# Stability

Example 1: Consider $\{ x' = ax$. Then any two solutions $\phi_1(t)$ and $\phi_2(t)$ of $x' = ax$ satisfies

$$|\phi_1(t) - \phi_2(t)| = |\phi_1(0) - \phi_2(0)| e^{at}.$$

Then $$\lim_{t\to+\infty} |\phi_1(t) - \phi_2(t)| = \begin{cases} 0 & \text{if } a<0 \\ +\infty & \text{if } a>0. \end{cases}$$

So when $a<0$, any two sols. starting at points close to one another, remains close. And, when $a>0$ sols. tend to diverge from one another.

Def: Consider the sol. $\gamma(t; t_0, x_0)$ of $\begin{cases} x' = F(t,x) \\ x(0) = x_0 \end{cases}$

Here $F \in C(U, \mathbb{R}^n)$, $U \subset \mathbb{R}^n$ and $F$ is locally Lipschitz in the $x$ variable uniformly w.r.t $t$ variable.

→ The sol $\gamma(t, x_0)$ is said to be stable (in the sense of Lyapunov) if $\forall\ \varepsilon>0$, $\exists\ \delta>0$ s.t for any sol. $\gamma(t, y_0)$ of $\begin{cases} x' = F(t,x) \\ x(t_0) = y_0 \end{cases}$

with $|y_0 - x_0| < \delta$ we have that

$\gamma(t, y_0)$ exists $\forall\ t \geq 0$ (in particular $\gamma(t, x_0)$) and satisfies

$$|\gamma(t, y_0) - \gamma(t, x_0)| < \varepsilon \quad \forall\ t \geq 0.$$

→ The sol. $\gamma(t, x_0)$ is said to be asymptotically stable, if it is stable and $\exists\ r_0$ s.t if $|y_0 - x_0| < r_0$ then for any sol $\gamma(t, y_0)$ we have

$$\lim_{t\to+\infty} |\gamma(t, y_0) - \gamma(t, x_0)| = 0.$$

→ The sol. $x(t, x_0)$ is said to be unstable if it is not stable.

→ The sol. $x(t, x_0)$ is said to be exponentially stable if $\exists\ r_0 > 0$ s.t. if $|y_0 - x_0| < r_0$ then any sol $x(t, y_0)$ satisfies

$$|x(t, y_0) - x(t, x_0)| \leq C e^{-\alpha t} |y_0 - x_0|. \quad \text{for some } C > 0, \alpha > 0.$$

Def: Consider
$$\begin{cases} X' = F(X) \\ X(0) = \xi_0 \end{cases}$$

where $F \in C(U, \mathbb{R}^n)$ is locally Lipschitz.

If $F(\xi_0) = 0$, then the sol $X(t) \equiv \xi_0$ is said to be an equilibrium or steady state sol, and also called a fixed point of the given system.

→ The fixed point $\xi_0$ of the system is said to be stable, asymptotically stable or exponentially stable if the equilibrium sol. is stable, asymptotically stable or exponentially stable respectively.

Example 2 : Consider the linear system $\{ X' = AX$. (Recall from chapter 3.)

Here $A \in \mathbb{R}^{n \times n}$ is a $n \times n$ matrix with real entries.

Let $\sigma_{max} := \max \{ Re(\lambda) : \lambda$ is an eigenvalue of $A \}$.

The trival sol $X(t) \equiv 0$ of $\{ X' = AX$ is

a.) Asymptotically stable if $\sigma_{max} < 0$.

b.) Unstable if $\sigma_{max} > 0$.

c.) Stable if $\sigma_{max} \leq 0$ with the corresponding algebraic and geometric multiplicity being equal for all eigenvalues $\lambda$ with $Re(\lambda) = 0$.

Remark: We have the bound on the sol given by the bound on the $\exp(tA)$ as follows

a.) If $\sigma_{max} < 0$, then $\| \exp(tA) \| \leq C e^{t \sigma_{max}} \quad \forall \, t \geq 0$, for some $C > 0$.

b.) If $\sigma_{max} \leq 0$ with the corresponding algebraic and geometric multiplicity being equal for all eigenvalues $\lambda$ with $Re(\lambda) = 0$, then

$$\| \exp(tA) \| \leq C \quad \forall \, t \geq 0, \text{ for some } C > 0.$$

Here $\| \cdot \|$ is the operator norm for matrices.

## Linearization.

Consider the autonomous system $\{X' = F(X)$ where $F : \underbrace{U \subset \mathbb{R}^n}_{\text{open}} \to \mathbb{R}^n$ is continuously differentiable, i.e. $C^1$-function.

Suppose $\xi_0 \in U$ is a fixed point of the given system, i.e. $F(\xi_0) = 0$.

We linearize the (non-linear) function $F(X)$ around $\xi_0$ by considering its derivative at $\xi_0$ which is given by the Jacobian matrix $DF(\xi_0)$.

The linearized system at $\xi_0$ corresponding to $\{X' = F(X)$ becomes $\{Y' = DF(\xi_0)Y$

We write $\{X' = F(X)$ as $\{X' = \underbrace{F(\xi_0)}_{=0} + \underbrace{DF(\xi_0)\cdot(X - \xi_0)}_{\text{Derivative}} + \underbrace{o(|X - \xi_0|)}_{\text{remainder}}$

$$\Rightarrow \underbrace{(X(t) - \xi_0)'}_{Y(t)} = \underbrace{DF(\xi_0)\cdot(X - \xi_0)}_{DF(\xi_0)\cdot Y(t)} + o(|X - \xi_0|)$$

Theorem The equilibrium sol. $X(t) \equiv \xi_0$ of the nonlinear system $\{X' = F(X)$ is asymptotically stable if the corresponding linearized system $\{Y' = DF(\xi_0) Y$ is asymptotically stable.

The nonlinear system $\{X' = F(X)$ is unstable if the corresponding linearized system $\{Y' = DF(\xi_0) Y$ is unstable.

Def:

Let $F(\xi_0)=0$. Then the equilibrium sol $X(t)\equiv\xi_0$ or the fixed point is called hyperbolic if for all eigenvalues $\lambda$ of $DF(\xi_0)$ one has $Re(\lambda)\neq 0$.

Hartman-Grobman Theorem (An important result (proof see the text))

Suppose $F: U\to\mathbb{R}^n$ is $C^1$ and $\xi_0\in U$ is a hyperbolic equilibrium point of $\{X'=F(X)$.

Then $\exists$ $\Omega_1$ a neighbourhood of $\xi_0$ and $\Omega_2$ a neighbourhood of $0$ and a homeomorphism $\psi:\Omega_2\to\Omega_1$ that transforms the (sols) trajectories of the corresponding linearized system to the sol. curves or trajectories of the nonlinear system $\{X'=F(X)$.

Example: $\{u''+\sin u=0$ which can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -\sin x \end{pmatrix} := F(x,y)$$

$$F(0,0)=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad F(\pi,0)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$DF(x,y)=\begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$$

The linearizations around $(0,0)$ and $(\pi,0)$ are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

respectively.

## Stability via the method of Lyapunov

Consider the autonomous system $\{X' = F(X)$ where $F$ is (Lipschitz) continuous in some open set $\mathcal{U} \subset \mathbb{R}^n$.

Suppose $\xi_0 \in \mathcal{U}$ is a fixed point of the given system, i.e. $F(\xi_0) = 0$. Then the equilibrium sol. or steady state sol. $X(t) \equiv \xi_0$ solves $\begin{cases} X' = F(X). \\ X(0) = \xi_0 \end{cases}$

**Def:** $\mathcal{L} : \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ a $C^1$ function (continuously diff'able) is said to be a Lyapunov function for $\{X' = F(X)$ if:

a) $\mathcal{L}(\xi_0) = 0$

b) $\mathcal{L}(X) > 0 \quad \forall\, X \in \mathcal{U} / \{\xi_0\}$.

c) $\frac{d}{dt}\mathcal{L}(\nu(t)) \leq 0$ where $\{\nu'(t) = F(X)$ and $\forall\, t$ s.t the sol. $\nu(t)$ is defined.

or equivalently $\langle \nabla\mathcal{L}(X), F(X) \rangle \leq 0 \quad \forall\, X \in \mathcal{U}$

(This means $\mathcal{L}$ is non-increasing along the integral (sol.) curves $\gamma(t)$.)

Note: $\frac{d}{dt}\mathcal{L}(\nu(t)) = \langle \nabla\mathcal{L}(\nu(t)), \nu'(t) \rangle = \langle \nabla\mathcal{L}(\nu(t)), F(\nu(t)) \rangle$
(by Chain-rule)

**Theorem** (Stability via Lyapunov) $\quad \{X' = F(X)$, $F(\xi_0) = 0$.

a) If $\exists$ a Lyapunov fn. $\mathcal{L}$ for $\{X' = F(X)$ then the equilibrium sol $X(t) \equiv \xi_0$ is stable.

b) Moreover, if $\langle \nabla\mathcal{L}(X), F(X) \rangle < 0 \quad \forall\, X \in \mathcal{U} / \{\xi_0\}$, then the equilibrium sol. $X(t) \equiv \xi_0$ is asymptotically stable.

Proof: a) We first show stability of the equilibrium sol. $\{X(t) = \xi_0$

Note, $\mathcal{L}$ is non increasing along the sol. curves $\gamma(t)$ of $X' = F(X)$ since $(\mathcal{L}(\gamma(t)))' \leq 0$. Therefore upon integration we have that

$\mathcal{L}(\gamma(t)) \leq \mathcal{L}(\gamma(0))$ for all time $t$ s.t the sol. curve $\gamma(t)$ exists.

Given $\varepsilon > 0$ small we can find $m > 0$ s.t. $\overline{B(\xi_0, \varepsilon)} \subset U$ and

$\mathcal{L}(x) \geq m$ for $|x - \xi_0| = \varepsilon$. (Otherwise $\exists$ a seq $(x_\alpha)$ s.t $|x_\alpha - \xi_0| = \varepsilon$ but $\mathcal{L}(x_\alpha) \to 0$)

We can choose $\delta > 0$ s.t $\mathcal{L}(x) \leq \frac{m}{2}$ for all $x$ s.t $|x - \xi_0| < \delta$.

(Otherwise we can find a seq $(x_\alpha)$ s.t $\lim_{\alpha \to +\infty} x_\alpha = \xi_0$ and $\mathcal{L}(x_\alpha) \geq \frac{m}{2} > 0$

This is a contradiction since $\mathcal{L}(\xi_0) = 0$.)

To show stability of $\xi_0$ we need to show that for any $\varepsilon > 0$ $\exists\, \delta > 0$ s.t if $|\gamma(0) - \xi_0| < \delta$ then $|\gamma(t) - \xi_0| < \varepsilon$ $\forall\, t \geq 0$, for any sol. $\gamma(t)$.

We have obtained that $\forall\, \varepsilon > 0$ small, we can find $\delta > 0$ s.t

$\mathcal{L}(\gamma(t)) \leq \mathcal{L}(\gamma(0)) \leq \frac{m}{2}$ if $|\gamma(0) - \xi_0| \leq \delta$.

If $|\gamma(t) - \xi_0| \geq \varepsilon$ for $t$ then for some $\tilde{t} \in (0, t)$ and we have

$|\gamma(\tilde{t}) - \xi_0| = \varepsilon$, which implies $\mathcal{L}(\gamma(\tilde{t})) \geq m$ a contradiction since $\mathcal{L}(\gamma(\tilde{t})) \leq \mathcal{L}(\gamma(0)) \leq \frac{m}{2}$.

Therefore $|\gamma(t) - \xi_0| < \varepsilon$ $\forall\, t \geq 0$ s.t the sol. $\gamma(t)$ exists.

But then $\overline{B(\xi_0, \varepsilon)}$ is a compact set and the sol. $\gamma(t)$ can be extended (or exists) for all $t \geq 0$.

Hence, the existence of a Lyapunov fn. for $\{X = F(X)$ implies existence $\forall\, t \geq 0$ and stability.

b.) We now show that if $\langle \nabla \mathcal{L}(x), F(x) \rangle < 0 \ \forall \ x \in U / \{\xi_0\}$, then the equilibrium sol. is asymptotically stable.

From (a) we have that for some $\delta > 0$, $\mathcal{L}(\gamma(t)) \leq \mathcal{L}(\gamma(0))$ if $|\gamma(0) - \xi_0| \leq \delta$ and the sol. exists for all $t \geq 0$ with $|\gamma(t) - \xi_0| < \varepsilon$.

So $\lim\limits_{t \to +\infty} \mathcal{L}(\gamma(t)) = l$ exists ($\mathcal{L}(\gamma(t))$ is positive and non-increasing)

Then $l = 0$ iff $\lim\limits_{t \to +\infty} \gamma(t) = \xi_0$.

( Otherwise, for a seq. $(\gamma(t_\alpha))$ s.t. $t_\alpha \to +\infty$ we will have $|\gamma(t_\alpha) - \xi_0| > \varepsilon$ for some $\varepsilon > 0$, and this then implies $\lim\limits_{\alpha \to +\infty} \mathcal{L}(\gamma(t_\alpha)) > 0$. as $\mathcal{L}(x) > 0 \ \forall \ x \neq 0$ )

Suppose: $\lim\limits_{t \to +\infty} \gamma(t) \neq \xi_0$.

Assume $|\gamma(t) - \xi_0| \geq \tilde{\delta} \ \forall \ t \geq 0$. Then on the compact set $\overline{B(\xi_0, \varepsilon)} | B(\xi_0, \tilde{\delta})$ we have $\langle \nabla \mathcal{L}(x), F(x) \rangle \leq -a \ \forall \ x$ s.t $\tilde{\delta} \leq |x - \xi_0| \leq \varepsilon$.

(as max exists on compact sets)

This gives
$$\mathcal{L}(\gamma(t)) - \mathcal{L}(\gamma(0)) = \int_0^t \frac{d}{dt}(\gamma(s))\, ds = \int_0^t \underbrace{\langle \nabla \mathcal{L}(\gamma(s)), F(\gamma(s)) \rangle}_{\leq -a} ds$$

since $\tilde{\delta} \leq |\gamma(s) - \xi_0| \leq \varepsilon$.

so, $\mathcal{L}(\gamma(t)) \leq -at + \mathcal{L}(\gamma(0))$

This $\Rightarrow$ $\mathcal{L}(\gamma(t)) < 0$ for some large $t$, which is impossible.

Hence $\lim\limits_{t \to +\infty} \gamma(t) = \xi_0$.

**Theorem** (Instability condition of Lyapunov.) Consider $\begin{cases} X' = F(X), \quad F(\xi_0) = 0. \\ F \text{ is Lipschitz} \end{cases}$

Suppose $\mathcal{L} : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ is a $C^1$ function satisfying

a) $\mathcal{L}(\xi_0) = 0$

b) $\mathcal{L}(x) > 0 \quad \forall\, x \in \mathcal{U} \setminus \{\xi_0\}$.

and

c) $\langle \nabla \mathcal{L}(x), F(x) \rangle > 0 \quad \forall \quad x \in \mathcal{U} \setminus \{\xi_0\}$.

Then the equilibrium sol $X(t) \equiv \xi_0$ is unstable.

**Proof:** Suppose $\gamma(t)$ solves $\{ X' = F(X)$ with $\gamma(0) \neq \xi_0$. (To show that $\xi_0$ is unstable, it suffices to show that for some $\tilde{\varepsilon} > 0$, we have for $t$ large $|\gamma(t) - \xi_0| \geq \tilde{\varepsilon}$ with $\gamma(0) \neq \xi_0$.)

We have $\mathcal{L}(\gamma(t))' \geq 0$ and hence $\mathcal{L}(\gamma(t)) \geq \mathcal{L}(\gamma(0)) \;\forall\, t \geq 0$

Let $\delta_0 > 0$ be small s.t. $\mathcal{L}(x) \leq \mathcal{L}(\gamma(0)) \quad \forall\, x$ s.t. $|x - \xi_0| \leq \delta_0$

Hence $|\gamma(t) - \xi_0| \geq \delta_0$. Let $r > \delta_0$ be s.t. $B(\xi_0, r) \subset \mathcal{U}$. Then

$\tilde{m} := \min\{\langle \nabla \mathcal{L}(x), F(x) \rangle : \delta_0 \leq |x - \xi_0| \leq r\}$. Then $\tilde{m} > 0$ and

$$\mathcal{L}(\gamma(t)) \geq \mathcal{L}(\gamma(0)) + \tilde{m}t \quad \text{if } |\gamma(t) - \xi_0| \leq r$$

For $t$ large the right hand side arbitrary large, and so for $t$ large we must have $|\gamma(t) - \xi_0| > r$ as $\mathcal{L}(\gamma(t)) \leq \max\{\mathcal{L}(x) : |x - \xi_0| \leq r\}$.

Hence, $\gamma(t)$ leaves every ball $B(\xi_0, r) \subset \mathcal{U}$ for some large $t$ and so $X(t) \equiv \xi_0$ is unstable.

Example: Consider $\{ X' = Ax + b(x)$ where $A$ is a $n \times n$ real matrix and $b: \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Let $\mathcal{L}(x) = \frac{1}{2}|x|^2$ : the distance from the origin.
Then $\mathcal{L}(x) \geq 0$ and $\mathcal{L}(x) = 0$ iff $x = 0$. And we have $\nabla \mathcal{L}(x) = x \in \mathbb{R}^n$.

So $\langle \nabla \mathcal{L}(x), Ax + b(x) \rangle = \langle x, Ax \rangle + \langle x, b(x) \rangle$

Then by the Lyapunov's stability theorem, then the $0$ equilibrium sol. is

stable, if $\langle x, Ax \rangle \leq 0$ and $\langle x, b(x) \rangle \leq 0$ $\forall x$ in a neighbourhood of the origin.

Suppose $b(x) = o(|x|)$, then the $0$ equilibrium sol. is
a) asymptotically stable if $\max\limits_{|x|=1} \langle x, Ax \rangle < 0$.
b) Unstable if $\min\limits_{|x|=1} \langle x, Ax \rangle > 0$.

Example:
Consider the 2nd order ODE:
$$\begin{cases} u'' + h(u) = 0 \\ \text{where } h \in C^1(\mathbb{R}) \text{ with } u h(u) > 0 \\ \text{for } u \neq 0, \text{ and } h(0) = 0 \end{cases}$$

This can be written as
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} y(t) \\ -h(x) \end{pmatrix}.$$

A choice for the Lyapunov fn. here is

$$\mathcal{L}(x,y) = \underbrace{\frac{1}{2} y^2}_{\text{Kinetic energy}} + \underbrace{\int_0^x h(s)\,ds}_{\text{Potential energy}} \qquad \text{Total energy}$$

Then, $\left\langle \nabla \mathcal{L}(x,y), (y, -h(x)) \right\rangle = \left\langle (h(x), y), (y, -h(x)) \right\rangle = 0$

Hence $\mathcal{L}$ is const. along the sol. curves

We have $\mathcal{L}(x,y) = \underbrace{\frac{1}{2} y^2}_{>0} + \underbrace{\int_0^x h(x)\,ds}_{> \int_\varepsilon^x \frac{s h(s)}{s} ds > 0 \text{ if } x > 0} \geq 0$

and $\mathcal{L}(x,y) = 0$ iff $(x,y) = (0,0)$.

Therefore, $\mathcal{L}$ is a Lyapunov fn. for the given system and to 0 equilibrium sol is stable.