Systems of Linear Differential Equations

Consider $\left\{ x_i'(t) = \sum_{j=1}^{n} a_{ij}(t) x_j(t) + \sum_{j=1}^{n} b_j(t) \right.$,

where $x_1(t), \dots x_n(t)$ are the unknown functions; $a_{ij}(t); b_i(t) \in \mathbb{R}$.
It is convinient to write this in the matrix form, as

$\rightarrow$ $X'(t) = A(t)X(t) + B(t)$, where $X = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $A(t) = (a_{ij}(t))_{1 \le i,j \le n}$ (real coefficients)

$B(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$, and $(A(t)X(t))_i = \sum_{j=1}^{n} a_{ij} x_j$

Example 1: Consider the homogeneous, linear ODE

$$x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = 0,$$

where $x^{(n)} = \frac{d^n}{dt^n} x(t)$ (nth derivative.); $a_i \in \mathbb{R}$, $1 \le i \le n-1$.

This can be written in the above (matrix) form: $X'(t) = AX(t)$,
where, $x_i(t) = x^{(i-1)}(t)$ ($(i-1)$th derivative) for $i = 1, \dots n$; and

$$A = \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & 0 & & \\ \vdots & & & & 0 \\ \vdots & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix}$$

Example 2 (The exponential ansatz)

Consider $\{ X'(t) = AX(t)$ where $A = (a_{ij})_{1\le i,j\le n}$ is a constant matrix,

and $X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ is the unknown soln.

→ In case, $n=1$ and $A \equiv \lambda \in \mathbb{R}$ is a const., then the sol. $X(t) = X(0)e^{\lambda t}$.

→ For $n \geq 1$, consider $X(t) = \begin{pmatrix} x_1(0)e^{\lambda_1 t} \\ \vdots \\ x_n(0)e^{\lambda_n t} \end{pmatrix}$. Differentiating componentwise,

$$X'(t) = \begin{pmatrix} \lambda_1 x_1(0)e^{\lambda_1 t} \\ \vdots \\ \lambda_n x_n(0)e^{\lambda_n t} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1(t) \\ \vdots \\ \lambda_n x_n(t) \end{pmatrix}.$$ So, $x_i'(t) = \lambda_i x_i(t)$ for all $1 \leq i \leq n$.

Then $X'(t) = AX(t)$ iff $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ (diagonal matrix).

and then in that case the soln. $X(t) = \begin{pmatrix} x_1(0)e^{\lambda_1 t} \\ \vdots \\ x_n(0)e^{\lambda_n t} \end{pmatrix}$,

which can be written as $X(t) = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} X(0)$.

Example 3: Consider $\begin{cases} X'(t) = A X(t) \\ X(0) = X_0 \end{cases}$, where $A = (a_{ij})_{1 \le i,j \le n}$ is a const. matrix.

→ We look at the Picard's iteration

$$X_{i+1}(t) = X_0 + \int_0^t A X_i(s)\, ds \quad \text{(Componentwise integration)}$$

This gives,

$$X_1(t) = X_0 + \int_0^t A X_0 \, ds = X_0 + t A X_0$$

$$X_2(t) = X_0 + \int_0^t A X_1(s) ds = X_0 + \int_0^t A(X_0 + s A X_0) ds.$$

$$= X_0 + t A X_0 + \frac{t^2}{2} A^2 X_0,$$

continuing by induction,

$$X_m = X_0 + t A X_0 + \frac{t^2}{2} A^2 X_0 + \cdots + \frac{t^m}{m!} A^m X_0,$$

$$X_m = \sum_{i=0}^{m} \frac{t^i}{i!} A^i X_0.$$

Here $A^i = A^{i-1} A$, $A^0 = I$ (identity matrix).

→ The above limit exists as $m \to +\infty$, and we denote (like in one-dim)

$$\lim_{m \to +\infty} \sum_{i=0}^{m} \frac{t^i}{i!} A^i X_0 = \sum_{i=0}^{+\infty} \frac{t^i}{i!} A^i X_0 := \exp(tA) X, \quad \text{where}$$

$$\exp(A) := \sum_{i=0}^{+\infty} \frac{A^i}{i!} \quad \text{which is also a } (n \times n) \text{ matrix.}$$

(exponential of a matrix A)

* The convergence of the sum defining exp(A) and the existence of the limit is shown next.

The the solution of the initial value p'blm is given by

$$X(t) = exp(tA)\, X_0 .$$

The Exponential of a Matrix

Let $A \in \mathbb{C}^{n \times n}$ be a $n \times n$ matrix with complex entries. (The usefulness of considering matrix with complex entries is the fact that $\mathbb{C}$ is algebraically closed but not $\mathbb{R}$.)

The matrix norm or operator norm is defined as: $\|A\| := \sup_{\substack{x \in \mathbb{C}^n, \\ \|x\| \leq 1}} \|Ax\|$,

where $\|X\|^2 = \sum_{i=1}^{n} |X_i|^2$, $X = (x_1, \; x_n) \in \mathbb{C}^n$.

We have by definition $\|Ax\| \leq \|A\| \|x\|$ for all $x \in \mathbb{C}^n$.

The operator norm is sub-multiplicative

$$\|AB\| := \sup_{\substack{x \in \mathbb{C}^n, \\ \|x\| \leq 1}} \|ABx\| \leq \sup_{\substack{x \in \mathbb{C}^n, \\ \|x\| \leq 1}} \|A\| \|Bx\|$$

so, $\|AB\| \leq \|A\| \|B\|$.

Iterating, we obtain $\|A^k\| \leq \|A\|^k \quad \forall \; k \geq 1$.

Fact: The set of matrices $A \in \mathbb{C}^{n \times n}$ is a Banach space with respect to the above matrix norm, that is, The set of matrices $A \in \mathbb{C}^{n \times n}$ forms a vector sp., $\| \cdot \|$ defined above is a norm and Cauchy sequences of matrices with complex entries converges with respect to the above matrix norm.

Note, for a matrix $A \in \mathbb{C}^{n\times n}$,

$$\| \exp(A) \| = \left\| \lim_{m\to+\infty} \sum_{i=0}^{m} \frac{A^i}{i!} \right\| \leq \lim_{m\to+\infty} \sum_{i=0}^{m} \frac{\|A^i\|}{i!}$$

$$\leq \sum_{i=0}^{+\infty} \frac{\|A\|^i}{i!} = e^{\|A\|},$$ and hence the infinite sum defining $\exp(A)$ converges.

Example 4: Consider a diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \vdots \\ 0 & \cdots & & \lambda_n \end{pmatrix}$.

Then $D^2 = \begin{pmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$, and by induction $D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ 0 & \lambda_2^k & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix}$

for all $k \geq 1$. Term by term addition gives for all $t \in \mathbb{R}$

$$\exp(tD) = \sum_{i=0}^{+\infty} \frac{t^i}{i!} D^i = \begin{pmatrix} \sum_{i=0}^{+\infty} \frac{t^i}{i!}\lambda_1^i & & 0 \\ 0 & \sum_{i=0}^{+\infty} \frac{t^i}{i!}\lambda_2^i & \\ 0 & \cdots & \sum_{i=0}^{+\infty} \frac{t^i}{i!}\lambda_n^i \end{pmatrix}$$

And so, $\exp(D) = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$, also a diagonal matrix.

Example 5: Consider $A = UDU^{-1}$, where $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

and $\det U \neq 0$.

$\to$ It follows by induction $A^k = (UDU^{-1})^k = UD^kU^{-1}, \forall\, k \geq 1$.

and so $\exp(A) = U \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} U^{-1} = U \exp(D) U^{-1}$.

Proposition 1: Let $A, B \in \mathbb{C}^{n\times n}$ be two $(n\times n)$ commuting matrices, that is $AB = BA$, then

$$\exp(A+B) = \exp(A) \cdot \exp(B) \qquad \text{(matrix product)}$$

Proof: We have since $AB = BA$

$$(A+B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \qquad \text{by using the Binomial theorem}$$

Then $\frac{1}{k!}(A+B)^k = \sum_{i=0}^{k} \frac{1}{(k-i)!\, i!} A^i B^{k-i}$, and

$$\sum_{k=0}^{2m} \frac{1}{k!}(A+B)^k = \sum_{k=0}^{2m} \sum_{i=0}^{k} \frac{1}{(k-i)!\, i!} A^i B^{k-i} = \sum_{\substack{i+j \leq 2m, \\ i\geq 0, j \geq 0}} \frac{1}{i!\, j!} A^i B^j$$

Hence $\sum_{k=0}^{2m} \frac{1}{k!}(A+B)^k - \left(\sum_{i=0}^{m} \frac{1}{i!} A^i\right)\left(\sum_{j=0}^{m} \frac{1}{j!} B^j\right)$

$$= \sum_{i,j\geq 0;\ i+j \leq 2m;\ \max\{i,j\} > m} \frac{1}{i!\, j!} A^i B^j$$

This then implies, taking the matrix norm

$$\left\| \sum_{k=0}^{2m} \frac{1}{k!}(A+B)^k - \left(\sum_{i=0}^{m} \frac{1}{i!}A^i\right)\left(\sum_{j=0}^{m}\frac{1}{j!}A^j\right)\right\|$$

$$\leq \left(\sum_{i=m+1}^{2m} \frac{1}{i!}\|A\|^i\right)\left(\sum_{j=m+1}^{2m}\frac{1}{j!}\|A\|^j\right)$$

Letting $m \to +\infty$, since the sums in the right hand side $\to 0$ as $m \to +\infty$, we obtain

$$\sum_{k=0}^{+\infty} \frac{1}{k!}(A+B)^k = \left(\sum_{i=0}^{+\infty}\frac{1}{i!}A^i\right)\left(\sum_{j=0}^{+\infty}\frac{1}{j!}A^j\right),$$

that is $\exp(A+B) = \exp A \cdot \exp B$.

Corollary: a) $\exp(A)\exp(-A) = \exp(A-A) = \exp 0 = I$ since $(A)(-A) = (-A)A$

hence, $(\exp(A))^{-1} = \exp(-A)$.

b) $\exp(tA)\cdot\exp(sA) = \exp((t+s)A)$, since $(tA)\cdot(sA) = (sA)\cdot(tA)$, where $t,s \in \mathbb{R}$.

Planar Linear Systems

Consider the linear system: $X' = AX$, where $X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

<u>Proposition</u>. If $\det A \neq 0$, then $X(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only sol. s.t $X'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. (steady state / equilibrium sol.)

If $\det A = 0$ and if $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then there exists a sol $X(t) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ s.t $X'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

In particular if $a \neq 0$ then $X(t) = \begin{pmatrix} -\frac{b}{a}\lambda \\ \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}$ solves $X'(t) = AX$ and $X'(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The proof of this proposition follows from linear algebra.

Eigenvalues and Eigenvectors

Def: $V \in \mathbb{R}^2$, $V \neq 0$ is called an eigenvector of $A = (a_{ij})_{1 \le i,j \le 2}$, if $AV = \lambda V$ for some λ.

λ is called an eigenvalue of A.

Theorem: Suppose λ is an eigenvalue of A and $V \in \mathbb{R}^2$, $V \neq 0$ is an eigenvector corresponding to λ, that is $AV = \lambda V$.

Then $X(t) = e^{\lambda t} V$ is a sol. of $X' = AX$.

Proof: $X(t) = e^{\lambda t} V$, $X'(t) = \lambda e^{\lambda t} V$ (componentwise differentiation)

$$= e^{\lambda t} A V$$
$$= A X(t)$$

Given $A = (a_{ij})_{1 \le i,j \le 2}$, to find its eigenvalues and eigenvectors, we consider the linear eqn:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

The vetor $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector with eigenvalue λ.

We write the eqn: $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ as $(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

The linear eqn $(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has only the trivial sol $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when $\det(A - \lambda I) \neq 0$.

Conversely, if $\det(A-\lambda I)=0$, then the eqn $A\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}$ has non zero sols. The eqn $\det(A-\lambda I)=0$ is a quadratic eqn. in λ and is called the characteristic eqn, and as a function of λ $\det(A-\lambda I)$ is a quadratic eqn in λ called the <u>characteristic polynomial</u>.

First we find the roots of the characteristic eqn $\det(A-\lambda I)=0$, and then find the corresponding eigenvalues.

<u>Example 1</u> Consider $A=\begin{pmatrix}1 & 3\\1 & -1\end{pmatrix}$

The characteristic eqn is $\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda & 3\\1 & -1-\lambda\end{pmatrix}=0.$

That is, $(1-\lambda)(-1-\lambda)-3=0$, or $-1-\lambda+\lambda+\lambda^2-3=0$,
or $\lambda^2-4=0$. so $\lambda=\pm 2$.

A has eigenvalues $\lambda_1=-2$, $\lambda_2=2$.

An eigenvector $V_1=\begin{pmatrix}x\\y\end{pmatrix}$ corresponding to $\lambda_2=2$ satisfies: $\begin{pmatrix}1 & 3\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=2\begin{pmatrix}x\\y\end{pmatrix}$.

or, $\begin{pmatrix}x+3y\\x-y\end{pmatrix}=\begin{pmatrix}2x\\2y\end{pmatrix}$ or $\begin{pmatrix}-x+3y\\x-3y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ or $x-3y=0$.

Choosing $x=3$, $y=1$, we obtain that $V_1=\begin{pmatrix}3\\1\end{pmatrix}$ is an eigenvector of A with eigenvalue 2. Note any scalar multiple $\alpha V_1, \alpha\in\mathbb{R}$, is also an eigenvector with eigenvalue $\lambda_2=2$.

Similarly, an eigenvector corresponding to $\lambda_1=-2$ satisfies

$\begin{pmatrix}1 & 3\\1 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=-2\begin{pmatrix}x\\y\end{pmatrix}$ or $\begin{pmatrix}3x+3y\\x+y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ or $x+y=0$.

Choosing $x=1, y=-1$, we obtain that $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda_1 = -2$.

So, we have obtained three sols. to the system: $X' = AX$,

$$X_0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; ; \; X_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \; ; \; X_2(t) = e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

<u>Remark</u>: Are these all the sols. to $X' = AX$.
Are the sols. we obtained linearly independent

<u>Note</u>: $\lim_{t\to+\infty} |X_1(t)| = 0$ and $\lim_{t\to+\infty} |X_2(t)| = +\infty$.

<u>Linear Independance</u>. The eigenvectors V_1, V_2 corresponding to eigenvalues $\lambda_1 < \lambda_2$ are linearly independent

($V_1 = \alpha V_2 \Rightarrow AV_1 = \alpha AV_2 \Rightarrow \lambda_1 V_1 = \alpha \lambda_2 V_2 = \lambda_2 V_1$.
A contradiction since $\lambda_1 < \lambda_2$).

Consider $Z(t) = \alpha X_1(t) + \beta X_2(t)$ where $X_1(t), X_2(t)$ are as above.

Then, $Z'(t) = \alpha X_1'(t) + \beta X_2'(t) = \alpha A X_1(t) + \beta A X_2(t)$.

so, $Z'(t) = A Z(t)$.

<u>Theorem</u> : Suppose $A = (a_{ij})_{1 \le i,j \le 2}$ has real eigenvalues $\lambda_1 \neq \lambda_2$ and corresponding eigenvectors V_1 and V_2. Then the general solution of the linear system $X' = AX$ is given by

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2,$$

where $\alpha, \beta \in \mathbb{R}$.

<u>Proof</u>: Differentiating $X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2$, we get

$$X'(t) = \alpha e^{\lambda_1 t} \underbrace{\lambda_1 V_1}_{= AV_1} + \beta e^{\lambda_2 t} \underbrace{\lambda_2 V_2}_{AV_2} = A(\alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2) = AX.$$

The matrix A has distinct real eigenvalues λ_1 and λ_2 and suppose V_1 and V_2 are the corresponding eigenvectors. Then V_1 and V_2 are linearly independent. Let T be the matrix whose colums are V_1 and V_2. Then $\det T \neq 0$ and

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = V_1, \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = V_2.$$

We can then write $AT = T\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

The general sol of the system $Y' = (T^{-1}AT)Y$ is given by

$$Y(t) = \alpha e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the general sol. of the system $X' = AX$ is

$$X(t) = TY(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2.$$

Example 2: Consider $x'' + 3x' + 2x = 0$.

We can write this as $X' = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{A} X$,

where $X = \begin{pmatrix} x \\ x' \end{pmatrix}$. We look for the general sol. of the above system.

We need to find the eigenvalues of A and the corresponding eigenvectors.

The characteristic eqn is $\det(A - \lambda I) = 0$ or $\det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = 0$,

so $(3+\lambda)\lambda + 2 = 0$ or $\lambda^2 + 3\lambda + 2 = 0$
$\Rightarrow \lambda = -2, -1$.

So the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -1$.

The eigenvector V_1 corresponding to the eigenvalue $\lambda_1 = -2$ satisfies $AV_1 = -2V_1$ or $V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies $\begin{pmatrix} y \\ -2x-3y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$

or $2x + y = 0$. hence $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$x+y$

Similary $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda_2 = -1$.

Therefore, using our previous result, the general sol. of the given system is

$$X(t) = \alpha e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The case of Real Distinct Eigenvalues.

Consider the linear system in $\mathbb{R}^2$: $X'(t) = AX(t)$ where the matrix $A = (a_{ij})_{1\leq i,j\leq 2}$ has two real eigenvalues $\lambda_1 < \lambda_2$.

Example 3: (Saddle) Consider $X' = AX$ where $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, with $\lambda_1 < 0 < \lambda_2$.

A has eigenvalues λ_1, λ_2 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

The general sol of the above system is given by

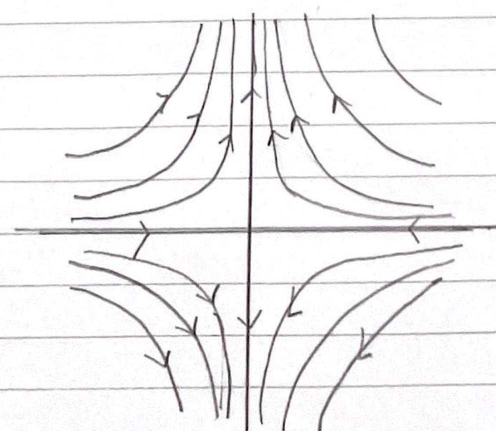
$$X(t) = \alpha e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lambda_1 < 0 < \lambda_2$$

$\alpha, \beta \in \mathbb{R}$.

Sol. of the form $\alpha e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ lie on the x-axis and tend to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to +\infty$.

Sol. of the form $\beta e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ lie on the y-axis and $\left| e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right| \to +\infty$ as $t \to +\infty$.

The general sol (with $\alpha, \beta \neq 0$) tend to $\pm\infty$ in the direction of the y-axis



Phase Portrait for $\begin{cases} x' = -1 \\ y' = 1 \end{cases}$ (Saddle)

→ In the general case where A has a positive and a negative eigenvalue, we can always find a stable line and an unstable line on which sol. tend towards or away from the origin. In general, sols approach the unstable line as $t \to +\infty$ and tend towards the stable line as $t \to -\infty$.

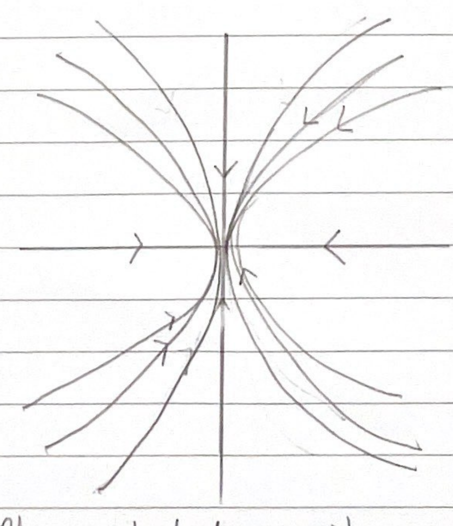
<u>Example 4</u>: (Sink) Consider $X' = AX$ where $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, with $\lambda_1 < \lambda_2 < 0$.

The general sol. is given by $X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\lambda_1 < \lambda_2 < 0$ and $\alpha, \beta \in \mathbb{R}$.

In this case all solutions tend to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to +\infty$. Since $|\lambda_1| > |\lambda_2|$, the x-coordinate of the general sol. tends to 0 much faster.

We have $\begin{cases} x(t) = \alpha e^{\lambda_1 t} \\ y(t) = \beta e^{\lambda_2 t} \end{cases}$ so $\frac{dy}{dx} = \frac{(\beta e^{\lambda_2 t})'}{(\alpha e^{\lambda_1 t})'} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t}$

Thus the sols become "vertical" and tends to the origin tangentially to the y-axis.



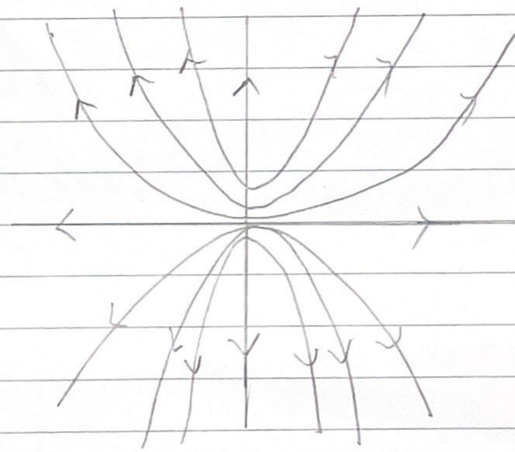
Phase portrait for a sink.

<u>Example 5</u>: (source) Consider $X' = AX$ where $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

where $\lambda_2 > \lambda_1 > 0$.

The general sol. is given by $X(t) = \alpha e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In this case all sols. tend away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to +\infty$



Phase - potrait for a source

The case of Complex Eigenvalues

<u>Example</u> (Center) Consider $X' = AX$ with $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$, $\beta \neq 0$

The characteristic equation is $\det(A - \lambda I) = 0$, that is, $\lambda^2 + \beta^2 = 0$. So $\lambda = \pm i\beta$. We write $\lambda_1 = i\beta$, $\lambda_2 = -i\beta$.

The eigenvector $V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to $\lambda_1 = i\beta$ satisfies

$$AV_1 = i\beta V_1 \quad \text{or} \quad \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = i\beta\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \beta y \\ -\beta x \end{pmatrix} = i\beta\begin{pmatrix} x \\ y \end{pmatrix}$$

so $i\beta x = \beta y$ and $V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Similarly a eigenvector corresponding to $\lambda_2 = -i\beta$ is given by $V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

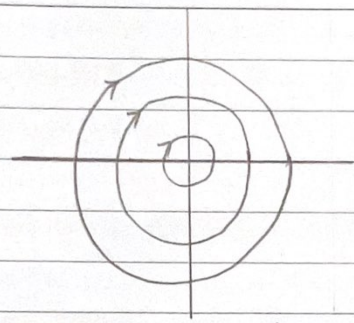
Then the general sol. of the linear system $X' = AX$ is given by

$X(t) = C_1 e^{i\beta t}\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-i\beta t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$, which can be written using Euler's formula.

$$X(t) = (C_1 + C_2)\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + i(C_1 - C_2)\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$$

$X(t) = C\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + iD\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$ where C, D are constants.

The sol. is periodic with period $2\pi/\beta$.



The sol's lie on circles centered at the origin. The circles are traversed in the clockwise direction if $\beta > 0$, counterclockwise if $\beta < 0$

Phase potrait for a center.

<u>Example</u> (Spiral Sink / Source) Consider $X' = AX$, where $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\alpha, \beta \neq 0$.

The characteristic eqn is given by $\det \begin{pmatrix} \alpha-\lambda & \beta \\ -\beta & \alpha-\lambda \end{pmatrix} = 0$, or $(\alpha-\lambda)^2 + \beta^2 = 0$

So $\lambda = \alpha \pm i\beta$, and the eigenvalues of A are given by $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$.

The eigenvector V_1 corresponding to λ_1 satisfies $\begin{pmatrix} \alpha x + \beta y \\ -\beta x + \alpha y \end{pmatrix} = \alpha + i\beta \begin{pmatrix} x \\ y \end{pmatrix}$,

or $\beta y = i\beta x$. So, $V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = \alpha + i\beta$.

Similarly $V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector corresponding to $\lambda_2 = \alpha - i\beta$.

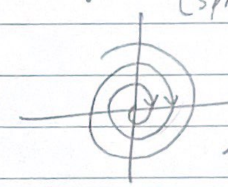
The general sol to $X' = AX$ is then given by

$$X(t) = C_1 e^{(\alpha + i\beta)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(\alpha - i\beta)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Then the general sol. can be written in the form

$X(t) = C e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + i D e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}$ where C, D are constants.

The $e^{\alpha t}$ makes the sols. spiral into the origin if $\alpha < 0$ (spiral sink) or away from the origin if $\alpha > 0$ (spiral source).



spiral sink

The case of Repeated Eigenvalues

Example. Consider $X' = AX$, where $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Here the general sol. is of the form $X(t) = e^{\lambda t} V$ where $V \in \mathbb{R}^2$.

Example. Consider $X' = AX$, where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}$.

In this case both the eigenvalues of A equals λ (repeated roots).

$V = \begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector of A if $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$, $\Rightarrow V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $X_1(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a sol, corresponding to

$x(t) = e^{\lambda t}, \; y(t) = 0.$

Note that the system of ODEs can be written $\begin{cases} x' = \lambda x + y \\ y' = \lambda y \end{cases}$

Thus, no-trivial sols to $y' = \lambda y$ are given by $y(t) = \beta e^{\lambda t}$, and in this case

$$x'(t) = \lambda x + \beta e^{\lambda t}.$$

The general sol to the above ODE is given by $x(t) = c\, e^{\lambda t} + \beta t e^{\lambda t}$.

Hence the general sol. to the given system $X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X$ is given by

$$X(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

If $\lambda < 0$, each term of the sol. tends to 0 as $t \to +\infty$.
If $\lambda > 0$ all sols. tend away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Example 5: Consider $A = UDU^{-1}$, where $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$

and $\det U \neq 0$.

→ It follows by induction $A^k = (UDU^{-1})^k = UD^kU^{-1}, \forall k \geq 1$.

and so $\exp(A) = U \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} U^{-1} = U \exp(D) U^{-1}$.

Proposition 1: Let $A, B \in \mathbb{C}^{n\times n}$ be two $(n\times n)$ commuting matrices, that is $AB = BA$, then

$$\exp(A+B) = \exp(A) \cdot \exp(B) \quad \text{(matrix product.)}$$

Proof: We have since $AB = BA$

$$(A+B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \quad \text{by using the Binomial theorem}$$

Then $\frac{1}{k!}(A+B)^k = \sum_{i=0}^{k} \frac{1}{(k-i)!\,i!} A^i B^{k-i}$, and

$$\sum_{k=0}^{2m} \frac{1}{k!}(A+B)^k = \sum_{k=0}^{2m} \sum_{i=0}^{k} \frac{1}{(k-i)!\,i!} A^i B^{k-i} = \sum_{\substack{i+j\leq 2m,\\ i\geq 0, j\geq 0}} \frac{1}{i!\,j!} A^i B^j$$

Hence $\sum_{k=0}^{2m} \frac{1}{k!}(A+B)^k - \left(\sum_{i=0}^{m} \frac{1}{i!} A^i\right)\left(\sum_{j=0}^{m} \frac{1}{j!} B^j\right)$

$$= \sum_{i,j\geq 0;\ i+j\leq 2m;\ \max\{i,j\}>m} \frac{1}{i!\,j!} A^i B^j$$

This then implies, taking the matrix norm

$$\left\| \sum_{k=0}^{2m} \frac{1}{k!}(A+B)^k - \left(\sum_{i=0}^{m} \frac{1}{i!}A^i\right)\left(\sum_{j=0}^{m}\frac{1}{j!}A^j\right)\right\|$$

$$\leq \left(\sum_{i=m+1}^{2m} \frac{1}{i!}\|A\|^i\right)\left(\sum_{j=m+1}^{2m}\frac{1}{j!}\|A\|^j\right)$$

Letting $m \to +\infty$, since the sums in the right hand side $\to 0$ as $m \to +\infty$, we obtain

$$\sum_{k=0}^{+\infty} \frac{1}{k!}(A+B)^k = \left(\sum_{i=0}^{+\infty}\frac{1}{i!}A^i\right)\left(\sum_{j=0}^{+\infty}\frac{1}{j!}A^j\right),$$

that is $\exp(A+B) = \exp A \cdot \exp B$.

Corollary: a) $\exp(A)\exp(-A) = \exp(A-A) = \exp 0 = I$ since $(A)(-A) = (-A)A$

hence, $(\exp(A))^{-1} = \exp(-A)$.

b) $\exp(tA)\cdot\exp(sA) = \exp((t+s)A)$, since $(tA)\cdot(sA) = (sA)\cdot(tA)$, where $t, s \in \mathbb{R}$.

Lemma Consider $U \in \mathbb{C}^{n\times n}$ s.t. $\det U \neq 0$ (non-singular matrix). Then

$$\exp(U^{-1}AU) = U^{-1}\exp A\, U.$$

Pf. $$\exp(U^{-1}AU) = \sum_{i=0}^{+\infty}\frac{1}{i!}(U^{-1}AU)^i = \sum_{i=0}^{+\infty}\frac{U^{-1}A^iU}{i!}$$

hence $$U\exp(U^{-1}AU)U^{-1} = \sum_{i=0}^{+\infty}\frac{A^i}{i!} = \exp A,$$

and $$\exp(U^{-1}AU) = U^{-1}\exp A\, U.$$

Example 6: Consider $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$

The eigenvalues of A are given by the roots of $\det(A - \lambda I) = 0$, that is
$(\alpha - \lambda)^2 + \beta^2 = 0$, $\Rightarrow \lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$.

$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector with eigenvalue λ_1, that is, $Av_1 = \lambda_1 v_1$.

$v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvalue with Eigenvalue λ_2, that is, $Av_2 = \lambda_2 v_2$.

Then, $A = U \begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix} U^{-1}$, where $U = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

Therefore $\exp(A) = U \begin{pmatrix} e^{\alpha + i\beta} & 0 \\ 0 & e^{\alpha - i\beta} \end{pmatrix} U^{-1} = e^{\alpha t} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$.

Jordan Canonical form.

Let $A \in \mathbb{C}^{n \times n}$ (complex $n \times n$ matrix.) Then $\exists$ non-singular matrix U st

$$U^{-1}AU = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix}$$

with each block B_i of the form $B_i = \lambda_i I + N$

$$= \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

N is a nilpotent matrix, with ones in the first diagonal above the main diagonal and zeros everywhere else.

λ_i's are the eigenvalues of A and the columns of U consists of generalized eigenvectors of A.

Calculating $\exp A$ for a general A

For any block $B = \lambda I + N$, $\exp B = \exp(\lambda I)\exp N$ since $IN = NI$ as I is the Id. matrix.

$$= e^{\lambda}(I \exp N)$$

$$(\text{since } \exp \lambda I = e^{\lambda} I)$$

$$= e^{\lambda} \exp N.$$

We note that $N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}_{k \times k}$, so (by induction) $N^k = 0$ matrix.

So $\exp N = \sum_{i=0}^{+\infty} \frac{N^i}{i!} = \sum_{i=0}^{k-1} \frac{N^i}{i!} = \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(k-1)!} \\ & 1 & 1 & \ddots & \frac{1}{(k-2)!} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & \frac{1}{2!} \\ & & & & 1 \\ & & & & 1 \end{pmatrix}_{k \times k}$

So,

$$\exp B = e^{\lambda}\begin{pmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(k-1)!} \\ & 1 & 1 & & \vdots \\ & & 1 & & \\ & & & \ddots & \frac{1}{2!} \\ & & & & 1 \\ & & & & 1 \end{pmatrix}$$

And then using the Jordan canonical form

$$\exp A = \mathcal{U}\begin{pmatrix} \exp B_1 & & 0 \\ & \ddots & \\ 0 & & \exp B_m \end{pmatrix}\mathcal{U}^{-1}.$$

Lemma. $\det(\exp A) = \exp(\mathrm{Tr}(A))$ where $\mathrm{Tr}\, A = \sum_{i=1}^{n} a_{ii}$ (sum of the diagonal)

Pf: Using the Jordan canonical form we have

$$\det(\exp A) = \det\left(\mathcal{U}\begin{pmatrix} \exp B_1 & & 0 \\ & \ddots & \\ 0 & & \exp B_m \end{pmatrix}\mathcal{U}^{-1}\right)$$

$$= \det\begin{pmatrix} \exp B_1 & & \\ & \ddots & \\ & & \exp B_m \end{pmatrix} = \prod_{i=1}^{m} \det(\exp B_i)$$

$$= \prod_{i=1}^{m} \exp(\mathrm{Tr}(B_i)) = \exp(\mathrm{Tr}(A)).$$

Lemma Let $A \in \mathbb{R}^{n\times n}$. The function $t \mapsto \exp(tA)$ is differentiable and $\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)\cdot A$

Proof: We have $\exp(tA) = \sum_{i=0}^{+\infty} \frac{(tA)^i}{i!}$. Then $\frac{1}{h}\left(\exp(hA) - I\right) = \sum_{i=1}^{+\infty} \frac{h^{i-1}}{i!} A^i$

Hence $\lim_{h\to 0} \frac{\exp(hA) - I}{h} = A$, and we get

$$\lim_{h\to 0} \frac{\exp((t+h)A) - \exp(tA)}{h} = \exp(tA)\left(\lim_{h\to 0} \frac{\exp hA - I}{h}\right) = \exp(tA)\cdot A$$

$$\left[\text{or } \left(\lim_{h\to 0} \frac{\exp hA - I}{h}\right)\exp(tA) = A\cdot \exp(tA)\right]$$

We state the existence and uniqueness of sol. for an autonumous linear system.

Theorem Given any (vector) $X_0 \in \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{n\times n}$, there exits a <u>unique solution</u> of the ODE: $\begin{cases} X'(t) = AX(t) \\ X(0) = X_0 \end{cases}$.

Moreover, the solution $X(t) = \exp(tA)X_0$.

Proof: Suppose $X(t)$ solves the given initial value p'bm then

$$\begin{aligned} \frac{d}{dt}\left(\exp(-tA)X(t)\right) &= \exp(-tA)X'(t) + \exp(-tA)'X(t) \\ &= \exp(-tA)X'(t) - \exp(-tA)\underbrace{AX(t)}_{X'(t)} \\ &= 0. \end{aligned}$$

Hence $\exp(-tA)X(t) \equiv \text{const}$. For $t=0$, $\exp(0A)X_0 = \text{const}$.

Therefore, $X(t) = \exp(tA)X(t)$ as $\exp(-tA)^{-1} = \exp(tA)$.

Corollary: Consider a matrix $A \in \mathbb{R}^{n\times n}$. Then the set of all sols. to the ODE: $\{ X'(t) = AX$ forms a vector sp. of dimension n

Pf:

Let $Z(t) = \alpha X_1(t) + \beta X_2(t)$ where $\alpha, \beta \in \mathbb{R}$ and $X_i'(t) = AX_i$, $i=1,2$.

Then
$$Z'(t) = \alpha X_1'(t) + \beta X_2'(t) = \alpha AX_1(t) + \beta AX_2(t)$$
$$= A(\alpha X_1(t) + \beta X_2(t)) = AZ(t).$$

(Principle of superposition)

Given any $X_0 \in \mathbb{R}^n$, $X(t) = \exp(tA)X_0$ solves the initial value p'blm $\begin{cases} X'(t) = AX(t) \\ X(0) = X_0 \end{cases}$.

Conversely, using our previous theorem, given any sol. $\phi(t)$ of $X'(t) = AX(t)$ we have $\phi(t) = \exp(tA)\phi(0)$.

Hence $X_0 \longmapsto \exp(tA)X_0$ is an isomorphism from $\mathbb{R}^n$ to the vector sp. of sols.

Dynamics of Planar Linear Systems.

Consider $\begin{cases} X'(t) = AX \\ X(0) = X_0 \end{cases}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A \in \mathbb{R}^{2\times n}$

A has two eigenvalues λ_1 and λ_2. Both λ_1 and λ_2 are real, or λ_1 and λ_2 are complex no. with $\lambda_2 = \overline{\lambda_1}$.

Case A: A is diagonalizable and hence there exists two linearly independent eigenvectors V_1 and V_2. We have
$$U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{where } U = (V_1, V_2)$$

With a change of coordinates $Y(t) = U^{-1}X(t)$, $Y_0 = U^{-1}X_0$, the eqn transforms as

$$Y'(t) = U^{-1}X'(t) = U^{-1}AX(t) = (U^{-1}AU)\,Y(t).$$

Hence, $Y(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} Y_0$ and transforming back we obtain the sol. to the original system $X(t) = \left[U \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} U^{-1} \right] X_0$, or

$$X(t) = y_{0,1} e^{\lambda_1 t} V_1 + y_{0,2} e^{\lambda_2 t} V_2, \quad \text{where} \quad \begin{pmatrix} y_{0,1} \\ y_{0,2} \end{pmatrix} = Y_0 = U^{-1} X_0.$$

<u>Case A(i)</u>: $\lambda_1, \lambda_2 \in \mathbb{R}$. Then the sol. to the initial value to the initial value p'blm $\begin{cases} X'(t) = AX \\ X(0) = X_0 \end{cases}$ is given as above by

$$X(t) = y_{0,1} e^{\lambda_1 t} V_1 + y_{0,2} e^{\lambda_2 t} V_2 \quad \text{where} \quad \begin{pmatrix} y_{0,1} \\ y_{0,2} \end{pmatrix} = U^{-1} X_0; \; AV_i = \lambda_i V_i \text{ for } i = 1,2.$$

<u>Case A(ii)</u>: $\lambda_1, \lambda_2 \in \mathbb{C}$. Then $\lambda_2 = \overline{\lambda_1}$, $V_2 = \overline{V_1}$ and we let $\begin{cases} \lambda_1 = \alpha + i\beta \\ \lambda_2 = \alpha - i\beta \end{cases}$

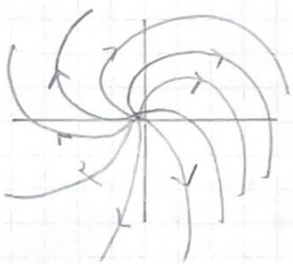
We have $\underbrace{(V_1, V_2)}_{U} \underbrace{\begin{pmatrix} y_{0,1} \\ y_{0,2} \end{pmatrix}}_{Y_0} = \underbrace{\begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}}_{X_0} \in \mathbb{R}^2.$

Taking conjugates this implies $\overline{y_{0,1}} V_2 + \overline{y_{0,2}} V_1 = y_{0,1} V_1 + y_{0,2} V_2$, therefore $y_{0,2} = \overline{y_{0,1}}$. As above the sol. to the initial value p'blm is then given by

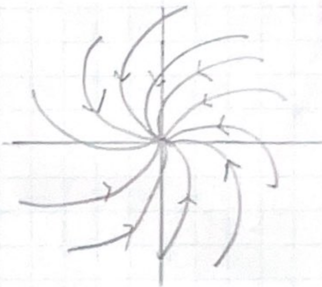
$$X(t) = y_{0,1} e^{(\alpha + i\beta)t} V_1 + y_{0,2} e^{(\alpha - i\beta)t} V_2$$
$$= e^{\alpha t} \left(y_{0,1} e^{i\beta t} V_1 + y_{0,2} e^{-i\beta t} V_2 \right) = 2e^{\alpha t} \operatorname{Re}\left(y_{0,1} e^{i\beta t} V_1 \right)$$
$$= 2e^{\alpha t} (\cos \beta t) \operatorname{Re}(y_{0,1} V_1) - 2e^{\alpha t} (\sin \beta t) \operatorname{Im}(y_{0,1} V_1).$$

Qualitative behavior for large t.

Source : Both eigenvalues have positive real part. Then all solutions grow exponentially as $t \to +\infty$



Source

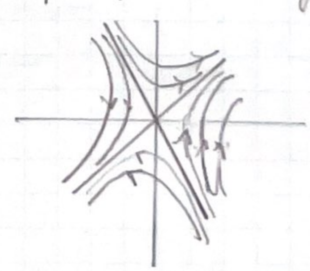


Sink

Sink : Both eigenvalues have negative real part. Then all solutions decay exponentially as $t \to +\infty$.

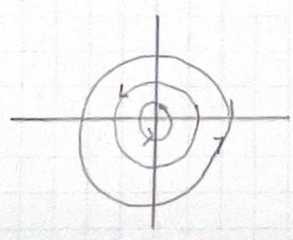
Saddle : If one eigenvalue is positive and the other eigenvalue is negative, the origin is called a saddle.

The long-time behaviour depends on the initial condition X_0. If X_0 lies in the eigenspace corresponding to the negative eigenvalue, the sol will decay exponentially as $t \to +\infty$. If X_0 lies in the eigenspace corresponding to the positive eigenvalue, the sol. will grow exponentially as $t \to +\infty$. If X_0 has components from both eigenspaces, it will grow exponentially as $t \to \pm\infty$.



Saddle.

Center : If both eigenvalues are purely imaginary, the origin is called a center. The solutions are periodic and encircle the origin. All sol. remain bounded.



Center.

Case B: A is not diagonalizable, then $\lambda_1 = \lambda_2 = \lambda$ and we have

$$U^{-1} A U = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{where} \quad U = (V_1, V_2)$$

V_1 is the eigenvector and V_2 is the generalised eigenvector

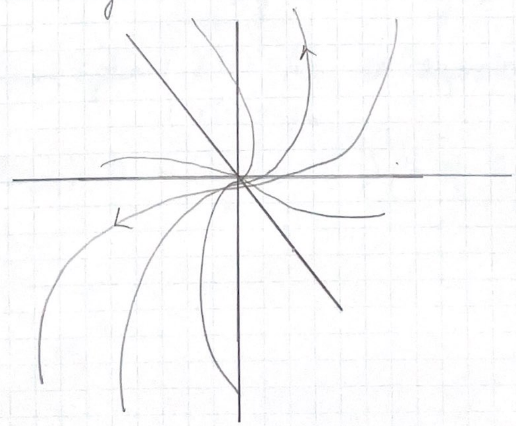
In this case, $U^{-1} X(t) := Y(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} Y_0$

and $X(t) = e^{\lambda t} (y_{0,1} + y_{0,2} t) V_1 + y_{0,2} e^{\lambda t} V_2$.

Qualitative Behaviour

If $\lambda < 0$ all sol. tends to 0, and if $\lambda > 0$ all sol. grow exponentially as $t \to +\infty$.

If $\lambda = 0$, the sol. is const if we start in the subspace spanned by the eigenvector and grows like t otherwise.



<u>Theorem</u> (without proof.) A sol. of the linear system $\begin{cases} X'(t) = AX \\ X(0) = X_0 \end{cases}$ converges to 0 as $t \to +\infty$ iff X_0 lies in the subspace spanned by the generalized eigenspaces corresponding to eigenvalues with negative real part.

The sol. will remain bounded as $t \to +\infty$ iff X_0 lies in the subspace spanned by generalized eigenspaces corresponding to eigenvalues with negative real part plus the eigenspaces corresponding to eigenvalues with vanishing real part.

<u>Definition</u> A linear system is said to be <u>stable</u> if all solutions remain bounded as $t \to +\infty$ and <u>asymptotically stable</u> if all solutions tend to 0 as $t \to +\infty$.

<u>Corollary</u>: a) The linear system $\begin{cases} X'(t) = AX \\ X(0) = X_0 \end{cases}$ is stable iff $Re(\lambda_i) \leq 0$ for all eigenvalues λ_i's of A, with the algebraic and geometric multiplicity being same for eigenvalues λ s.t. $Re(\lambda) = 0$.

b) The linear system $\begin{cases} X'(t) = AX \\ X(0) = X_0 \end{cases}$ is asymptotically stable iff $Re(\lambda_i) < 0$ $\forall$ eigenvalues λ_i's of A.

Theorem: (Inhomogeneous eqn.) Let $A \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix, and let $f : I \to \mathbb{R}^n$ be a continuous fn. defined on an open interval I containing 0. Then there exists a unique solution of the initial value p'blm
$$\begin{cases} X'(t) = AX(t) + f(t) \\ X(0) = X_0 \end{cases}.$$

Moreover, the sol. is given by
$$X(t) = \exp(tA)X_0 + \int_0^t \exp((t-s)A) f(s)\, ds.$$
(Duhamel's formula)
The sol. consists of the general sol. of the homogeneous eqn. plus a particular sol. of the inhomogeneous eqn.

Proof: Suppose $X(t)$ solves the initial value p'blm, then
$$\frac{d}{dt}\left(\exp(-tA)X(t)\right) = \exp(-tA)X'(t) - \exp(-tA)AX(t)$$
$$= \exp(-tA)f(t).$$

Integrating this,
$$\int_0^t \left(\exp(-sA)X(s)\right)' ds = \int_0^t \exp(-sA)f(s)\,ds$$
$$\Rightarrow \exp(-tA)X(t) - X_0 = \int_0^t \exp(-sA)f(s)\,ds$$
$$\Rightarrow X(t) = \exp(tA)X_0 + \int_0^t \exp((t-s)A)f(s)\,ds,$$ since $(\exp(tA))^{-1} = \exp(-tA)$.

Linear Autonomous equations of order n.

Consider the n-th order eqn: $\left\{ x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0, \right.$

where $a_i \in \mathbb{R}$, $0 \le i \le n-1$, with the initial conditions: $\left\{ x(0) = x_0, \ldots, x^{(n-1)}(0) = x_{n-1} \right.$.

Here $x^{(k)}(t) := \frac{d^k}{dt^k} x(t)$.

Letting $X(t) = \begin{pmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{pmatrix}$, the n-th order eqn becomes a first order linear system $X'(t) = AX$

where $A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix}$.

The characteristic eqn is given by $\det(A - \lambda I) =$

$$(-1)^n \left(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots a_1\lambda + a_0 \right) = 0$$

Theorem

Let λ_i, $1 \le i \le k$ be the roots of the characteristic eqn:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} = 0.$$

Here m_i is the multiplicity of λ_i. Then

$$x_{i,j}(t) = t^j \exp(t\lambda_i), \quad 0 \le j < m_j, \quad 1 \le i \le k$$

are n linearly independent solutions of: $\left\{ x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0. \right.$

Any other sol. can be written as a linear combination of $x_{i,j}$'s

If $\lambda_i = \alpha_i + i\beta_i$ a complex no. then taking real and imaginary parts

$$X_{i,j}(t) = t^j e^{t\alpha_i} \cos \beta_i t \ , \quad \tilde{X}_{i,j}(t) = t^j e^{t\alpha_i} \sin \beta_i t \ .$$

Inhomogeneous Case

Consider the inhomogeneous eqn:

$$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = f(t)$$

Then, transforming our eqn into a first order system, we have that the sol. of the above inhomogeneous eqn. is given by

$$x(t) = x_h(t) + \int_0^t u(t-s) f(s)\, ds \ ,$$

where $x_h(t)$ is a sol. of the homogeneous eqn, and $u(t)$ is the sol. of the homogeneous eqn. corresponding to the initial condition $u(0) = u'(0) = \cdots u^{(n-2)}(0) = 0, \ u^{(n-1)}(0) = 1$.

Algorithm for solving a linear n-th order ODE with constant coefficients

A) Consider the corresponding homogeneous eqn.

- Compute the roots of the characteristic eqn.
- Write the general sol of the homogeneous eqn as a linear combination of fundamental sols.

B.) Find a particular sol. of the inhomogeneous eqn

Example Consider $\{ x'' + \omega_0^2 x = \cos(\omega t)$, where $\omega_0, \omega \geq 0$.

We first find the sol. of the homogeneous eqn: $x'' + \omega_0^2 x = 0$. The characteristic eqn is given by $\lambda^2 + \omega_0^2 = 0$, and the roots are $\lambda_1 = i\omega_0$, $\lambda_2 = -i\omega_0$. Hence the general sol. of the homogeneous eqn is given by $x_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$, in case $\omega_0 > 0$, and $x_h(t) = c_1 + c_2 t$.

For the inhomogeneous sol. we make the ansatz $y(t) = a\cos\omega t + b\sin\omega t$.

Then, $y'' + \omega_0^2 y = a(\omega_0^2 - \omega^2)\cos\omega t + b(\omega_0^2 - \omega^2)\sin\omega t$.

Choosing $b = 0$, and $a = \frac{1}{\omega_0^2 - \omega^2}$ if $\omega \neq \omega_0$, we obtain that

$y(t) = \frac{1}{\omega_0^2 - \omega^2}\cos\omega t$ is a particular sol. of $x'' + \omega_0^2 x = \cos\omega t$.

(Note $|y(t)| \to +\infty$ as $\omega \to \omega_0$.)

So the solutions of $\{ x'' + \omega_0^2 x = \cos\omega t$, is of the form

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2}\cos\omega t \quad \text{if } \omega \neq \omega_0 \text{ and } \omega_0 > 0.$$

$$= c_1 + c_2 t - \frac{1}{\omega^2}\cos(\omega t) \quad \text{if } \omega_0 = 0 \text{ and } \omega > 0.$$

The inhomogeneous term is also called the forcing term and is <u>resonant</u> if $\omega = \omega_0$. The sol. blows up as the forcing term becomes resonant.

Example Consider $\{ x'' + 2\eta x' + \omega_0^2 x = 0$, where $\eta, \omega_0 > 0$

The characteristic eqn is given by $\lambda^2 + 2\eta\lambda + \omega_0 = 0$. The roots of the characteristic eqn and hence the eigenvalues of the corresponding system is given by

$$\lambda_1 = -\eta + \sqrt{\eta^2 - \omega_0^2} \quad \text{and} \quad \lambda_2 = -\eta - \sqrt{\eta^2 - \omega_0^2}.$$

· If $\eta > \omega_0$, both the eigenvalues are negative, and the solution of the homogeneous eqn is given by

$$x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \qquad \text{(Over-damping.)}$$

· If $\eta = \omega_0$, both eigenvalues equal $-\eta$, and the solution of the homogeneous eqn is given by

$$x_h(t) = (c_1 + c_2 t) e^{-\eta t} \qquad \text{(critical-damping)}$$

· If $\eta < \omega_0$, both the eigenvalues are complex conjugates, and the sol. of the homogeneous eqn is given by

$$x_h(t) = c_1 e^{-\eta t}\cos(\beta t) + c_2 e^{-\eta t}\sin(\beta t), \text{ where } \beta = \sqrt{\omega_0^2 - \eta^2} > 0.$$

(Under-damping.)

In each of the cases, the real part of both the eigenvalues are negative and the homogeneous eqn decays exponentially as $t \to +\infty$.

General Linear First-Order Systems.

We consider the general first-order linear system: $\boxed{X'(t) = A(t)X(t)}$ where $A(t) \in \mathbb{R}^{n\times n}$ is a $n\times n$ matrix, whose entries are continuous fn. of t.

Regarding existence, it follows from the Existence and Uniqueness theorem, that the first-order linear initial value p'blm $\begin{cases} X'(t) = A(t)X(t) \\ X(t_0) = X_0 \end{cases}$ has a unique sol, defined for all $t \in I$, where $I \subset \mathbb{R}$ is an interval with $A \in C(I, \mathbb{R}^{n\times n})$ (continuous)

<u>Remark</u>: If we write $X(t) = \exp\left(\int_{t_0}^{t} A(s)ds\right) X_0$, then the noncommutativity of matrices in general will prevent us in showing that $X(t)$ solves the ODE.

<u>Theorem</u> The solutions of $\{X'(t) = A(t)X(t)$, form a n-dimensional vector space. Moreover, the sol. satisfying $\begin{cases} X'(t) = A(t)X(t) \\ X(t_0) = X_0 \end{cases}$ is given by

$X(t) = \Pi(t,t_0)X_0$, where the matrix $\Pi(t,t_0)$ is given by

$$\Pi(t,t_0) := \left(\Phi(t,t_0,e_1), \ldots \Phi(t,t_0,e_n)\right).$$

Here, $\{e_i : 1 \le i \le n\}$ is the canonical basis vectors of $\mathbb{R}^n$, and $\Phi(t,t_0,e_i)$ is the unique sol. to $\begin{cases} X'(t) = A(t)X(t) \\ X(t_0) = e_i \end{cases}$, $1 \le i \le n$

<u>Pf</u>: We note that, for any $X_0 = \sum_{i=1}^{n} X_{0,i} e_i$, we have

$$\begin{aligned} \Phi(t,t_0,X_0) &= \sum_{i=1}^{n} \Phi(t,t_0,e_i) X_{0,i} \quad \text{(Superposition Principle)} \\ &= \Pi(t,t_0) X_0 \end{aligned}$$

The matrix $\Pi(t,t_0)$ is called the <u>principal matrix solution</u> at t_0 and it solves the matrix-valued initial value problem $\begin{cases} \Pi'(t,t_0) = A(t)\Pi(t,t_0) \\ \Pi(t_0,t_0) = I_{n\times n} \end{cases}$.

The principal matrix sol. $\Pi(t,t_0)$ satisfies: $\Pi(t,t_1)\Pi(t_1,t_0) = \Pi(t,t_0)$.
Both sides coincide for $t=t_1$ and $\left(\Pi(t,t_1)\Pi(t_1,t_0)\right)' = \Pi'(t,t_1)\Pi(t_1,t_0)$

$$= A(t)\Pi(t,t_1)\Pi(t_1,t_0)$$
$$= A(t)\Pi(t,t_0) = \Pi'(t,t_0).$$

In particular $\Pi(t_0,t)\Pi(t,t_0) = \Pi(t_0,t_0) = I_{n\times n}$
Therefore $\Pi(t,t_0)$ is invertible with $\Pi^{-1}(t,t_0) = \Pi(t_0,t)$.

<u>Example</u> $A(t) \equiv A$, a const. matrix. Then $\Pi(t,t_0) = \exp((t-t_0)A)$.

<u>Example</u> Consider $X'(t) = \begin{pmatrix} 1 & t \\ 0 & 2 \end{pmatrix} X(t)$.

letting $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, this can be written as $\begin{cases} x_1' = x_1(t) + t\,x_2(t) \\ x_2'(t) = 2x_2(t) \end{cases}$

The general sol. is given by $\begin{cases} x_1(t) = x_1(t_0)e^{(t-t_0)} + x_2(t_0)\left[e^{2(t-t_0)}(t-1) - e^{(t-t_0)}(t_0-1)\right] \\ x_2(t) = x_2(t_0)e^{2(t-t_0)} \end{cases}$

Therefore $\Phi(t,t_0,e_1) = e^{t-t_0}$, and

$$\phi(t,t_0,e_2) = \begin{pmatrix} e^{2(t-t_0)}(t-1) - e^{(t-t_0)}(t_0-1) \\ e^{2(t-t_0)} \end{pmatrix}$$

Hence the principal matrix sol is given by

$$\Pi(t,t_0) = \begin{pmatrix} e^{t-t_0} & e^{2(t-t_0)}(t-1) - e^{(t-t_0)}(t_0-1) \\ 0 & e^{2(t-t_0)} \end{pmatrix}$$

Generally, taking n solutions $\phi_1(t), \dots \phi_n(t)$ of $\{ X'(t) = A(t)X(t)$, we obtain a matrix sol $U(t) = (\phi_1(t), \dots \phi_n(t))$. Here $\phi_i(t) \in \mathbb{R}^n$.

The determinant of the matrix $U(t)$ is called the Wronskian $W(t)$

$$W(t) = \det U(t) = \det(\phi_1(t), \dots \phi_n(t)).$$

If $\det U(t) \neq 0$, the matrix sol $U(t)$ is called a fundamental matrix solution.

Suppose $U(t)$ and $V(t)$ are two fundamental matrix sols of $\{X'(t) = A(t)X(t)$

Then
$$\begin{aligned}(U^{-1}(t)V(t))' &= U^{-1}(t)V'(t) + (U^{-1}(t))'V(t) \\ &= U^{-1}(t)V'(t) - U^{-1}(t)U'(t)U^{-1}(t)V(t) \\ &= U^{-1}(t)[A(t)V(t) - (A(t)U(t))U^{-1}(t)V(t)] \\ &= U^{-1}(t)[A(t)V(t) - A(t)V(t)] = 0.\end{aligned}$$

Therefore $U^{-1}(t)V(t) \equiv U^{-1}(t_0)V(t_0)$, and hence $V(t) = U(t)U^{-1}(t_0)V(t_0)$

So in particular

Lemma : For any fundamental matrix sol. we have $\Pi(t,t_0) = U(t)U^{-1}(t_0)$, here $\Pi(t,t_0)$ is the principal matrix sol.

Then $U(t) = \Pi(t,s)U(s)$ hence $\det U(t) \neq 0$ iff $\det U(s) \neq 0$, and it suffices to check $\det U(t) \neq 0$ for a $t \in \mathbb{R}$.

Consider the inhomogeneous system $\{ X'(t) = A(t)X + g(t),$

where $A \in \underbrace{C(I, \mathbb{R}^n \times \mathbb{R}^n)}_{\substack{\text{matrix with} \\ \text{continuous coeff.}}}$ and $g \in C(I, \mathbb{R}^n)$.

<u>Theorem</u> (Variation of Parameters)

The sol. of the inhomogeneous initial value problem $\begin{cases} X(t) = A(t)X(t) + g(t), \\ X(t_0) = X_0 \end{cases}$

is given by
$$X(t) = \Pi(t,t_0)X_0 + \int_{t_0}^{t} \Pi(t,s)g(s)ds.$$

Here $\Pi(t,t_0)$ is the principal matrix sol. of the corresponding homogeneous system

<u>Proof</u>: Consider the following ansatz $X(t) \stackrel{!}{=} \Pi(t,t_0)Z(t)$ with $Z(t_0) = X_0$.

The differentiating
$$\begin{aligned} X'(t) &= \Pi'(t,t_0)Z(t) + \Pi(t,t_0)Z'(t) \\ &= A(t)\Pi(t,t_0)Z(t) + \Pi(t,t_0)Z'(t) \\ X'(t) &= A(t)X(t) + \Pi(t,t_0)Z'(t) \end{aligned}$$

Comparing this to the given eqn, gives $\Pi(t,t_0)Z'(t) = g(t)$, so then
$$Z'(t) = \Pi^{-1}(t,t_0)g(t) = \Pi(t_0,t)g(t).$$

Integrating we obtain
$$Z(t) = X_0 + \int_{t_0}^{t} \Pi(t_0,s)g(s)\,ds.$$

This gives
$$X(t) = \Pi(t,t_0)X_0 + \int_{t_0}^{t} \Pi(t,s)g(s)ds$$

Example Consider the system $\left\{ X'(t) = \begin{pmatrix} t^2 & -1 \\ 2t & 0 \end{pmatrix} X(t) \right.$

Writing $X_A(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ we have that $\begin{cases} x_1'(t) = t^2 x_1(t) - x_2(t) \\ x_2'(t) = 2t\, x_1(t) \end{cases}$

By a direct calculation we obtain that $X_A(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$ is a sol.

Let $Y(t) = \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} X(t)$

Then $Y'(t) = \left[\begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} \right]' X(t) + \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} X'(t)$

$$Y'(t) = -\begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} X(t) + \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} X'(t)$$

$$= \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} \left\{ \begin{pmatrix} t^2 & -1 \\ 2t & 0 \end{pmatrix} X(t) - \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} X(t) \right\}$$

$$= \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} \left\{ \begin{pmatrix} t^2 & -1 \\ 2t & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix} Y(t) - \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} Y(t) \right\}$$

$$= \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} \left\{ \begin{pmatrix} 0 & -1 \\ 2t & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} \right\} Y(t)$$

$$= \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} Y(t)$$

$$Y'(t) = \begin{pmatrix} 1 & 0 \\ -t^2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} Y(t) = \begin{pmatrix} 0 & -1 \\ 0 & t^2 \end{pmatrix} Y(t)$$

So $\begin{cases} y_1'(t) = -y_2(t) \\ y_2'(t) = t^2 y_2(t) \Rightarrow y_2(t) = C e^{t^3/3} \end{cases}$ and $y_1(t) = -\int e^{t^3/3} dx + \tilde{C}$

So knowing the sol. $\begin{pmatrix} 1 \\ t^2 \end{pmatrix}$ we are able to obtain another sol

$$X_B(t) = \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix} Y(t) = \begin{pmatrix} 1 & 0 \\ t^2 & 1 \end{pmatrix} \begin{pmatrix} -\int e^{t^3/3} dt \\ e^{t^3/3} \end{pmatrix} = \begin{pmatrix} -\int e^{t^3/3} dt \\ e^{t^3/3} - t^2 \int e^{t^3/3} dt \end{pmatrix}$$

Differential Calculus for Matrices.

Lemma: Let $A, B \in \mathbb{R}^{n \times n}$ and assume that A is invertible. Then

$$\frac{d}{dt} \det(A + tB) \Big|_{t=0} = \det(A) \cdot \operatorname{Tr}(A^{-1}B).$$

Proof: Since A is invertible, $\det(A+tB) = \det(A + t AA^{-1}B)$

$$= \det(A) \cdot \det(I + tA^{-1}B)$$

A direct computation gives $\det(I+tM) = 1 + t\operatorname{Tr}(M) + \underbrace{O(t^2)}_{\leq Ct^2}$

So, $\det(A+tB) = \det(A)\left[1 + t\operatorname{Tr}(A^{-1}B) + O(t^2)\right]$.

Then, $$\lim_{t\to 0} \frac{\det(A+tB) - \det A}{t} = \lim_{t\to 0} \det(A) \frac{\left[1 + t\operatorname{Tr}(A^{-1}B) + O(t^2) - 1\right]}{t}$$

$$= \det(A) \cdot (\det(\operatorname{Tr}(A^{-1}B)) + \lim_{t\to 0} \frac{O(t^2)}{t} = (\det(A))(\det(\operatorname{Tr}(A^{-1}B)))$$

Hence $\det(A+tB)'(0) = \det A \cdot \operatorname{Tr}(A^{-1}B)$.

Lemma (Abel's identity or Liouville's formula)

Let $M(t) \in \mathbb{R}^{n\times n}$ be a continuously differentiable matrix valued function satisfying

$$M'(t) = A(t)M(t).$$

Then, $$\det(M(t)) = \det(M(t_0)) \exp\left(\int_{t_0}^{t} Tr(A(s))\,ds\right).$$

Proof: We note that it suffices to show $\frac{d}{dt}\det(M(t)) = Tr(A(t))\det(M(t))$

We have $M(t+\varepsilon) = M(t) + \varepsilon M'(t) + O(\varepsilon^2) = M(t) + \varepsilon A(t)M(t) + O(\varepsilon^2)$.

Then $\det M(t+\varepsilon) = \det\left(I + \varepsilon A(t) + O(\varepsilon^2)\right)\det M(t)$

Then $$\frac{d}{dt}\det(M(t)) = \lim_{\varepsilon\to 0}\frac{\det(M(t+\varepsilon)) - \det M(t)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0}\frac{\left(\det(I + \varepsilon A(t) + O(\varepsilon^2)) - 1\right)}{\varepsilon}\det M(t)$$

$$= \lim_{\varepsilon\to 0}\det(I)\cdot\frac{Tr\left((\varepsilon A(t) + O(\varepsilon^2))\right)}{\varepsilon}\det M(t)$$

So, $$\frac{d}{dt}\det(M(t)) = Tr(A(t))\det M(t).$$

Corollary: The Wronskian satisfies associated to $X'(t) = A(t)X(t)$

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} Tr(A(s))\,ds\right).$$