Introduction

An ordinary differential equation (ODE) is an equation relating an unknown function, derivatives of the unknown function and an independent variable of the form:

$$\mathcal{F}(t, X, X^{(1)}, \ldots, X^{(k)}) = 0. \tag{0.1}$$

Here $X := X(t) \in C^k(I, \mathbb{R}^n)$ (k-times continuously differentiable function from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$ or a curve) is the unknown function, $X^{(i)}(t) := \dfrac{d^i X}{dt^i}(t) \in \mathbb{R}^n$ denotes the i-th derivative, and $\mathcal{F} \in C(U, V)$ is a continuous function from U to $V \subset \mathbb{R}^m$ with U a domain (open and connected) in $\subset \mathbb{R}^{1+(k+1)n}$. The variable t (time) is often called the *independent variable* and $X = X(t) = (x_1(t), \ldots, x_n(t))$ the *dependent variable*. The highest derivative appearing in (0.1) is called the *order of the differential equation*. A solution of the ODE (0.1) is a function $\gamma \in C^k(\mathcal{I}, \mathbb{R}^n)$, where $\mathcal{I} \subset I$ an interval, such that $\gamma(t)$ satisfies (0.1) for all $t \in \mathcal{I}$.

We will mostly consider [1] ODEs of the form:

$$\frac{dX^k}{dt^k} = F(t, X, X^{(1)}, \ldots, X^{(k-1)}), \tag{0.2}$$

where $F \in C(U, V)$ is a continuous function from a domain $U \subset \mathbb{R}^{1+kn}$ to $V \subset \mathbb{R}^n$. Writing in coordinates $X = (x_1, \ldots, x_n)$ and $F = (f_1, \ldots, f_n)$ gives a *system of ODEs* of the form:

$$\frac{dx_i^k}{dt^k} = f_i(t, X_1, X^{(1)}, \ldots, X^{(k-1)}), \quad 1 \leq i \leq n,$$

A system can always be reduced to a first-order system by changing to the new set of dependent variables: $Y = (X, X^{(1)}, \ldots, X^{(k-1)}) \in \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ copies}}$, yielding a new first-order system:

$$\begin{cases} \dfrac{dY_i}{dt} = Y_{i+1} \quad \text{for } 1 \leq i \leq k-1, \\[2ex] \dfrac{dY_k}{dt} = F(t, Y) \end{cases} \qquad \text{or} \qquad \left\{ \frac{dY}{dt} = \mathcal{G}(t, Y). \right.$$

The system (0.2) is called *autonomous* if it does not depend on t. We can add t to the dependent variables $Z = (t, Y)$, making the right-hand side of (0.2) independent of t converting it into an autonomous system:

$$\begin{cases} \dfrac{dZ_1}{dt} = 1 \\[2ex] \dfrac{dZ_i}{dt} = Z_{i+1} \quad \text{for } 2 \leq i \leq k, \\[2ex] \dfrac{dZ_{k+1}}{dt} = F(Z). \end{cases} \qquad \text{or} \qquad \left\{ \frac{dZ}{dt} = \mathcal{H}(Z). \right.$$

[1] we can do this at least locally by solving for the highest derivative X^k in (0.1) using the implicit function theorem.