Ordinary Differential Equations

Problem Set 5

1. Show that the equilibrium solution 0 of $x' = \beta x^3$ is:
   (a) asymptotically stable if $\beta < 0$,
   (b) stable if $\beta \le 0$,
   (c) unstable if $\beta > 0$.

2. Study the stability of the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the following systems (by linearization or otherwise):
   (a) $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y + x^3 \\ x \end{pmatrix}$.
   (b) $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 2\sin x - 4y \\ \sin x - 3y \end{pmatrix}$.
   (c) $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \sin(6x - 3y) \\ \sin(2x + y) \end{pmatrix}$.

3. Find all the equilibrium solutions of the following system and discuss the stability of each of them:
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 8x - y^2 \\ -y + x^2 \end{pmatrix}.$$

4. Consider the system:
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y + x^3 \\ x + y^2 \end{pmatrix}.$$
   (a) Show that the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the above system is unstable.
   (b) Show that the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for the associated linearized system is stable.

5. Consider the system:
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} (y-1)(y^2 - 2y + 2) \\ (1-x)(x^2 - 2x + 2) \end{pmatrix}.$$
   Show that the equilibrium solution $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is stable.

6. Study the stability of the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the following system (via Liapunov or otherwise):
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y \\ -\sin x \end{pmatrix}$$