# Initial Value Problems : Existence and Uniqueness Result for ODEs

We will show existence and uniqueness of sols. for the Initial Value P'blm

(IVP) $\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$

We suppose $f \in C(\Omega, \mathbb{R}^n)$ (continuous), where $\Omega \subset \mathbb{R}^{n+1}$ is a domain (open, connected).

$(t_0, x_0) \in \Omega$.

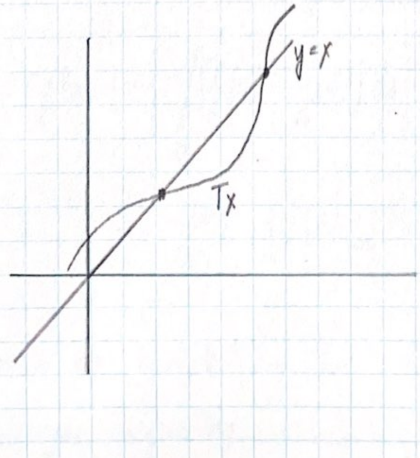
Integrating both sides w.r.t $t$ : $\displaystyle\int_{t_0}^{t} x'(s)\,ds = \int_{t_0}^{t} f(s, x(s))\,ds$ , gives

$$\boxed{x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\,ds.}$$

The sol. $x(t)$ satisfies this integral equation.

We can formulate this as an eqn. of the form $\boxed{X = TX}$ in a suitably chosen Banach space $X$ (space of fns.) and $T : C \subseteq X \to X$ is an mapping (operator). An element $x \in X$ is called a fixed point of $T$ if $Tx = x$, or if $x$ solves: $x - Tx = 0$. ($Tx$ denotes the image of $x$ by $T$)

For example if $X = \mathbb{R}$, and $T : \mathbb{R} \to \mathbb{R}$ is a continuous map, the the fixed pts of $T$ are points $x$ where the graph of $T$ ($y = Tx$) intersects the line $y = x$.



→ How to find Fixed-Points

Start with $x_0 \in C \subseteq X$, and iteratively define $x_1 = Tx_0, \ \dots \ x_{m+1} = Tx_m$.

→ When does such a seq $(x_m)$ converge. Note that if $x_\infty = \lim_{m \to +\infty} x_m$ exists in $X$, then $x_\infty \in C$ since the set $C$ is closed and $x_\infty$ is a fixed point as the map $T$ is continuous, hence passing to the limit gives $x_\infty = Tx_\infty$.

→ Suppose the map $T: C \to X$ satisfies $\|Tx - Ty\|_X \le \theta \|x-y\|_X$

$$\underbrace{\forall\, x, y \in C \subset X \text{ with } \theta < 1,}_{\text{contraction mapping}}$$

here $\|\cdot\|_X$ denotes the norm (distance) of the Banach sp. $X$

So for a contraction map $T$, the distance between the images $Tx$ and $Ty$ of the points $x, y \in X$ under the mapping is smaller than the distance between $x$ and $y$. (hence contraction).

Example 1 $\quad X = C([0,1])$: The space of continuous fns. on $[0,1]$

Consider $T: X \to X$ defined as $\quad Tf(x) := \int_0^x f(s)\,ds$

The operator (or mapping) is linear: $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$

for $\forall\, f, g \in X$ and $\alpha, \beta \in \mathbb{R}$.

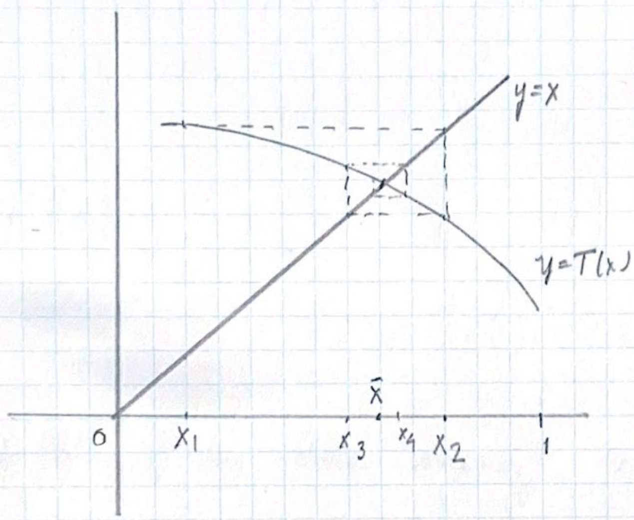
We have $\quad |Tf(x) - Tg(x)| = \left| \int_0^x [f(s) - g(s)]\,ds \right| \le \int_0^x |f(s) - g(s)|\,ds$

Hence $\quad \sup_{x \in [0,1]} |Tf(x) - Tg(x)| \le \sup_{x \in [0,1]} |f(x) - g(x)|$.

Example 2: In any Banach sp. $(X, \|\cdot\|)$, we have $\big| \|x\| - \|y\| \big| \le \|x - y\|$ by the triangle inequality.

→ <u>Banach Fixed Point Theorem</u> : Let $C$ be a nonempty closed subset of a Banach space $X$ and let $T : C \to X$ be a contraction. Then $T$ has a unique fixed point $\bar{x} \in C$.

→ <u>Example 3</u>




$x_{m+1} = Tx_m$

$T : [0,1] \to [0,1]$ is continuously diff'able with $-1 < T'(x) < 0$ in $[0,1]$

Coming back to our initial value p'bm: (Cauchy P'bm) $\begin{cases} x' = f(x(t)) \\ x(0) = x_0 \end{cases}$, which

can be written in the integral form: $x(t) = x_0 + \int_0^t f(x(s))\,ds$.

The sol. curve $x(t)$ is a fixed-pt of the mapping

$$\gamma \longmapsto \underbrace{x_0 + \int_0^t f(\gamma(s))\,ds}_{:= T\gamma}$$

To find a fixed-point of the above mapping, we consider the iterations

Picard Iterations

$$X_1(t) := Tx_0 = x_0 + \int_0^t f(x_0)\,ds$$

$$X_2(t) := Tx_1 = x_0 + \int_0^t f(x_1(s))\,ds$$

$$\vdots$$

$$X_{m+1}(t) := Tx_m = x_0 + \int_0^t f(x_m(s))\,ds$$

Defining everything appropriately, the map $T$ will have a fixed point if it is a contraction map, by the Banach fixed point theorem.

We have $$|T\gamma_1(t) - T\gamma_2(t)| = \left| \int_0^t \left( f(\gamma_1(s)) - f(\gamma_2(s)) \right) ds \right|$$

$$\leq \int_0^t |f(\gamma_1(s)) - f(\gamma_2(s))|\,ds$$

Assuming $f: \mathbb{R} \to \mathbb{R}$ satisfies: $\begin{cases} |f(x)| \leq M \quad \forall\, x \text{ in } \mathbb{R} \text{ (Bounded)} \\ |f(x) - f(y)| \leq K|x-y| \text{ (\underline{Lipschitz})} \end{cases}$

Then, $|T(\gamma_1(t)) - T(\gamma_2(t))| \leq K \int_0^t |\gamma_1(s) - \gamma_2(s)| \, ds$

$$\leq K \int_0^t \|\gamma_1 - \gamma_2\|_\infty \, ds$$

$\|\gamma\|_\infty := \sup_{t_0 \leq s \leq t} \{|\gamma(t)|\}$

Fixing some $\delta$, we have then $\forall\, t$ s.t $|t| \leq \delta$

$$|T(\gamma_1(t)) - T(\gamma_2(t))| \leq (K\delta) \|\gamma_1 - \gamma_2\|_\infty \text{ and hence}$$

$$\rightarrow \|T(\gamma_1(t)) - T(\gamma_2(t))\|_\infty \leq (K\delta) \|\gamma_1 - \gamma_2\|_\infty .$$

Choosing $\delta$ s.t $\underbrace{K\delta}_{:=\theta} < 1$ gives us that the mapping

$T : C([-\delta, \delta]) \rightarrow C([-\delta, \delta])$ is a contraction. Since $C([-\delta, \delta])$ is a Banach sp. w.r.t $\|\cdot\|_\infty$ norm, this implies that $T$ has a unique fixed pt. $\gamma \in C([-\delta, \delta])$, which satisfies

$T(\gamma) = \gamma$ that is $\gamma(t) = x_0 + \int_0^t f(\gamma(s)) \, ds$ , $t \in [-\delta, \delta]$.

Differentiating this w.r.t gives us a unique sol. of the initial value p'blm $\begin{cases} x'(t) = f(x(t)) & \text{for } -\delta \leq t \leq \delta \\ x(0) = x_0 \end{cases}$

We get existence and uniqueness for a initial value p'blm using the Banach fixed point theorem, provided $f$ is a Lipschitz function.
Note that the interval of existence $[-\delta, \delta]$ is determined by the Lipschitz const. $K$ of $f$, by $\delta < \frac{1}{K}$.

## Tools from Functional Analysis

Let $X$ be a vector space over $\mathbb{R}$. A norm on $X$, denoted by $\|\cdot\|$ is a map $\|\cdot\| : X \to [0,+\infty)$ satisfying the following:

a) $\|a\| \geq 0 \;\forall\, a \in X$ and $\|a\| = 0$ iff $a = 0$ (vector)

b) $\|\lambda a\| = |\lambda|\,\|a\| \;\forall\, \lambda \in \mathbb{R}$ and $a \in X$.

c) $\|a+b\| \leq \|a\| + \|b\|$ for all $a, b \in X$. (Triangle inequality)

The vector sp. $X$ with a norm $\|\cdot\|$ is called normed linear space (NLS) or normed vector space, and is denoted by $(X, \|\cdot\|)$.

→ A norm $\|\cdot\|$ defines a <u>metric</u> on $X$ given by $d(a,b) := \|a-b\|$

<u>Exercise 1</u>: Show that i) $\|\sum_{i=1}^{k} a_i\| \leq \sum_{i=1}^{k} \|a_i\| \quad \forall \; a_i \in X;\ 1 \leq i \leq k.$

ii) $\big|\|a\| - \|b\|\big| \leq \|a-b\| \quad \forall \; a, b \in X$

<u>Examples</u>: 1.) $\mathbb{R}^n$ with its usual Euclidean norm

$$\|a\| = \sqrt{\sum_{i=1}^{n} a_i^2} \quad \text{where} \quad a = (a_1, \dots, a_n) \in \mathbb{R}^n$$

2.) Consider the space of continuous fns. on $[0,1]$, denoted by $C([0,1])$.
(real valued)

For $f, g \in C([0,1])$, $f+g \in C([0,1])$ is defined as $(f+g)(x) = f(x) + g(x)$.

For $\lambda \in \mathbb{R}$, $f \in C([0,1])$ $\lambda f \in C([0,1])$ is defined as $(\lambda f)(x) = \lambda(f(x))$.

We can consider the <u>sup</u> <u>norm</u> ($L^\infty$ norm) on $C([0,1])$ defined as $\|f\| := \max\{f(x) : x \in (0,1)\}$.

<u>Def</u>: → Consider a normed linear sp. $(X, \|\cdot\|)$. We say a seq $(a_m)_{m\geq 1}$ in $X$ converges to $a$ in $X$, if

$\lim_{m\to+\infty} \|a_m - a\| = 0$, and we write $\lim_{m\to+\infty} a_m = a$.

→ A map $F: X \to Y$, two normed linear spaces, is continuous if for every $(a_m) \in X$ s.t $a_m \to a$ as $m \to +\infty$ in $X$, we have $F(a_m) \to F(a)$ as $m \to +\infty$ in $Y$.

→ A seq $(a_m)_{m\geq 1}$ in $X$ is a Cauchy Sequence if: $\forall\ \varepsilon > 0$ $\exists$ a positive integer $N_\varepsilon$ s.t $\|a_m - a_k\| < \varepsilon$ $\forall\ m, k \geq N_\varepsilon$, or

$\lim_{m,k\to+\infty} \|a_m - a_k\| = 0$.

→ A normed linear sp. $(X, \|\cdot\|)$ is said to be complete: if every Cauchy sequence in $X$ has a limit in $X$. A complete normed linear sp. is called a <u>Banach Space</u>.

<u>Examples</u> 1.) $\mathbb{R}^n$ with its Euclidean norm.

2) $C([0,1])$ with the sup norm.

<u>Exercise 2</u>: A closed <u>subspace</u> $C$ of a Banach sp. $X$ is a Banach space (w.r.t norm of $X$).

Banach spaces, which will often be infinite dimensional function spaces, are crucial to do analysis. One can even do Calculus on Banach spaces.

→ Consider a continuous linear map (operator) $T : X \to X$.

The quantity: $\sup_{u \in X} \frac{\|Tu\|}{\|u\|} = \sup \{ \|Tu\| : \|u\| \le 1 \} := \|T\|$

is called the operator norm of $T$.

→ The space of all continuous linear functionals $\ell : X \to \mathbb{R}$ is a Banach space w.r.t to the dual norm $\|\ell\|_* := \sup\{ \|\ell u\| : u \in X \text{ with } \|u\| \le 1 \}$. This space is called the dual of $X$ and is denoted by $X^*$.

→ <u>Def</u> : Let $X$ be a Banach sp. with norm $\|\cdot\|$. A map $F : X \to X$ is said to be a <u>contraction</u> if $\exists$ a const. $0 < \theta < 1$ s.t $\|F(x) - F(y)\| \le \theta \|x - y\| \quad \forall \; x, y \in X$.

→ A point $\xi \in X$ is a <u>fixed-point</u> of $F : X \to X$ if $F(\xi) = \xi$.

→ <u>Theorem</u> (Contraction Mapping Fixed point theorem)
(<u>Banach, Picard</u>)

Let $C$ be a nonempty closed set of a Banach space $X$ and let $F : C \to C$ be a contraction. Then $F$ has a unique fixed point $\xi_0 \in C$ s.t

$$\|F^m(x) - \xi_0\| \le \frac{\theta^m}{1-\theta} \|F(x) - x\| \quad \text{for any } x \in C,$$

where $F^m(x) = F(F^{m-1}(x)) = \underbrace{(F \circ \cdots \circ F)}_{m \text{ times}}(x)$

Proof: Uniqueness of Fixed-Point: Suppose $F$ has two fixed point $\xi_1$ and $\xi_2$ in $C$, so $\begin{cases} F(\xi_1)=\xi_1 \\ F(\xi_2)=\xi_2 \end{cases}$. Then since $F$ is a contraction

$$\|F(\xi_1)-F(\xi_2)\| \leq \theta\|\xi_1-\xi_2\| \Rightarrow \|\xi_1-\xi_2\| \leq \theta\|\xi_1-\xi_2\|$$

$\Rightarrow \|\xi_1-\xi_2\|=0 \Rightarrow \xi_1=\xi_2$. Hence $F$ can have at most one fixed point in $C$
(since $0<\theta<1$)

Existence of Fixed-Point

Take $x \in C$ and consider the seq $(x_m)$ in $C$ defined by $x_m := F^m(x)$ with $x_1 := F(x)$. That is $x_{m+1} = F(x_m)$.

We have $\|x_{m+1}-x_m\| = \|F(x_m)-F(x_{m-1})\| \leq \theta\|x_m - x_{m-1}\|$

$$\leq \theta^2\|x_{m-1}-x_{m-2}\|$$

Hence (using an induction argument) $\|x_{m+1}-x_m\| \leq \theta^m\|x_1 - x\|$.

Using the triangle inequality we get for $k>m$

$$\|x_k - x_m\| = \left\|\sum_{i=m}^{k-1}(x_{i+1}-x_i)\right\| \leq \sum_{i=m}^{k-1}\|x_{i+1}-x_i\|$$

$$\leq \sum_{i=m}^{k-1}\theta^i\|x_1-x\| = \underbrace{\left(\sum_{i=m}^{k-1}\theta^i\right)}_{\text{(Geometric series)}}\|F(x)-x\|$$

Therefore we get the estimate $\|x_k - x_m\| \leq \theta^m\left(\dfrac{1-\theta^{k-m}}{1-\theta}\right)\|F(x)-x\|$

$$\leq \frac{\theta^m}{1-\theta}\|F(x)-x\| \quad \text{(since } 0<\theta<1)$$

Note $\theta^m \to 0$ as $m\to+\infty$, since $0<\theta<1$. Therefore given any $\varepsilon>0$ $\exists$ a positive integer $N_\varepsilon$ s.t $\theta^m < \underbrace{(1-\theta)\varepsilon}_{>0}$, that is $\dfrac{\theta^m}{1-\theta}<\varepsilon$, $\forall\ m\geq N_\varepsilon$.

So given any $\varepsilon > 0$ $\exists$ a positive integer $N_\varepsilon$ s.t $\forall$ $k, m \geq N_\varepsilon$ we have $\quad \| x_k - x_m \| < \varepsilon \| F(x) - x \|$.

This shows that the seq $(x_m)$ is a Cauchy-Sequence in C and hence in X. X is a Banach Space and by the completeness property, every Cauchy-seq. converges in X. So the $\lim_{m \to +\infty} x_m := \xi_0$ exists in X. As $C \subset X$ is closed the (limit) point $\xi_0 \in C$, and hence the Cauchy seq $(x_m)$ converges in C (showing C is also complete) to $\xi_0 \in C$.

We have $\quad \| F(x_m) - x_m \| = \| x_{m+1} - x_m \| \leq \theta^m \| F(x) - x \|$

Passing to the limit as $m \to +\infty$ we have $\lim_{m \to +\infty} \| F(x_m) - x_m \| = 0$. Also $x_m \to \xi_0$ as $m \to +\infty$ and by the continuity of F, $\lim_{m \to +\infty} F(x_m) = F(\xi_0)$. Hence

$$\| F(\xi_0) - \xi_0 \| \leq \| F(\xi_0) - F(x_m) \| + \| F(x_m) - x_m \|$$

Passing to the limit gives $\| F(\xi_0) - \xi_0 \| = 0$ or $F(\xi_0) = \xi_0$. Hence $\xi_0$ is a fixed point.

Coming back to the estimate $\| x_k - x_m \| \leq \frac{\theta^m}{1-\theta} \| F(x) - x \|$, that is $\| x_k - F^m(x) \| \leq \frac{\theta^m}{1-\theta} \| F(x) - x \|$ and letting $k \to +\infty$ gives the rate of convergence $\| \xi_0 - F^m(x) \| \leq \frac{\theta^m}{1-\theta} \| F(x) - x \|$.

→ Optimality of the hypothesis in the Fixed Point theorem

- Consider $T:[0,1]\to[1,3/2]$ given by $T(x)=1+\frac{x}{2}$

  T does not map the sp. $X=[0,1]$ to itself.

- Consider $T:[0,+\infty)\to[0,+\infty)$ given by $T(x)=2x+1$.

  T is not contracting

- Consider $T:(0,1)\to(0,1)$ with $T(x)=x/2$.

  The space $X=(0,1)$ is not complete.

→ Many p'blms in analysis can be reduced to finding fixed-points. Consider $F:X\to X$ and given $y\in X$ we want to find $x\in X$ s.t $F(x)=y$ This can be reduced to finding fixed points of $G:X\to X$ defined by $G(x):=x-F(x)+y$. For example, the proof the Inverse function theorem proceeds by this approach of finding a fixed point.

Theorem (Existence and Uniqueness result for ODEs)
(Picard-Lindelöf / Cauchy-Lipchitz)
Consider the initial value problem (IVP) $\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases}$, where

$f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous fn, $U$ is open and $(t_0, x_0) \in U$.
→ If $f = f(t,x)$ is locally Lipschitz in the $x$-variable uniformly w.r.t the $t$ variable, that is, for every open set $V \subset U$, $\sup_{\substack{(t,x) \neq (t,y) \\ \text{in } V}} \frac{|f(t,x) - f(t,y)|}{|x-y|} := \underbrace{K}_{\text{depends on } V} < +\infty$.

Then there exists a unique (local) solution $\phi : [t_0 - T_0, t_0 + T_0] \to U$ of the IVP for some $T_0 > 0$.

Proof: We write the IVP in the integral form: $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$. Then $\phi(t)$ solves the IVP iff $\phi$ is a fixed point of the map $F$ given by $x(t) \xmapsto{F} x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$. This fixed-point will be obtained using the Banach fixed point theorem.

$(t_0, x_0) \in U$ and $U$ is open. Then $\exists$ $\delta > 0$ and $r > 0$ s.t $[t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, r)} \subset U$, where $B(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$; denotes the ball of radius $r$ centered at $x_0$.
Note that the set $[t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, r)}$ is compact (closed and bounded). We have

$$|F(x(t)) - x_0| \le \int_{t_0}^{t} |f(s, x(s))|\,ds \le |t - t_0| M, \text{ where } M := \max_{[t_0-\delta, t_0+\delta] \times \overline{B(x_0, r)}} |f(t,x)| < +\infty.$$

for $|t - t_0| \le \delta$ and $|x(t) - x_0| \le r$. Let $\tilde{T}_0 := \min\{\delta, \frac{r}{M}\}$.
Then $|F(x)(t) - x_0| \le r$ for $|t - t_0| < \tilde{T}_0$.

Let $\underline{T_0} := \min\left\{\frac{1}{K}, \frac{r}{M}, \delta\right\}$, and we choose our $X := C([t_0-T_0, t_0+T_0], \mathbb{R}^n)$ as our Banach sp, with the supremum norm $\|x(t)\|_\infty = \max_{[t_0-T_0, t_0+T]} |x(t)|$.

Let $C := \{\phi \in X : \|\phi - x_0\|_\infty \leq r\}$, a closed subset of $X$.

Then by our choice of $T_0$, it follows that $F: C \to C$. We need to show that $F$ is a contraction to apply the fixed point theorem. We have for $\phi_1, \phi_2 \in C$

We have
$$|F(\phi_1)(t) - F(\phi_2)(t)| \leq \int_{t_0}^{t} |f(s, \phi_1(s)) - f(s, \phi_2(s))|\, ds$$
$$\leq K \int_{t_0}^{t} |\phi_1(s) - \phi(s)|\, ds$$
(since $f$ is locally lipschitz in $x$)

or $|F(\phi_1)(t) - F(\phi_2)(t)| \leq (K|t-t_0|) \max_{[t_0,t]} |\phi_1(s) - \phi_2(s)|$.

Then for all $|t-t_0| \leq T_0$, we have $|F(\phi_1)(t) - F(\phi_2)(t)| \leq \underbrace{KT_0}_{<1} \|\phi_1 - \phi_2\|_\infty$ by our choice of $T_0$. That is

$$\|F(\phi_1) - F(\phi_2)\|_\infty \leq \underbrace{KT_0}_{<1} \|\phi_1 - \phi_2\|_\infty \quad \forall \ \phi_1, \phi_2 \in C \subset C([t_0-T_0, t_0+T_0], \mathbb{R}^n).$$

This gives that the map $F: C \to C$ is a <u>contraction</u>, and hence $F$ has a <u>unique</u> fixed point $\phi$, a continuous map $\phi: [t_0-T, t_0+T] \longrightarrow \mathbb{R}^n$ s.t

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds \quad \forall \ |t-t_0| \leq T_0.$$ Hence $\phi$ is diff'able

and is the <u>unique sol.</u> of the IVP in an interval around $t_0$.

This completes the proof of the existence and uniqueness result.

<u>Remarks</u> 1) $x_{m+1}(t) := x_0 + \int_0^t f(s, x_m(s))\,ds$ are called the <u>Picard iterations</u>

In the existence and uniqueness result we show that the seq of Picard iterations $(x_m)$ converges to a fixed-point $\phi$ in a closed subset of $C([-T_0+t_0, T_0+t_0], \mathbb{R}^n)$ containing $x_0$.

ii) If $f \in C^1(U, \mathbb{R}^n)$ (continuously diff'able fn of $(n+1)$ variables) then by the mean-value theorem it follows that <u>$f$ is locally-lipschits</u> in the $x$-variable uniformly w.r.t the $t$ variable.

iii) Using the existence and uniqueness theorem we get the following result:

<u>Proposition</u>: Suppose $f \in C^k(U; \mathbb{R}^n)$ ($k$-times continuously diff'ble) where $U \subset \mathbb{R}^{n+1}$ is open and $(t_0, x_0) \in U$. Then there a unique (local) solution $\phi : [t_0 - T_0, t_0 + T_0] \to \mathbb{R}^n$ to the IVP
$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$
for some $T_0 > 0$. Moreover, $\phi \in C^k([t_0 - T_0, t_0 + T_0]; \mathbb{R}^n)$.

## Dependence on the Initial condition

We first show the following inequality.

Lemma (Gronwall's inequality). Suppose $\psi(t)$ satisfies

$$\rightarrow \psi(t) \leq A + \int_0^t (B\psi(s)+C)ds, \quad \text{for } t \in [0,T],$$

for constants $A, C \in \mathbb{R}$ and $B>0$

$$\rightarrow \quad \psi(t) \leq Ae^{Bt} + \frac{C}{B}(e^{Bt}-1) \quad \text{for } t \in [0,T].$$

Pf: One has $\left(e^{-Bt}\int_0^t \psi(s)ds\right)' = \left(\psi(t) - B\int_0^t \psi(s)ds\right)e^{-Bt} \leq (A+Ct)e^{-Bt}$

Integrating $e^{-Bt}\int_0^t \psi(s)ds \leq \int_0^t (A+Cs)e^{-Bs}ds = \frac{C}{B}\int_0^t e^{-Bs}ds - \frac{1}{B}\left[(A+Ct)e^{-Bt} - A\right]$

$$\Rightarrow e^{-Bt}\underbrace{\left(A + \int_0^t (B\psi(s)+C)ds\right)}_{\geq \psi(t)} \leq A - \frac{C}{B}(e^{-Bt}-1)$$

$\Rightarrow e^{-Bt}\psi(t) \leq A + \frac{C}{B}(1-e^{-Bt})$, hence

$$\Rightarrow \psi(t) \leq Ae^{Bt} + \frac{C}{B}(e^{Bt}-1).$$

## Gronwall's Inequality.

$\to \psi'(t) \leq \alpha(t)\psi(t) + \beta(t)$ in $[0,T]$

Then $\psi(t) \leq \exp\left(\int_0^t \alpha(s)\,ds\right)\left[\psi(0) + \int_0^t \beta(s)\exp\left(-\int_0^s \alpha(r)\,dr\right)ds\right]$

In particular if $\psi'(t) \leq \alpha(t)\psi(t)$ and $\psi(0)=0$

then $\psi \equiv 0$ in $[0,T]$

Proof. $\left(\psi(t)\exp\left(-\int_0^t \alpha(s)\,ds\right)\right)' \leq \beta(t)\exp\left(-\int_0^t \alpha(s)\,ds\right)$

Integrating we get

$$\psi(t)\exp\left(-\int_0^t \alpha(s)\,ds\right) - \psi(0) \leq \int_0^t \beta(s)\exp\left(-\int_0^s \alpha(r)\,dr\right)ds$$

so, $\psi(t) \leq \exp\left(\int_0^t \alpha(s)\,ds\right)\left[\psi(0) + \int_0^t \beta(s)\exp\left(-\int_0^s \alpha(r)\,dr\right)ds\right]$.

→ The initial value problem is <u>well-posed</u>, that is, small changes in the data will result in small changes of the solution.

<u>Theorem</u>. Suppose $f, g \in C(\mathcal{U}, \mathbb{R}^n)$ (continuous fn. from an open set $\mathcal{U} \subset \mathbb{R}^{n+1}$ to $\mathbb{R}^n$); $f = f(t,x)$, $g = g(t,x)$ are moreover <u>locally Lipschitz</u> in the $x$ variable uniformly w.r.t to the $t$ variable.

Assume $x = x(t)$ and $y = y(t)$ solves respectively the IVPs

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dy}{dt} = g(t,x) \\ y(t_0) = y_0 \end{cases}.$$

Then, $$|x(t) - y(t)| \leq |x_0 - y_0| e^{K|t-t_0|} + \frac{M}{K}\left(e^{K|t-t_0|} - 1\right),$$

where $$K := \sup_{\substack{(t,x) \neq (t,y) \\ \in V}} \frac{|f(t,x) - f(t,y)|}{|x-y|}, \quad M := \sup_{(t,x) \in V} |f(t,x) - g(t,y)|,$$

with $x(t), y(t) \in V \subset \mathcal{U}$.

<u>Proof</u>: We have $x(t) = x_0 + \int_0^t f(s, x(s))\,ds$ and $y(t) = y_0 + \int_0^t g(s, y(s))\,ds$

Then, $$x(t) - y(t) = x_0 - y_0 + \int_0^t \left(f(s, x(s)) - g(s, y(s))\right) ds,$$

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |f(s, x(s)) - g(s, y(s))|\,ds,$$

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t \left[|f(s, x(s)) - f(s, y(s))| + |f(s, y(s)) - g(s, y(s))|\right] ds,$$

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t \left(K|x(s) - y(s)| + M\right) ds$$

Then using the Gronwall's inequality gives

$$|x(t)-y(t)| \leq |x_0-y_0| e^{K|t-t_0|} + \frac{M}{K}\left(e^{K|t-t_0|}-1\right).$$

We denote the solution of the initial value problem: $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ as

$\phi(t,t_0,x_0)$ to emphasize the dependence on the initial condition $x(t_0)=x_0$.

By our previous result, we have in the case $f=g$.

$$\left|\phi(t,t_0,x_0)-\phi(t,t_0,y_0)\right| \leq |x_0-y_0| e^{K|t-t_0|}.$$

Note, we have equality in the above inequality for the linear eqn: $\dot{x}=x$, showing the b'dd is optimal.

<u>Theorem</u> (continuous dependence on data) Suppose $f: U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous, $f=f(t,x)$ is locally Lipschitz in the $x$-variable, uniformly w.r.t to the $t$ variable. Given any $(t_0,x_0) \in U$ (open in $\mathbb{R}^n$), there exists $T_0>0$ and $r_0>0$ s.t $[t_0-T_0/2, t_0+T_0/2] \times B(x_0, r_0/2) \subset U$, and $\Phi(t,s,y) \in C\left([t_0-T_0/2, t_0+T_0/2] \times [t_0-T_0/2, t_0+T_0/2] \times B(x_0, r_0/2); \mathbb{R}^n\right)$ (that is the sol. of the IVP with $X(s)=y$ is continuous). Moreover the fn. $\Phi(t,t_0,x_0)$ is Lipschitz continuous, and we have the estimate:

$$|\Phi(t,t_0,x_0) - \phi(s,s_0,y_0)| \leq M|t-s| + |x_0-y_0| e^{K|t-t_0|} + M|s_0-t_0| e^{K|t-s_0|},$$

where $M := \max\limits_{[t_0-T_0, t_0+T_0] \times B(x_0,r)} |f(t,x)|$ and $K := \sup\limits_{\substack{(t,x)\neq(t,y) \\ \text{in } [t_0-T_0,t_0+T_0]\times B(x_0,r)}} \dfrac{|f(t,x)-f(t,y)|}{|x-y|}$,

for $|t-t_0| \leq T_0/2$ and $(s, s_0, y_0) \in [t_0-T_0/2, t_0+T_0/2] \times [t_0-T_0/2, t_0+T_0/2] \times B(x_0, r_0/2)$.

In the proof of the existence and uniqueness theorem, we have shown that a unique sol. $\phi(t, t_0, x_0)$ of the IVP: $\begin{cases} \dot{x} = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$ exists for $|t - t_0| \leq T_0$.

for some $T_0 > 0$, $\phi(t, t_0, x_0) \in \overline{B(x_0, r_0)}$ for some $r_0 > 0$. Similarly for any $x_1 \in B(x_0, r_0/2)$, the IVP: $\begin{cases} \dot{x} = f(t, x(t)) \\ x(t_1) = x_1 \end{cases}$ has a unique sol.

$\phi(t, t_0, x_1) \in \overline{B(x_1, r_0/2)}$.

Then for $\phi(t, t_0, x_0)$ and $\phi(s, s_0, y_0)$, with $|t - t_0| \leq T_0/2$, and $(s, s_0, y_0) \in [t_0 - T_0/2, t_0 + T_0/2] \times [t_0 - T_0/2, t_0 + T_0/2] \times B(x_0, r_0/2)$, we have:

$$|\phi(t, t_0, x_0) - \phi(s, s_0, y_0)| \leq |\phi(t, t_0, x_0) - \phi(t, t_0, y_0)| + |\phi(t, t_0, y_0) - \phi(t, s_0, y_0)| + |\phi(t, s_0, y_0) - \phi(s, s_0, y_0)|.$$

We estimate each of the terms separately

We have $\underset{\text{(3rd-term)}}{|\phi(t, s_0, y_0) - \phi(s, s_0, y_0)|} \leq \left| \int_{s_0}^{t} f(r, \phi(r, s_0, y_0)) dr - \int_{s_0}^{s} f(r, \phi(r, s_0, y_0)) dr \right|$

$$\leq \left| \int_{s}^{t} f(r, \phi(r, s_0, y_0)) \, dr \right| \leq \int_{s}^{t} |f(r, \phi(r, s_0, y_0))| \, dr$$

$$\leq M|t - s| \quad \text{where } M := \max_{[t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, r)}} |f(t, x)|$$

For the 1st term we use the estimate from our previous theorem (in case $f = g$)

Then we have $|\phi(t, t_0, x_0) - \phi(t, t_0, y_0)| \leq |x_0 - y_0| e^{K|t - t_0|}$, where

$$K := \sup_{\substack{(t,x) \neq (t,y) \\ \text{in } [t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, r)}}} \frac{|f(t, x) - f(t, y)|}{|x - y|}.$$

We have $|\phi(t,t_0,y_0)-\phi(t,s_0,y_0)| \le \left|\int_{t_0}^{t} f(r,\phi(r,t_0,y_0))\,dr - \int_{s_0}^{t} f(r,\phi(r,s_0,y_0))\,dr\right|$

(2nd term)

$$\le \left|\int_{t_0}^{s_0} f(r,\Phi(r,t_0,y_0))\,dr\right| + \left|\int_{s_0}^{t}\left[f(r,\phi(r,t_0,y_0))-f(r,\phi(r,s_0,y_0))\right]dr\right|$$

$$\le \int_{t_0}^{s_0} |f(r,\Phi(r,t_0,y_0))|\,dr + \int_{s_0}^{t} |f(r,\phi(r,t_0,y_0))-f(r,\Phi(r,s_0,y_0))|\,dr$$

$$\le M|s_0-t_0| + K\int_{s_0}^{t} |\phi(r,t_0,y_0)-\phi(r,s_0,y_0)|\,dr$$

Then using the Gronwall's inequality, we obtain

$$|\phi(t,t_0,y_0)-\phi(t,s_0,y_0)| \le M|s_0-t_0|e^{K|t-s_0|}.$$

Combining the three estimate, we obtain

$$|\Phi(t,t_0,x_0)-\Phi(s,s_0,y_0)| \le M|t-s| + |x_0-y_0|e^{K|t-t_0|} + M|s_0-t_0|e^{K|t-s_0|}.$$

This shows the continuity of the sol. $\Phi(t,t_0,x_0)$ with respect to the initial data $(s,y)$; $\Phi(s)=y$ and the variable $t$.

## Interval of Existence and Extensibility of Solutions

Consider the initial value problem (IVP): $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$

where $f: \mathcal{U} \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is locally Lipschitz in the x-variable uniformly w.r.t the t variable. Then the IVP has a unique solution for some interval $|t-t_0| \le T_0$.

Theorem (Maximal interval of existence and maximal solution) The above IVP has a unique (maximal) solution defined on some maximal interval $(T_-, T_+)$ containing $t_0$.

Let $S$ be the set of solutions of the IVP $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ defined on an open interval.

$$S := \{ \phi : I_\phi \to \mathbb{R}^n \text{ solves the IVP} : I_\phi \text{ is an interval containing } t_0 . \}$$

Let $I_{max} := \bigcup_{\phi \in S} I_\phi$. Then $I_{max}$ being an union of open intervals is open. If $t_1 > t_0 \in I_{max}$, then $t_1 \in I_\phi$ for some solution $\phi$, and thus $[t_0, t_1) \subseteq I_\phi \subseteq I_{max}$, as $I_\phi$ is an interval. Similarly, if $t_2 < t_0 \in I_{max}$, then $(t_2, t_0] \subseteq I_\phi \subseteq I_{max}$. Therefore,

$$I_{max} = (T_-, T_+), \text{ for some } T_+ \in (t_0, +\infty] \text{ and } T_- \in [-\infty, t_0)$$

We define the maximal solution $\Phi_{max} : I_{max} \to \mathbb{R}^n$ by $\Phi_{max}(t) = \phi(t)$ for any $\phi \in S$ and $t \in I_\phi$.

Claim: $\Phi_{max}$ is well-defined: If $\phi_1, \phi_2 \in S$ then $\Phi_1(t) = \Phi_2(t)\ \forall t \in I_{\phi_1} \cap I_{\phi_2}$.

Proof of the claim: Let $\phi_1, \phi_2 \in S$ and let $I_{\phi_1} \cap I_{\phi_2} = (T_-, T_+)$.

Let $t_+ := \inf \{ t \in I_{\phi_1} \cap I_{\phi_2} \cap (t_0, +\infty) : \Phi_1(t) \neq \Phi_2(t) \}$

We have $t_0 < t_+$ and by continuity $\Phi_1(t_+) = \Phi_2(t_+) := \xi_+$ and we have $\xi_+ \in U$. Consider the initial value p'blm starting at $t_+$ with initial value $\xi_+$: $\begin{cases} \dot{x} = f(t,x) \\ x(t_+) = \xi_+ \end{cases}$. ( By the (local) uniqueness of ODE solutions, $\exists$ a $\delta > 0$ s.t. $\Phi(t) = \Phi(t)$ $\forall$ $t \in [t_+, t_+ + \delta]$. This contradicts the definition of $t_+$. Hence $t_+ = T_+$. Similarly let $t_- := \sup \{ t \in I_{\phi_1} \cap I_{\phi_2} \cap (-\infty, t_0) : \Phi_1(t) \neq \Phi_2(t) \}$, and we obtain that $t_- = T_-$. )

OR

We have $t_+ \in I_{\phi_1} \cap I_{\phi_2}$ an interval and so $[t_+, t_+ + \delta] \in I_{\phi_1} \cap I_{\phi_2}$ for some $\delta > 0$. Then using the ODE

$$\frac{d}{dt} |\phi_1(t) - \phi_2(t)|^2 = 2 \langle \phi_1(t) - \phi_2(t), \phi_1'(t) - \phi_2'(t) \rangle$$
$$= 2 \langle \phi_1(t) - \phi_2(t), f(t, \phi_1(t)) - f(t, \phi_2(t)) \rangle$$
$$\leq 2K |\phi_1(t) - \phi_2(t)|^2,$$

where $K : \sup_{[0, t_+ + \delta] \times B[\xi_0, r)} \frac{|f(t,x) - f(t,y)|}{|x - y|} < +\infty$.

Consequently, the fn. $t \longmapsto e^{-2Kt} |\phi_1(t) - \phi_2(t)|^2$ is monotone decreasing on the interval $[t_+, t_+ + \delta]$. So,

$e^{-2Kt} |\Phi_1(t) - \Phi_2(t)| \leq 0$ $\forall$ $t \in [t_+, t_+ + \delta]$, hence $\Phi_1(t) = \phi_2(t)$ $\forall$ $t \in [t_+, t_+ + \delta]$.

This contradicts the definition of $t_+$. Hence $t_+ = T_+$. Similarly

$t_- = T_-$, where $t_- := \sup\{t \in I_{\phi_1} \cap I_{\phi_2} \cap (-\infty, t_0) : \Phi_1(t) \neq \Phi_2(t)\}$.

Hence $\Phi_1(t) = \Phi(t)$ $\forall\, t \in I_{\phi_1} \cap I_{\phi_2} = (T_-, T_+)$.

This proves the claim.

<u>Note</u>: (Gluing / Patching solutions)

Consider $\left\{ \phi_i' = f(t, \phi_i) \right.$ for $t \in I_i$ (an interval), $i = 1,2$

Then $\Phi(t) := \begin{cases} \Phi_1(t) & \text{for } t \in I_1 \\ \Phi_2(t) & \text{for } t \in I_2 \end{cases}$ is (well) defined in $I = I_1 \cup I_2$.

Lemma: Let $\phi(t)$ be a solution of IVP $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ defined ($f: U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is locally Lipschitz.) on $(t_-, t_+)$. There exists an extension of $\Phi$ to the interval $(t_-, t_+ + \varepsilon)$ for some $\varepsilon > 0$ iff there exists a sequence $(t_m)_m$, $t_m \in (t_-, t_+)$ s.t $\lim_{m \to +\infty} t_m = t_+$ and $(t_+, \lim_{m\to+\infty} \Phi(t_m)) \in U$.

Similarly, $\exists$ an extension of $\Phi$ to the interval $(t_- - \varepsilon, t_+)$ for some $\varepsilon > 0$ iff $\exists$ a seq. $(t_m)_m$, $t_m \in (t_-, t_+)$ s.t $\lim_{m \to +\infty} t_m = t_-$

and $(t_-, \lim_{m \to +\infty} \Phi(t_m)) \in U$.

Proof: If there is an extension of $\Phi$ to $(t_-, t_+ + \varepsilon)$, for some $\varepsilon > 0$ then by continuity, for every seq. $(t_m)_m$ in $(t_-, t_+)$ s.t $\lim_{m \to +\infty} t_m = t_+$ we have $(t_+, \lim_{m\to+\infty} \Phi(t_m)) \in U$.

Now suppose there is a seq $t_- < t_m < t_m$ s.t $\lim_{m \to +\infty} t_m = t_+$ and $(t_+, \lim_{m \to +\infty} \Phi(t_m)) \in U$. Then we claim that

Claim: $\lim_{\substack{t \to t_+, \\ t < t_+}} \phi(t) = \xi$ exists with $(t_+, \xi) \in U$.

Assuming the claim holds, consider the initial value p'blm:

$\begin{cases} \dot{x} = f(t,x) \\ x(t_+) = \xi \end{cases}$. By the existence and uniqueness theorem, this IVP has an unique sol $\psi$ defined on an interval $(t_+ - \varepsilon, t_+ + \varepsilon)$ around $t_+$. We can glue the sols. $\phi$ and $\psi$ at $t_+$ to obtain an unique sol $\Phi_E$ on the interval $(t_-, t_+ + )$.

$$\phi_E(t) := \begin{cases} \phi(t) & \text{for } t \in (t_-, t_+) \\ \psi(t) & \text{for } t \in (t_+ - \varepsilon, t_+ + ) \text{ (or } [t_+ + \varepsilon)\text{)} \end{cases}$$

$\phi_E$ is $C^1$ by construction and uniquely solves the original IVP on $(t_-, t_+ + \varepsilon)$, and hence extends the sol. $\Phi$ to a larger interval.

Proof of the Claim: Since $\mathcal{U}$ is open, we have $[t_+ - \delta, t_+] \times \overline{B(\xi, \delta)} \subset \mathcal{U}$, for some $\delta > 0$. Suppose on the contrary $\exists$ a seq $\tau_m \in (t_-, t_+)$ with $\lim_{m \to +\infty} \tau_m = t_+$ s.t. $|\phi(\tau_m) - \xi| \geq \theta > 1$, for some $\theta > 0$. We can choose $\tau_m \in [t_+ - \delta, t_+]$, by using the continuity of $\Phi$ along with $\phi(t_m) \to \xi$, is s.t. $|\phi(\tau_m) - \xi| = \delta$.

Thus,
$$\delta = |\phi(\tau_m) - \xi| \leq |\phi(\tau_m) - \phi(t_m)| + |\phi(t_m) - \xi|$$
$$\leq \int_{t_m}^{\tau_m} |f(s, \phi(s))| ds + |\phi(t_m) - \xi|$$
$$\leq M|\tau_m - t_m| + |\phi(t_m) - \xi|$$
where, $M = \sup_{[t_+ - \delta, t_+] \times \overline{B(\xi, \delta)}} |f(t, x)| \; (< +\infty)$.

So, $\delta \leq M|\tau_m - t_m| + |\phi(t_m) - \xi| \leq M|\tau_m - t_+| + M|t_m - t_+| + |\phi(t_m) - \xi|$

but each of the three terms tends to $0$ as $m \to +\infty$. This gives us a contradiction and therefore for every sequence with $\lim_{\substack{m \to +\infty \\ \tau_m < t_+}} \tau_m = t_+$ we must have $\lim_{m \to +\infty} \Phi(t_m) = \xi$ exists with $(t_+, \xi) \in \mathcal{U}$. This proves the claim.

→ Some consequences of the extension lemma.

Corollary: Consider the IVP $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$

Suppose $\phi$ solves the above IVP on $[t_0, t_+)$ and there exists a compact set $C$ (possibly depending on $t_+$) s.t $\phi(t_m) \in C$ for some sequence $t_m \in (t_0, t_+)$ with $\lim_{m\to+\infty} t_m = t_+$. Then there exists an unique extension of the sol $\Phi$ to $[t_0, t_+ + \varepsilon)$ for some $\varepsilon > 0$. Here $[t_0, t_+] \times C \subset \mathcal{U}$.

Proof: The seq $(\Phi(t_m))_m$ of points in the compact $C$ has a subsequence converging to a limit $\xi \in C$, that is, there exists a (sub) seq. $t_m \in (t_0, t_+)$ s.t $\lim_{m\to+\infty} (t_m, \phi(t_m)) = (t_+, \xi) \in C \subset \mathcal{U}$.

Then we can apply the extension lemma to conclude that there is an unique extension of the sol. $\Phi$ to $[t_0, t_+ + \varepsilon)$ for some $\varepsilon > 0$.

Corollary: If for every $\bar{t} > t_0$, there is a compact set $C$ (possibly depending on $\bar{t}$) s.t $\phi(t_m) \in C$ for some seq $t_m \in (t_0, t_+)$ with $\lim_{m\to+\infty} t_m = t_+$, then the sol. $\phi$ can be extented to $[t_0, +\infty)$.

Note: Similar conclusions for the left end-point $t_+$.

Corollary: Let $(T_-, T_+)$ be the maximal interval of existence for the sol of the IVP $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$.

If $T_+ < +\infty$, then the sol is not contained in any compact set $C$ with $[t_0, t_+] \times C \subset \mathcal{U}$.

Corollary: Consider $\begin{cases} X'(t) = f(t, X(t)) \\ X(0) = X_0 \end{cases}$ where $f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is locally lipschitz.

Suppose $(T_-, T_+)$ is the maximal interval of existence of the sol. of the above IVP. If $T_+ < +\infty$, or $T_- > -\infty$, then

$\lim_{\substack{t\to T_+ \\ t<T_+}} |X(t)| = +\infty$, or $\lim_{\substack{t\to T_- \\ t>T_-}} |X(t)| = +\infty$ respectively.

That is the sol. $X(t)$ blows-up (becomes infinity) as $t \overset{<}{\to} T_+ < +\infty$, or $t \overset{>}{\to} T_- > -\infty$.

Proof: If $\lim_{\substack{t\to T_+ \\ t<T_+}} |X(t)| < +\infty$, then $X(t)$ is contained in

some closed ball of radius $R$, $\overline{B(0,R)}$, that is $X(t) \in \overline{B(0,R)}$ ($|X(t)| \leq R$)

for some $R$, $\forall\ t \in [t_0, T_+]$. $\overline{B(0,R)}$ is compact, being closed and bounded, hence by our previous corollaries it is possible to extend the sol. $X(t)$ to a larger interval $[t_0, T_+ + \varepsilon)$, $\varepsilon > 0$. But this contradicts the definition of $T_+$, showing that this is not possible, and hence $\lim_{\substack{t\to T_+ \\ t<T_+}} |X(t)| = +\infty$.

Similarly, we must have blow up as $t \overset{>}{\to} T_+$.

→ A solution to a initial value p'blm defined for all $t \in \mathbb{R}$, that is $-\infty < t < +\infty$, is called a Global Solution.

Theorem (Existence of Global solution)

Consider the initial value p'blm (IVP) $\begin{cases} X'(t) = f(t, X(t)) \\ X(0) = X_0 \end{cases}$, where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

is a locally Lipschitz function st: $|f(t,x)| \leq M_T + K_T |x|$

$\forall$ $(t,x)$ in $[-T,T] \times \mathbb{R}^n$.

$M_T$ and $K_T$ depends only on $T$. Then any sol. of the above IVP is a global sol, that is any sol of the above IVP is defined for all $t \in \mathbb{R}$.

Proof: We have $X(t) = X_0 + \int_0^t f(s, X(s))\, ds$

Then $|X(t)| \leq X_0 + \int_0^t |f(s, X(s))|\, ds$

$\leq X_0 + \int_0^t (M_T + K_T |X(s)|)\, ds$, by using the growth condition on $f$

$\forall$ $t \in [0,T]$

Then using Gronwall's inequality, we obtain

$$|X(t)| \leq |X_0| e^{K_T T} + \frac{M_T}{K_T} \left( e^{K_T T} - 1 \right)$$

Hence for any $T$, $|X(t)|$ is contained in some compact set, the closed ball $B(0, R_T)$ of radius $R_T := |X_0| e^{K_T T} + \frac{M_T}{K_T} \left( e^{K_T T} - 1 \right)$ (say). Therefore by our previous corollaries we can extended the sol. $X(t)$ to be defined for all $t \in \mathbb{R}$.