A. First Order ODEs

We consider the 1st order ODE: $\left\{\frac{dx}{dt} = f(t,x)\right.$, where $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a real valued continuous fn. defined on a "domain" (open and connected set) "Ω" in $\mathbb{R}^2$.

A fn. $\phi: I \to \mathbb{R}$, where I is an interval, is called a solution of the above ODE, if: ϕ is differentiable in I, $(t, \phi(t)) \in \Omega$ and $\phi'(t) = f(t, \phi(t))$ for all $t \in I$.

The solution $\phi(t)$ can be interpreted as a curve in the plane ($\mathbb{R}^2$) s.t its slope at a point $(t, \phi(t)) \in \mathbb{R}^2$ is given by $f(t, \phi(t))$. The solution can also be interpreted as a path or the trajectory of a particle s.t its velocity at time t is $f(t, \phi(t))$.

Finding Explicit Solutions

Example 1: Consider $\left\{\frac{dx}{dt} = f(t)\right.$, where $f: I \to \mathbb{R}$ is a given continuous fn. defined on an interval $I \subset \mathbb{R}$.

A fn. ϕ is a solution of the above ODE iff

$$\phi(t) = \int_{t_0}^{t} f(s)\,ds + C, \quad \text{for some const. } C \text{ and } t_0 \in I.$$

The solution ϕ of the above form is called a general solution. The constant C

Example 2: (1st order linear ODE with const. coefficients)

Consider $\left\{ \frac{dx}{dt} = a x(t) \right.$, with $a \equiv$ const.

A fn. ϕ is a sol. of the above ODE iff $\phi(t) = Ce^{at}$, for some const. $C \in \mathbb{R}$.

→ If $\phi(t) = Ce^{at}$, then $\phi'(t) = a(Ce^{at}) = a\phi(t)$, and hence ϕ solves the ODE.

Conversely, if ϕ is a sol, then : $\frac{d}{dt}\left(e^{-at}\phi(t)\right) = e^{-at}\left(\phi'(t) - a\phi(t)\right) = 0.$

Consequently, $e^{-at}\phi(t) = C$ (a const.) and therefore $\phi(t) = Ce^{at}$.

Example 3: (1st order linear ODE)

Consider $\left\{ \frac{dx}{dt} = \alpha(t)x(t) + \beta(t) \right.$, where $\alpha, \beta : I \to \mathbb{R}$ are continuous. A fn. ϕ is a sol. of the above ODE iff

$$\phi(t) = e^{\psi(t)}\left(\int_{t_0}^{t} e^{-\psi(s)}\beta(s)\,ds + C\right), \text{ for some const. } C$$

and $\psi(t) \equiv \int_{t_0}^{t} \alpha(s)\,ds$.

→ Suppose $\phi(t) = e^{\psi(t)}\left(\int_{t_0}^{t} e^{-\psi(s)}\beta(s)\,ds + C\right)$. Then by differentiating using the fundamental theorem of calculus and using $\psi'(t) = \alpha(t)$

$$\phi'(t) = \psi'(t)e^{\psi(t)}\left(\int_{t_0}^{t} e^{-\psi(s)}p(s)\,ds + C\right) + e^{\psi(t)}e^{-\psi(t)}p(t)$$

$$= \alpha(t)\,\phi(t) + p(t).$$

Conversely, suppose ϕ is a sol. Then

$$\frac{d}{dt}\left(e^{-\psi(t)}\phi(t)\right) = -\psi'(t)e^{-\psi(t)}\phi(t) + e^{-\psi(t)}\phi'(t)$$

$$= -e^{-\psi(t)}\left(\alpha(t)\,\phi(t) - \phi'(t)\right) = e^{-\psi(t)}p(t).$$

Integrating, $\displaystyle\int_{t_0}^{t}\left(e^{-\psi(s)}\phi(s)\right)'ds = \int_{t_0}^{t} e^{-\psi(s)}p(s)\,ds$, for some $t_0 \in I$.

so, $\displaystyle e^{-\psi(t)}\phi(t) - \underbrace{e^{-\psi(t_0)}\phi(t_0)}_{\text{Const.} \equiv C} = \int_{t_0}^{t} e^{-\psi(s)}p(s)\,ds$

hence $\displaystyle \phi(t) = e^{\psi(t)}\left(\int_{t_0}^{t} e^{-\psi(s)}p(s)\,ds + C\right).$

Example 4: (The method of Seperation of variables) We say that a ODE is separable if it can be written in the form: $\left\{ \dfrac{dx}{dt} = f(x)g(t), \right.$

where f and g are given functions.

Let's assume the f and g are continuous fns. defined on intervals $\tilde{I}$ and I respectively. Moreover, assume that $|f(\xi)| > 0 \ \forall \ \xi \in \tilde{I}$. A fn ϕ satisfies the above ODE iff

$$\phi(t) = \psi^{-1}\left(\int_{t_0}^{t} g(s)\,ds + C\right), \text{ with } \psi(u) := \int_{u_0}^{u} \frac{1}{f(y)}\,dy,$$

for some constant C and $t_0 \in I$, $u_0 \in \tilde{I}$.

Note: $\psi'(u) = f(u)^{-1} \neq 0$. Hence ψ is strictly monotone (increasing or decreasing) and has a diff'able inverse ψ^{-1}! Also $f(\phi(t)) \neq 0 \ \forall \ t$, and then by continuity $f(\phi(t))$ is either strictly positive or negative. Hence $\psi(\phi(t))$ is strictly monotone and ψ can be inverted along the solution (flow lines).

→ We have $\psi'(u) = f(u)^{-1}$! Assuming for ψ as defined above a fn. ϕ satisfies

$$\psi(\phi(t)) = \int_{t_0}^{t} g(s)\,ds + C,$$ we obtain by differentiating

$\psi'(\phi(t))\phi'(t) = g(t)$ or $f(\phi(t))^{-1}\phi'(t) = g(t)$. Hence $\left\{\phi'(t) = f(\phi(t))g(t)\right.$.

Conversely, suppose ϕ solves the given ODE. Then

$\frac{\phi'(t)}{f(t)} = g(t)$ or $\psi'(\phi(t))\phi'(t) = g(t)$. Integrating we obtain

$$\int_{t_0}^{t} \psi'(\phi(s))\phi(s)\,ds = \int_{t_0}^{t} g(t)\,ds.$$

A change of variable and the Fundamental theorem of calculus gives

$$\psi(\phi(t)) - \underbrace{\psi(\phi(t_0))}_{\equiv C} = \int_{t_0}^{t} g(s)\,ds, \text{ or } \psi(\phi(t)) = \int_{t_0}^{t} g(s)\,ds + C.$$

since ψ has a inverse, we obtain

$$\phi(t) = \psi^{-1}\left(\int_{t_0}^{t} g(s)\,ds + C \right).$$

Note: If $f(\xi_0) = 0$ for some $\xi_0 \in \tilde{I}$. Then $\phi(t) \equiv \xi_0$ is a sol. of the ODE: $\{ x'(t) = f(x) g(t)$. However, there maybe many other solutions.

Example 5: Consider the ODE $\left\{ \frac{dx}{dt} = x^3 \right.$.

→ We look for the general sol. of the above ODE. Using example 4, we have $\psi(u) = \int_{u_0}^{u} \frac{1}{y^3}\,dy = \frac{1}{2}\left(\frac{1}{u_0^2} - \frac{1}{u^2} \right)$

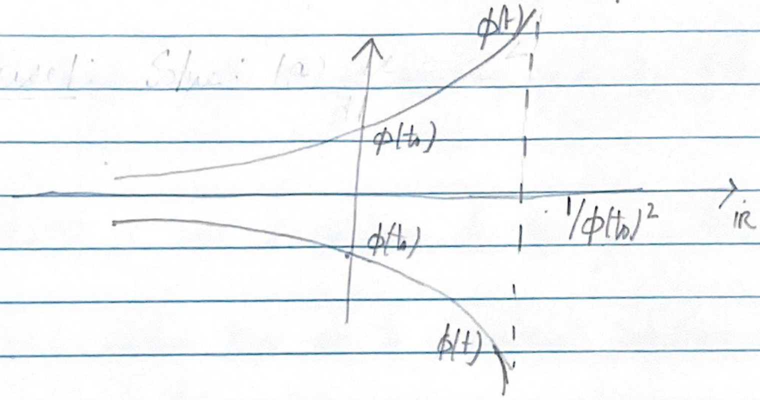
and hence the general sol. ϕ is given by

$$\psi(\phi(t)) = t + C, \Rightarrow \underbrace{\frac{1}{\phi(t_0)^2}}_{\text{const}} - \frac{1}{\phi(t)^2} = t$$

or, $\phi(t)^2 = \dfrac{\phi(t_0)^2}{1 - \phi(t_0)^2 t}$ This tells that the sol. is only defined till $t = \dfrac{1}{\phi(t_0)^2}$

and the sol. might blow up ($|\phi(t)| \to +\infty$ as $t \to t_0$).

Then, $\phi(t) = \dfrac{\phi(t_0)}{\sqrt{1-\phi(t_0)^2 t}}$. The sol. stays positive or negative, untill the time it blows up.



Exercise 1: Solve: (a) $\dfrac{dx}{dt} = x^2$; (b) $\dfrac{dx}{dt} = x(1-x)$; (c) $\dfrac{dx}{dt} = x(1-x) - c$

Example 6: Consider $\left\{ \dfrac{dx}{dt} = \sqrt{|x|} \right.$

$\rightarrow$ We look for a general sol. satisfying the above ODE.

By our example 4, $\psi(u) = \displaystyle\int_{u_0}^{u} \frac{1}{\sqrt{|y|}}\, dy = 2\left(\sqrt{|u|} - \sqrt{|u_0|}\right)$. And hence the general sol. is given by $2\left(\sqrt{|\phi(t)|} - \sqrt{|\phi(t_0)|}\right) = t$

or $\sqrt{|\phi(t)|} = \left(\dfrac{t}{2} + \sqrt{|\phi(t_0)|}\right)$. This is defined for $t > -2\sqrt{|\phi(t_0)|}$

Considering the positive branch of the sol, we have

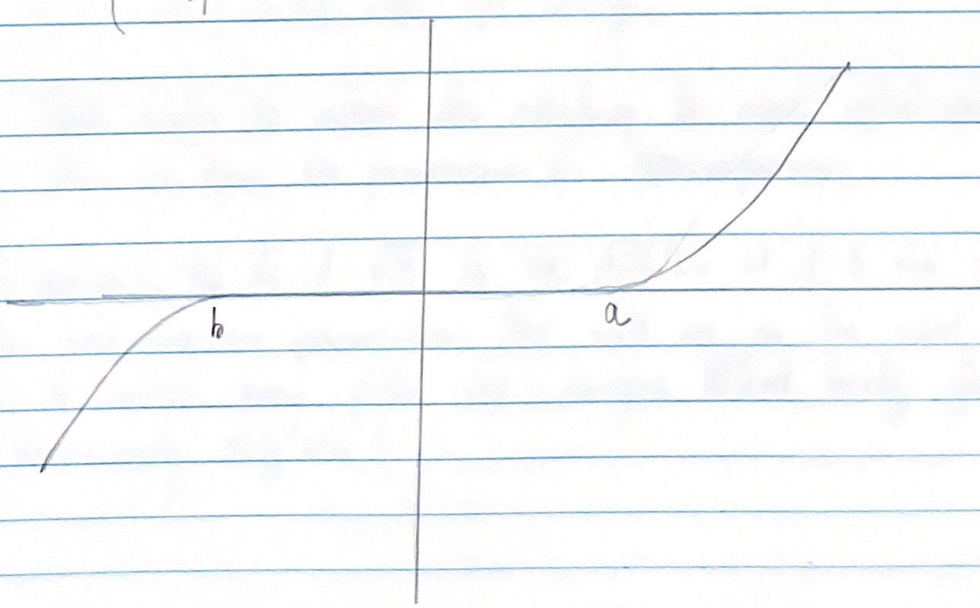
$$\phi(t) = \left(\frac{t + 2\sqrt{|\phi(t_0)|}}{2}\right)^2 \text{, which is defined for } t > -2\sqrt{|\phi(t_0)|}$$

The negative branch of the sol. is similarly given by $\phi(t) = -\left(\frac{t + 2\sqrt{|\phi(t_0)|}}{2}\right)^2$

<u>Note</u>: $\phi(t) \equiv 0$ is also a solution to the given ODE.

We can patch together these sol. to construct solutions that exists for all t.

$$\phi(t) = \begin{cases} \frac{(t-a)^2}{4} & \text{for } t \geq a \\ 0 & \text{for } b \leq t < a \\ -\frac{(t-b)^2}{4} & \text{for } t < b \end{cases}$$



Exercise 2: $\phi(t)$ defined as above is differentiable and satisfies the given ODE

Def: (Initial Value Problem)

ODEs where the value of the sol. is specified at some point. Often we will consider the case where the value of the sol. at $t=0$ (initial time) is specified to us:

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(0) = x_0 \end{cases}$$

We look for $\phi(t)$ satisfying the above ODE with $\phi(0) = x_0$.

→ The sol. might exists only for values of t in some neighbourhood to $t=0$ (locally in t), even for nice f.

→ There might be several sols satisfying the same initial value p'bm. Then we have the phenomenon of Non-uniqueness

In example 6, the fn. $f = \sqrt{|x|}$ is not diff'ble at $x=0$ and maybe causing the non-uniqueness phenomenon. We will see in the next chapter that sol to initial-value p'bms are unique at least locally for $f \in C^1(\mathbb{R})$ (continuously diff'able).

Example 7: $\frac{dx}{dt} = f(at+bx+c)$, $b \neq 0$

→ We look for a sol. of the form $u = at+bx+c$. Then $\frac{du}{dt} = a + b\frac{dx}{dt}$,

and we have a transformed eqn: $\frac{du}{dt} = a + bf(u)$

This is of the form of the ODE in example

Example 8: (Homogeneous ODE) $\frac{dx}{dt} = f\left(\frac{x}{t}\right)$

→ We let, $u = \frac{x}{t}$ $(t \neq 0)$ which transforms the given ODE into

$$u' = -\frac{x}{t^2} + \frac{x'}{t} = -\frac{u}{t} + \frac{1}{t}f(u)$$, therefore we get the ODE

$u' = \frac{f(u)-u}{t}$, which is a separable eqn.

Example 9: $\frac{dx}{dt} = f\left(\frac{at+bx+c}{\alpha t+\beta x+\gamma}\right)$

Case 1: If $a\beta = b\alpha$, then the ODE reduces to $\frac{dx}{dt} = g(\alpha t + \beta x)$

Case 2: If $\alpha\beta \neq b\alpha$, the system of linear eqns $\begin{cases} at+bx+c = 0 \\ \alpha t + \beta x + \gamma = 0 \end{cases}$

has a unique sol. (t_0, x_0). We use a change of coordinates as

$y = x - x_0$ and $s = t - t_0$.

So, the sol. in the new coordinate system: $y(s) = x(s) - x_0 = x(t-t_0) - x_0$

The given ODE transforms as: $y' = f\left(\dfrac{at+by}{\alpha t+\beta y}\right) = f\left(\dfrac{a+b\left(\frac{y}{t}\right)}{\alpha+\beta\left(\frac{y}{t}\right)}\right)$

or of the form $\left\{ y' = g\left(\frac{y}{t}\right)\right.$ as in example

Example 10: $\left\{ \dfrac{dx}{dt} = \dfrac{x+1}{t+2} - e^{\left(\frac{x+1}{t+2}\right)}\right.$

Using our previous example, the system of eqns $\begin{cases} x+1=0 \\ t+2=0 \end{cases}$ has a unique sol given by $(t,x) = (-2,-1)$. Then the change of variable

$\begin{cases} y = x+1 \\ s = t+2 \end{cases}$ transforms our ODE into $\left\{ \dfrac{dy}{ds} = \dfrac{y}{s} - e^{y/s}\right.$

We let $u = \dfrac{y}{s}$. Then the eqn transforms as $\dfrac{du}{ds} = -\dfrac{y}{s^2} + \dfrac{y'}{s}$

$= -\dfrac{u}{s} + \dfrac{u-e^u}{s}$, or $\left\{ \dfrac{du}{ds} = -\dfrac{e^u}{s}\right.$. Then u is given by

$e^{-u} = \log|s| + C$, or $e^{-u} = \log(C|s|)$. One then obtains

$u = \log\left(\log(C|s|)\right)^{-1}$, which is defined for $C|s| > 1$

transforming back to the original variables we get that the sol of the original ODE is given by

$(x+1) = (t+2)\log\left(\log(C|t+2|)\right)^{-1}$ or $\begin{cases} x = -1-(t+2)\log\left(\log(C|t+2|)^{-1}\right) \\ \text{for } C|t+2| > 1 \end{cases}$

Example 11: $\left\{ \dfrac{dx}{dt} = e^{x} \sin t \right.$

$\rightarrow$ This is a separable ODE, and the general sol. is given by

$e^{-x(t)} = \cos t + c$, or $x(t) = -\log(\cos t + c)$

which is defined for $\cos t + c > 0$.

Note: The sol. can have different behaviour, depending on the value of c. For $c > 1$, $\cos t + c > 0$ for all $t \in \mathbb{R}$, and hence the sol. exists $\forall\, t \in \mathbb{R}$. Also since $c - 1 \leq \cos t + c \leq c + 1$, the sol. is bounded for all $t \in \mathbb{R}$.

For $-1 < c \leq 1$, the sol. exists only in finite intervals and blows up (becomes $\pm\infty$).

In particular if $x(0) = -\log 2$, then $c = 1$ and

$x(t) = -\log(1 + \cos t)$, which is defined only for $-\pi < t < \pi$,

and $\lim\limits_{t \to \pm\pi} x(t) = +\infty$.

Example 12: (The Logistic eqn) $\left\{ \dfrac{dx}{dt} = x(b - cx) \right.$ with $b, c > 0$.

$\rightarrow$ This is a separable eqn and the sol. is given by

$$\phi(t) = \frac{b}{c} \frac{1}{1 + Ce^{-bt}} \quad \text{for } C \neq 0.$$

Also $\phi \equiv 0$, and $\phi \equiv \dfrac{b}{c}$ are sols. (stationary sols.)

Note: Every sol. ϕ of the above ODE with $\phi(0) > 0$ remains positive for all $t > 0$ and tends to $\frac{b}{c}$ as $t \to +\infty$.

Example 13 (Bernoulli equation)

$$\begin{cases} \frac{dx}{dt} = f(t)x + g(t)x^n \end{cases}, \quad n \neq 0, 1$$

→ We let $y = x^{1-n}$, then $y' = (1-n)x^{-n}x'$, and the given ODE is transformed to:

$$x^{-n}x' = f(t)x^{1-n} + g(t)$$

or $y' = (1-n)f(t)y + (1-n)g(t)$, which is a linear eqn.

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Example 14 (Riccati Equation.)

$$\left\{ \frac{dx}{dt} = f(t)x + g(t)x^2 + h(t) \right.$$

→ Solving this eqn is possible if a particular sol $\phi(t)$ is known.

Suppose $x(t)$ is a sol, then $u(t) = x(t) - \phi(t)$ satisfies

$$u'(t) = f(t)u(t) + g(t)u(t)(u + 2\phi) \text{ and so}$$

$$u'(t) = \left(f(t) + 2\phi(t)g(t)\right)u(t) + g(t)u^2(t)$$, which is a Bernoulli eqn.

Then letting $y(t) = \frac{1}{u(t)}$, we obtain a linear eqn

$$y' = -(f(t) + 2\phi(t)g(t))y - g(t)$$

Example 15 (Exact eqns.)

An ODE of the form $\left\{ P(x,y)\dfrac{dy}{dx} + Q(x,y) = 0 \right.$ is called exact (in Ω) if

$\exists$ a fn. $F(x,y) \in C^2(\Omega)$ s.t $\begin{cases} \partial_y F(x,y) = P(x,y) \\ \partial_x F(x,y) = Q(x,y) \end{cases}$ $\forall\ (x,y) \in \Omega$.

Ω denotes a domain (open, connected set) in $\mathbb{R}^n$.

Note that then $y = y(x)$ is a sol. of the above exact eqn, then

$$\partial_x F(x,y(x)) = \underbrace{\partial_x F(x,y(x))}_{Q(x,y(x))} + \underbrace{\partial_y F(x,y(x))}_{P(x,y(x))} \frac{dy}{dx} = 0$$

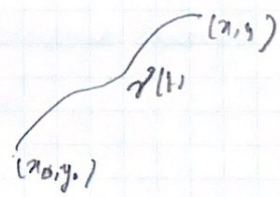
, and so the fn

F is const. along the sol. curve $y(x)$. Furthermore (by C^2

differentiability), $\underbrace{\partial_x P(x,y)}_{=\partial_{xy}F} = \underbrace{\partial_y Q(x,y)}_{\partial_{yx}F}$

Suppose now $y(x)$ satisfies $\left\{ P(x,y)\dfrac{dy}{dx} + Q(x,y) = 0 \right.$ with $\partial_x P(x,y) = \partial_y Q(x,y)$ $\forall\ (x,y)$ in Ω.

Consider the fn $F(x,y) := \displaystyle\int_\gamma \langle (Q,P), \gamma'(t) \rangle\, dt$ (line integration along the curve γ)



where γ is a curve (C^1)

joining a fixed point $(x_0,y_0) \in \Omega$.

to (x,y). F is well defined for <u>simply connected</u> domains Ω.

And we have $\begin{cases} \partial_x F(x,y) = Q(x,y) \\ \partial_y F(x,y) = P(x,y) \end{cases}$ in Ω.

Example 16 (Integrating Factors)

Consider $\begin{cases} P(x,y)\dfrac{dy}{dx} + Q(x,y) = 0. \end{cases}$

A fn. $\mu(x,y)$ is called an integrating factor if the eqn:

$$\mu(x,y)\,P(x,y)\frac{dy}{dx} + \mu(x,y)\,Q(x,y) = 0 \quad \text{is exact.}$$

$\rightarrow$ Then $\partial_x(\mu P)(x,y) = \partial_y(\mu Q)(x,y)$ in simply connected domain Ω

so, $P\,\partial_x\mu + \mu\,\partial_x P = Q\,\partial_y\mu + \mu\,\partial_y Q$ which a PDE

If now $\mu = \mu(x)$ (depends on only a single variable)

Then $\dfrac{d\mu}{dx} = \mu\dfrac{(\partial_y Q - \partial_x P)}{P}$. This is linear ODE for μ.

Example 17

Consider the ODE: $\begin{cases}(2x^2 + 2xy^2 + 1)y + (3y^2 + x)\dfrac{dy}{dx} = 0\end{cases}$

$\rightarrow$ Since $\underset{=1}{\partial_x(3y^2+x)} \neq \underset{= 2x^2 + 6xy^2 + 1}{\partial_y\left((2x^2+2xy^2+1)y\right)}$

We look for an integrating factor $\mu = \mu(x)$. Then by our previous example μ solves

$$\frac{d\mu}{dx} = \mu\left(\frac{2x^2 + 6xy^2}{3y^2 + x}\right) \Rightarrow \frac{d\mu}{dx} = (2x)\mu.$$

Hence $\mu = e^{x^2}$ is an integrating factor.

Multiplying the given ODE by the integrating factor

$$(3y^2 + x)e^{x^2}\frac{dy}{dx} + (2x^2 + 2xy^2 + 1)y\,e^{x^2} = 0.$$

Since the last eqn is exact, we are looking for $F(x,y)$ s.t

$$\begin{cases} \partial_x F = (2x^2+2xy^2+1)ye^{x^2}, \\ \partial_y F = (3y^2+x)e^{x^2} \end{cases}$$

Integrating $\{\partial_y F = (3y^2+x)e^{x^2}$ w.r.t y we get $F(x,y) = (y^3+xy)e^{x^2} + \phi(x)$

where ϕ needs to be found. Using $\{\partial_x F = (2x^2+2xy^2+1)ye^{x^2}$

we get $\phi(x) \equiv 0$. Thus, $F(x,y) = (y^3+xy)e^{x^2}$, and the sols.

of the given ODE $y=y(x)$ satisfies $\{ \; ye^{x^2}(x+y^2) = C$

which is an eqn for a curve in $\mathbb{R}^2$.

Qualitative Analysis

Phase Plane and Phase Portraits: Consider the system of two ODEs

$$\begin{cases} \frac{dx}{dt} = f(x,y) \; ; \; \frac{dy}{dt} = -g(x,y) \end{cases}$$

This is an autonomous system. Note that if $(x(t), y(t))$ solves the above ODE then also $(x(t+c), y(t+c))$ for any const. C.

A sol. $(x(t), y(t))$ of the above system of ODEs can be interpreted as a parametrization of the corresponding sol curve in the xy plane, which is also referred to as the phase plane. The solution curve are called the trajectories (or the orbits) of the ODEs. A sketch of several of these trajectories is called a phase portrait or phase diagram of the above system of ODEs. Arrows are added to the trajectories to give the orientation of the curve in the sense of increasing t. The phase portrait gives an overview of the qualitative behaviour of the solutions. Sometimes we can determine the trajectories with first determining the solutions, We need a fn. $F(x,y)$ which is const. on each trajectory (along each sol. curve), which are written implicitly as $F(x,y) = c$. The trajectories are then given as level sets $F^{-1}(c) := \{ (x,y) \in \mathbb{R}^2 : F(x,y) = c \}$

Example 1:
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$$

→ Along the sol. curve, $\left\{ y \frac{dy}{dt} + x \frac{dx}{dt} = 0 \right.$. Considering $y = y(x)$

this becomes $\left\{ y \frac{dy}{dx} + x = 0 \right.$. Integrating we obtain $y^2 + x^2 = C$

The sols. or trajectories are circles centered at the origin.

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Example 2
(Prey-Predator)

$$\begin{cases} \dfrac{dx}{dt} = x(A - By) \\ \\ \dfrac{dy}{dt} = y(-C + Dx) \end{cases}$$

where $A, B, C, D > 0$ are constants

$\to$ $(x(t), y(t)) \equiv (0,0)$ is a sol. Another stationary sol is given by $(x(t), y(t)) \equiv \left(\frac{C}{D}, \frac{A}{B}\right)$.

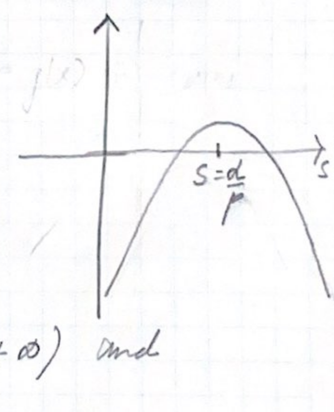
Let us try to eliminate t from the eqns to get a single 1st order eqn. Writing $y = y(x)$ we get

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{y(-C + Dx)}{x(A - By)}$$ which is a separable eqn

then, $\frac{dy}{dx} = \frac{y(-C+Dx)}{x(A-By)}$, and the orbits (sol.) are given by

$$F(x,y) := (C \log x - Dx) + (A \log y - By) \equiv C \text{ (const.)}$$

let $g(s) = \alpha \log s - \beta s$. We have $g'(s) = \frac{\alpha}{s} - \beta$,
$\alpha, \beta > 0$
$g''(s) = -\frac{\alpha}{s^2}$



$g(s)$ is increasing in $(0, \alpha/\beta)$, decreasing in $(\frac{\alpha}{\beta}, +\infty)$ and

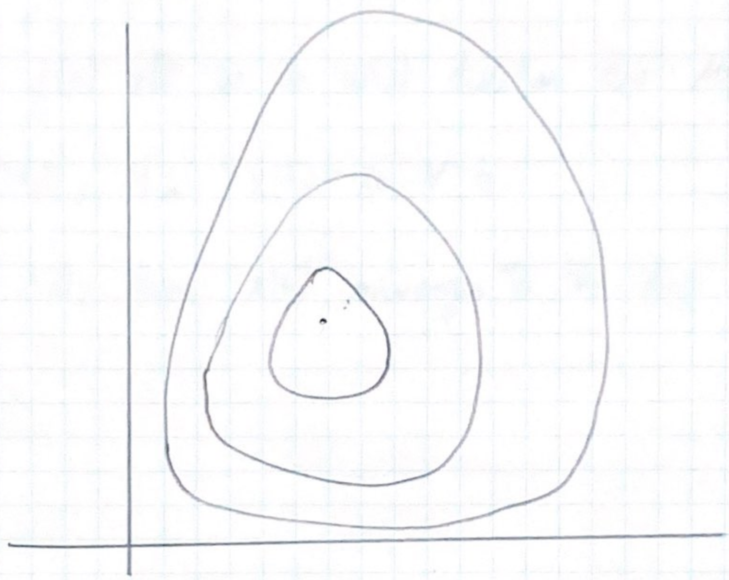
$\lim_{s \to +\infty} g(s) = -\infty$, $\lim_{s \to 0^+} g(s) = -\infty$. So g attains its global max

$s = \frac{\alpha}{\beta}$

Hence, it follows that F has a global maximum at $\left(\frac{C}{D}, \frac{A}{B}\right)$.

The sol $(x(t), y(t))$ traces a curve given by

$F(x(t), y(t)) = C$



Phase-Portrait of the Prey-Predator system (Volterra-Lotka)

All orbits of the Prey-Predator system are closed (Jordan) curves that surround the stationary point $\left(\frac{C}{D}, \frac{A}{B}\right)$