 0$.

Hence we've $|\phi(t)| \leq 1$ on I.

Again since $\overline{I} \times [-1, 1]$ is compact in $\mathbb{R} \times \mathbb{R}$ and $(t, \phi(t))$ is bounded on $\overline{I} \times [-1, 1]$, hence the solution $\phi(t)$ of ① exists ~~globally~~ for all $t \in \mathbb{R}$.

7\. Let $x(t)$ satisfy the differential equation

$$2\dot{x} = (2\cos 2x + 3)x \quad \cdots \text{①}$$

Let $z(t) = x(t)^2$. Then $z(t) \geq 0 \ \forall t \in \mathbb{R}$

Then $\dot{z}(t) = 2x(t)\dot{x}(t)$

So ① becomes $\quad \dot{z} = (2\cos 2\sqrt{z} + 3)z$.

$$\cancel{\Rightarrow |\dot{z}| \leq \tfrac{5}{2}|z|} \Rightarrow \dot{z} \leq 5z$$

Then by Gronwall's inequality, we have

$$z(t) \leq \exp\left(\int_0^t 5\,ds\right) z(0) \qquad \text{for } t \in [0,T], T>0$$

i.e. $\quad z(t) \leq e^{5t} z(0)$

i.e. $\quad x(t)^2 \leq e^{5t} x(0)^2$