Partial Differential Equations I

Notes for the course MA 817

Saikat Mazumdar

Department of Mathematics, IIT Bombay
Email address: saikat@math.iitb.ac.in, saikat.mazumdar@iitb.ac.in

These notes[1] are not original and are strongly influenced by the books cited in the references. These notes are meant for a PhD level introduction to Partial Differential Equations (PDEs, for short), focusing on the second-order linear equations.

Our notation and terminologies are mostly as in the book of Evans. In the appendix we collect various important results and useful formulas, without proofs, that we shall use throughout in our notes.

Prerequisite: Topology (metric spaces), Multi-variable Calculus, Introduction to Functional Analysis and Measure Theory.

[1]version of November 23, 2020

Contents

Notation and Preliminaries

Sets and Points

- $\mathbb{R}^n$ will denote the n-dimensional real Euclidean space. $\mathbb{R}^1 = \mathbb{R}$.
- We will denote a point in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$. Depending on the context, we will regard x as a row or column vector.
- $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ will denote the i^{th} coordinate vector of the standard Euclidean basis.
- The standard Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ will be denoted by
$$|x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}.$$
- The inner product of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is given by
$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$
This is the standard Euclidean metric.
- The open upper half-space is defined as
$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$
- A point $x \in \mathbb{R}^n$ will sometimes be written as $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.
- Open ball in $\mathbb{R}^n$ with center ξ and radius $r > 0$
$$B(\xi, r) = \{x \in \mathbb{R}^n : |x - \xi| < r\}.$$
- For $\mathcal{O} \subset \mathbb{R}^n$ nonempty, we denote

  the complement by $\mathcal{O}^c = \mathbb{R}^n \setminus \mathcal{O}$, the interior by $int(\mathcal{O})$ or $\mathcal{O}^o$

  and the closure by $\overline{\mathcal{O}}$.

  The boundary is given by $\partial\mathcal{O} = \overline{\mathcal{O}} \cap \overline{\mathcal{O}^c}$.
- Sphere of radius $r > 0$ in $\mathbb{R}^n$ with center ξ
$$\partial B(\xi, r) = \{x \in \mathbb{R}^n : |x - \xi| = r\}.$$
- The unit sphere in $\mathbb{R}^n$
$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$
$\mathbb{S}^{n-1}$ is a $(n-1)-$dimensional compact manifold.
- ω_{n-1} will denote the area of $\mathbb{S}^{n-1}$, the standard unit sphere in $\mathbb{R}^n$, that is $|\mathbb{S}^{n-1}| = \omega_{n-1}$.
- The volume of the unit ball $B(0,1)$ in $\mathbb{R}^n$ is given by $|B(0,1)| = \omega_{n-1}/n$.

- Let $\Omega', \Omega \subset \mathbb{R}^n$ be (nonempty) open sets. We write $\Omega' \Subset \Omega$ to denote that Ω' is compactly contained in Ω, that is, $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact.
- For $\xi \in \mathbb{R}^n$ and $\mathcal{O} \subset \mathbb{R}^n$ nonempty, we denote the distance from ξ to $\mathcal{O}$ by
$$d(\xi, \mathcal{O}) = \inf_{x \in \mathcal{O}} |\xi - x|.$$
Similarly for two nonempty sets $\mathcal{O}, \mathcal{O}' \subset \mathbb{R}^n$
$$d(\mathcal{O}, \mathcal{O}') = \inf\{|x - x'| : x \in \mathcal{O}, x' \in \mathcal{O}'\}.$$
- $\mathcal{O} \subset \mathbb{R}^n$ is bounded if $\sup\limits_{x \in \mathcal{O}} |x| < +\infty$.
- $\mathcal{O} \subset \mathbb{R}^n$ is connected if it cannot be written as a disjoint union of two nonempty relatively open subsets.
- A connected open set will sometimes be called a *domain*.
- The characteristic function of $\mathcal{O} \subset \mathbb{R}^n$ is defined as
$$\chi_{\mathcal{O}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O} \\ 0 & \text{if } x \notin \mathcal{O}. \end{cases}$$

?

Derivatives and Multi-index Notation

Let $\Omega \subset \mathbb{R}^n$ be an nonempty open set . Consider $f : \Omega \to \mathbb{R}$ and let $\xi \in \Omega$

- The partial derivative of f at ξ
$$\partial_i f(\xi) = f_{x_i}(\xi) = \frac{\partial f}{\partial x_i}(\xi) = \lim_{t \to 0} \frac{f(\xi + te_i) - f(\xi)}{t}$$
provided the limit exists. And so on
$$\partial_{ij} f(\xi) = f_{x_i x_j}(\xi) = \frac{\partial^2 f}{\partial x_i x_j}(\xi).$$
- The gradient of f
$$\nabla f = Df = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$
- The Hessian matrix
$$D^2 f = (\partial_{ij} f)_{1 \le i,j \le n}\,.$$
- The *Laplacian*
$$\Delta f = \sum_{i=1}^{n} \partial_{ii} f = trace\left(D^2 f\right) = div(\nabla f).$$
- *Multi-index notation*
An n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called an *multi-index* of order $|\alpha| := \sum\limits_{i=1}^{n} \alpha_i$.
$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ denotes a monomial of degree $|\alpha|$.
$D^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$.
We also denote
$$\alpha! := \alpha_1! \cdots \alpha_n!$$

For two multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n)$, we define the multi-index $\alpha + \beta$ as:

$$\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n).$$

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n)$ we say $\beta \le \alpha$ provided $\beta_i \le \alpha_i$ for all $1 \le i \le n$. And if $\beta \le \alpha$ we denote

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

- Multinomial expansion

$$\left(\sum_{i=1}^{m} x_i \right)^k = \sum_{|\alpha|=k} \begin{pmatrix} k \\ \alpha \end{pmatrix} x^\alpha,$$

where

$$\begin{pmatrix} k \\ \alpha \end{pmatrix} = \frac{k!}{\alpha_1! \cdots \alpha_n!}$$

- Leibniz formula

$$D^\alpha (fg)(\xi) = \sum_{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta f(\xi) D^{\alpha-\beta} g(\xi).$$

Functions

Let $\Omega \subset \mathbb{R}^n$ be an open set.

- $C(\Omega) = \{f : \Omega \to \mathbb{R} \text{ s.t } f \text{ is continuous}\}$.
- $C(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{R} \text{ s.t } f \text{ is continuous}\}$.
  An Extension result: If $f \in C(\Omega)$ is bounded and uniformly continuous on Ω, then it can be extended uniquely to a bounded continuous function on $\overline{\Omega}$.
- Let Ω be a bounded open set in $\mathbb{R}^n$. $C(\overline{\Omega})$ is a *Banach space* with respect to the supremum norm

$$||f||_\infty := \sup_{x \in \Omega} |f(x)|.$$

- We define the *support* of $f : \Omega \to \mathbb{R}^n$ to be the set

$$\text{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$$

- $$C_c(\Omega) = \{f \in C(\Omega) \text{ s.t supp}(u) \Subset \Omega\}.$$
- $C_0(\mathbb{R}^n)$ will denote the space of continuous functions on $\mathbb{R}^n$ that vanishes at infinity. $C_0(\mathbb{R}^n)$ is the completion of $C_c(\mathbb{R}^n)$ with respect to the sup-norm $|| \cdot ||$.
- $C^k(\Omega)$ will denote the space of functions with continuous partial derivatives of order $\le k$ in Ω.
- $C^\infty(\Omega)$ will denote the space of functions with continuous partial derivatives of all orders in Ω, that is, the space of smooth functions in Ω.
- $C^k(\overline{\Omega}) = \{f \in C^k(\Omega) \text{ s.t } D^\alpha f \in C(\overline{\Omega}) \text{ for all } |\alpha| \le k\}$.
- Let Ω be a bounded open set in $\mathbb{R}^n$. $C^k(\overline{\Omega})$ is a *Banach space* with respect to the norm

$$||f||_{C^k(\overline{\Omega})} := \sum_{|\alpha| \le k} \|D^\alpha f\|_\infty .$$

- Taylor's expansion: Suppose $f \in C^k(B(\xi,r))$, then for $x \in B(\xi,r)$

$$f(x) = u(\xi) + \sum_{|\alpha|=1}^{k-1} \frac{1}{\alpha!} D^\alpha f(\xi)(x-\xi)^\alpha + R_k(\xi, x-\xi),$$

with $R_m(\xi, x-\xi)$ is the remainder term given by

$$R_m(\xi, x-\xi) = \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha f\left((1-\theta)\xi + \theta x\right)(x-\xi)^\alpha$$

for some $0 < \theta < 1$.

- Borel's theorem: Given an arbitrarily family of constants $\{C_\alpha\}$, there exists $f \in C^\infty(\mathbb{R}^n)$ such that

$$\frac{D^\alpha f(0)}{\alpha!} = C_\alpha \quad \text{for all multi-index } \alpha.$$

- $C^\omega(\Omega)$ will denote the space of real analytic functions on Ω. A function is real-analytic in Ω if it has a power-series expansion that converges to the function in a ball of non-zero radius about every point in Ω.

- For $1 \le p < +\infty$ the space

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is Lebesgue measurable s.t } \int_\Omega f \ dx < +\infty \right\}$$

is a *Banach space* with respect to the norm

$$\|f\|_{L^p} := \left(\int_\Omega f \ dx \right)^{1/p}.$$

- The space $L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \text{ is Lebesgue measurable s.t } \|f\|_\infty < +\infty\}$ is a *Banach space* with respect to the norm $\|f\|_\infty = \ \text{ess-}\sup_\Omega |f|$, where the essential supremum is given by

$$\text{ess-}\sup_\Omega |f| = \inf\{M \in \mathbb{R} : |f| \le M \quad \text{a.e in } \Omega\}$$

- For $1 \le p \le +\infty$ the space

$$L^p_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ s.t } f \in L^p(\Omega') \text{ for any } \Omega' \Subset \Omega\}$$

.

- If f is continuous at ξ, then the average of f over small balls almost has the value $f(\xi)$.

$$\lim_{r\to 0} \frac{n}{\omega_{n-1} r^n} \int_{B(\xi,r)} f \ dx = f(\xi).$$

Hölder Functions.

- Let Ω be a domain in $\mathbb{R}^n$ and let $0 < \alpha \leq 1$. $f : \Omega \to \mathbb{R}$ is said to be Hölder continuous in Ω with exponent α if the quotient
$$[f]_{\alpha,\Omega} := \sup_{x,y\in\Omega,\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < +\infty.$$
The quantity $[f]_{\alpha,\Omega}$ is a seminorm.

- $f : \Omega \to \mathbb{R}$ is locally Hölder continuous in Ω with exponent α if the seminorm $[f]_{\alpha,\Omega'}$ is finite for all $\Omega' \Subset \Omega$.

- $C^{0,\alpha}(\Omega)$ will denote the the space of locally Hölder continuous functions in Ω with exponent α.

- Let Ω be bounded. $C^{0,\alpha}(\overline{\Omega})$ will denote the the space of Hölder continuous functions in Ω with exponent α.
  $C^{0,\alpha}(\overline{\Omega})$ is a Banach space with respect to the norm:
$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} = \sup_\Omega |f| + [f]_{\alpha,\Omega}.$$

- $C^{k,\alpha}(\Omega)$ will denote the space of functions with continuous partial derivatives of order $\leq k$ in Ω whose k-th partial derivatives are in $C^{0,\alpha}(\Omega)$.

- For Ω bounded, $C^{k,\alpha}(\overline{\Omega})$ will denote the space of functions with continuous partial derivatives of order $\leq k$ in Ω whose k-th partial derivatives are in $C^{0,\alpha}(\overline{\Omega})$.
  $C^{k,\alpha}(\overline{\Omega})$ is a Banach space with respect to the norm:
$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta|\leq k} \sup_\Omega |\partial^\beta f| + \sum_{|\beta|=k} [\partial^\beta f]_{\alpha,\Omega}.$$

CHAPTER 1

Laplace's Equation and Harmonic Functions

We start with the definition of the Laplacian or the Laplace operator which will be the fundamental object in our study of PDEs.

Definition 1.1 (Laplacian).

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

In this chapter we consider the Laplace's equation:

$$\Delta u = 0 \text{ in } \Omega,$$

where Ω will throughout denote a connected domain in $\mathbb{R}^n$. Laplace's equation is a linear, scalar equation, and is the prototype of an second order elliptic PDE.

Definition 1.2. *$u \in C^2(\Omega)$ is said to be harmonic in Ω if $\Delta u = 0$ in Ω.*

We will also consider the Poisson's equation which is the non-homogeneous version of the Laplace's equation:

$$\Delta u = f,$$

where f is a given function.

Exercise 1.3. *Show that the Laplace's equation is rotationally invariant.*

Exercise 1.4. *Find the expression of the Laplacian Δ in Polar-coordinates.*

1. Mean Value Properties

The goal here is to describe the properties of harmonic functions. We start by showing that the average over a ball of a harmonic function is equal to the value of the function at the center of the ball.

Definition 1.5. *Let $u \in C(\Omega)$. We say*

(1) *u satisfies the first mean value property if*

$$u(\xi) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{\partial B(\xi,r)} u(x)\, d\sigma(x) \qquad \text{for any } B(\xi,r) \subset \Omega.$$

(2) *u satisfies the second mean value property if*

$$u(\xi) = \frac{n}{\omega_{n-1} r^{n}} \int\limits_{B(\xi,r)} u(x)\, dx \qquad \text{for any } B(\xi,r) \subset \Omega.$$

Exercise 1.6. *Show that the two definitions above are equivalent.*

Exercise 1.7. *Show the following:*

(1) *u satisfies the first mean value property if*

$$u(\xi) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} u(\xi + rx)\ d\sigma(x) \qquad \text{for any } B(\xi, r) \subset \Omega.$$

(2) *u satisfies the second mean value property if*

$$u(\xi) = \frac{n}{\omega_{n-1}} \int_{B(0,1)} u(\xi + rx)\ dx \qquad \text{for any } B(\xi, r) \subset \Omega.$$

Theorem 1.8. *Let $u \in C^2(\Omega)$ be harmonic in Ω, then u satisfies the mean value property in Ω.*

Proof. Take a ball $B(\xi, r)$ in Ω and let $0 < \rho < r$. Integration by parts and change of variable formula gives us

$$\begin{aligned} 0 &= \int_{B(\xi,\rho)} \Delta u(x)\ dx = \int_{\partial B(\xi,\rho)} \frac{\partial u}{\partial \nu}(x)\ d\sigma(x) = \int_{\partial B(\xi,\rho)} \left(\nabla u(x), \frac{x-\xi}{\rho}\right)\ d\sigma(x) \\ &= \rho^{n-1} \int_{\mathbb{S}^{n-1}} (\nabla u(\xi + \rho x), x)\ \ d\sigma(x) = \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\mathbb{S}^{n-1}} u(\xi + \rho x)\ d\sigma(x). \end{aligned}$$

Hence

$$\frac{\partial}{\partial \rho} \int_{\mathbb{S}^{n-1}} u(\xi + \rho x)\ d\sigma(x) = 0.$$

Integrating from 0 to r gives

$$\int_{\mathbb{S}^{n-1}} u(\xi + rx)\ d\sigma(x) = \omega_{n-1} u(\xi).$$

So we have obtained for any $B(\xi, r) \subset \Omega$

$$u(\xi) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} u(\xi + rx)\ d\sigma(x) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B(\xi,r)} u(y)\ d\sigma(y).$$

Hence u satisfies the mean value property. □

Remark 1.9. *To satisfy the mean value property u need not be C^2. However a harmonic function is required to be C^2. It turns out that these two conditions are equivalent*

Theorem 1.10 (Regularity). *Let $u \in C(\Omega)$ satisfy the mean value property in Ω, then u is smooth and harmonic in Ω.*

Proof. Fix $\xi \in \Omega$ and let $\varepsilon < d(\xi, \partial\Omega)$. Consider the mollifiers $\eta_\varepsilon \in C_c^\infty(B(0, \varepsilon))$

$$(\eta_\varepsilon * u)(\xi) = \int_\Omega \eta_\varepsilon(\xi - y) u(y)\ dy.$$

Step 1: We show $u(\xi) = (\eta_\varepsilon * u)(\xi)$ for all $\xi \in \Omega_\varepsilon := \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$ This proves that u is smooth in Ω.

Proof of Step 1:

$$\begin{aligned}(\eta_\varepsilon * u)(\xi) &= \int\limits_\Omega u(x)\eta_\varepsilon(\xi - x)\ dx = \int\limits_\Omega u(x)\eta_\varepsilon(x-\xi)\ dx \\ &= \int\limits_{|x|<1} u(\xi+\varepsilon x)\eta(x)\ dx = \int\limits_0^1 r^{n-1} \int\limits_{\mathbb{S}^{n-1}} u(\xi+\varepsilon rx)\eta(rx)\ d\sigma\ dr \\ &= \int\limits_0^1 r^{n-1}\eta(r) \int\limits_{\mathbb{S}^{n-1}} u(\xi+\varepsilon rx)\ d\sigma\ dr = \omega_{n-1}u(\xi)\int\limits_0^1 r^{n-1}\eta(r)\ d\sigma\ dr \\ &= u(\xi).\end{aligned}$$

□

Step 2: u is harmonic in Ω since by the mean value property

$$\int\limits_{B(\xi,r)} \Delta u(x)\ dx = r^{n-1}\frac{\partial}{\partial r}\int\limits_{\mathbb{S}^{n-1}} u(\xi+rx)\ d\sigma(x) = 0 \qquad \text{for any } B(\xi,r)\subset\Omega.$$

□

Lemma 1.11. *Suppose $u \in C(\overline{B(\xi,r)})$ is harmonic in the ball $B(\xi,r)$. Then there holds*

$$|\partial_i u(\xi)| \le \frac{n}{r}\max_{B(\xi,r)}|u| \quad \textit{for all } 1\le i\le n. \tag{1.1}$$

Proof. u is smooth in $B(\xi,r)$ as u is harmonic. Then differentiating the Laplace's equation we obtain that $\partial_i u$ is harmonic in $B(\xi,r)$ for all $1\le i\le n$ and hence satisfies the mean value property. This together with Integration by parts gives us for all $1 \le i \le n$

$$\begin{aligned}\partial_i u(\xi) &= \frac{n}{\omega_{n-1}r^n}\int\limits_{B(\xi,r)} \partial_i u(x)\ dx = \frac{n}{\omega_{n-1}r^n}\int\limits_{\partial B(\xi,r)} u\nu_i\ d\sigma, \\ |\partial_i u(\xi)| &\le \frac{n}{\omega_{n-1}r^n}\max_{\partial B(\xi,r)}|u|\ \omega_{n-1}r^{n-1} \le \frac{n}{r}\max_{B(\xi,r)}|u|.\end{aligned}$$

□

Corollary 1.12. *Suppose $u \in C(\overline{B(\xi,r)})$ is a nonnegative harmonic in the ball $B(\xi,r)$. Then there holds*

$$|\partial_i u(\xi)| \le \frac{n}{r}u(\xi) \quad \textit{for all } 1\le i\le n.$$

Exercise 1.13 (Liouville's theorem)**.** *A bounded harmonic function in $\mathbb{R}^n$ is constant.*

Proposition 1.14. *Suppose $u \in C(\overline{B(\xi,r)})$ is harmonic in the ball $B(\xi,r)$. Then for any multi-index α*

$$|D^\alpha u(\xi)| \le \frac{n^{|\alpha|}e^{|\alpha|-1}|\alpha|!}{r^{|\alpha|}}\max_{B(\xi,r)}|u|.$$

PROOF. We prove the above estimate by induction on $|\alpha| = m$, which holds for $m = 1$ by lemma 1.11. Assume that the result holds for $|\alpha| = m$. Applying lemma 1.11 to $D^\alpha u$ on $B(\xi, \rho)$ with $\rho = (1-\theta)r$ and $0 < \theta < 1$, we have for all $1 \le i \le n$

$$|\partial_i D^\alpha u(\xi)| \le \frac{n}{\rho} \max_{B(\xi,\rho)} |D^\alpha u|.$$

By our induction hypothesis on the ball $B(x, r-\rho)$ for $x \in B(\xi, \rho)$

$$|D^\alpha u(x)| \le \frac{n^m e^{m-1} m!}{(r-\rho)^m} \max_{B(x,r-\rho)} |u| \le \frac{n^m e^{m-1} m!}{(r-\rho)^m} \max_{B(\xi,r)} |u|.$$

We then obtain for all $1 \le i \le n$

$$|\partial_i D^\alpha u(\xi)| \le \left(\frac{n}{\rho}\right) \frac{n^m e^{m-1} m!}{(r-\rho)^m} \max_{B(\xi,r)} |u| = \frac{1}{(1-\theta)\theta^m} \frac{n^{m+1} e^{m-1} m!}{r^{m+1}} \max_{B(\xi,r)} |u|.$$

Taking $\theta = \dfrac{m}{m+1}$ and using the inequality $\left(1 + \dfrac{1}{m}\right)^m \le e$, gives that

$$|\partial_i D^\alpha u(\xi)| \le \frac{n^{m+1} e^m (m+1)!}{r^{m+1}} \max_{B(\xi,r)} |u|.$$

This proves the hypothesis for $|\alpha| = m+1$ and the theorem follows. □

Theorem 1.15. *A harmonic function in Ω is real analytic.*

PROOF. Suppose u is harmonic in the domain $\Omega \subset \mathbb{R}^n$. Fix $\xi \in \Omega$ and take $r > 0$ such that $B(\xi, 2r) \subset \Omega$. Let By Taylor expansion we have for $|y| < r$

$$u(\xi + y) = u(\xi) + \sum_{|\alpha|=1}^{m-1} \frac{1}{\alpha!} D^\alpha u(\xi) y^\alpha + R_m(\xi, y)$$

where $R_m(\xi, y)$ is the remainder term given by

$$R_m(\xi, y) = \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha u(\xi + \theta y) y^\alpha \quad \text{for some } 0 < \theta < 1.$$

$$\begin{aligned}
|R_m(\xi, y)| &\le \sum_{|\alpha|=m} \left(\frac{|y^\alpha|}{\alpha!}\right) \frac{n^m e^{m-1} m!}{r^m} \max_{B(\xi,2r)} |u| \\
&\le \sum_{|\alpha|=m} \left(\frac{1}{\alpha!}\right) \frac{n^m e^{m-1} m!}{r^m} |y|^m \max_{B(\xi,2r)} |u| \\
&\le \frac{n^m}{m!} \cdot \frac{n^m e^{m-1} m!}{r^m} |y|^m \max_{B(\xi,2r)} |u| \\
&\le e^{-1} \left(\frac{n^2 e |y|}{r}\right)^m \max_{B(\xi,2r)} |u|
\end{aligned}$$

Then for any y such that $|y| \leq \dfrac{r}{2en^2}$ we have $\lim\limits_{m\to+\infty} |R_m(\xi, y)| = 0$. So the Taylor series of u at any $\xi \in \Omega$ converges to u in a ball of non-zero radius centered at ξ. □

Suppose u is harmonic in Ω, then by integration by parts

$$\int_\Omega u\Delta\varphi = 0 \qquad \text{for any } \varphi \in C_c^\infty(\Omega).$$

The converse is also true in a *weak sense*.

Theorem 1.16 (Weyl's lemma)**.** *Suppose $u \in L^1_{\mathrm{loc}}(\Omega)$ is weakly harmonic in the sense that*

$$\int_\Omega u\Delta\varphi = 0 \qquad \textit{for any } \varphi \in C_c^\infty(\Omega).$$

Then $u \in C^\infty(\Omega)$ and is harmonic in Ω.

Proof. To be completed ... □

2. Maximum Principle

Proposition 1.17 (Maximum Principle)**.** *Suppose $u \in C(\overline{\Omega})$ satisfies the mean value property in the domain Ω. Then u cannot assume its maximum and minimum values in the interior of Ω unless u is constant.*

Proof. Let $M := \max\limits_\Omega u$ and consider the level-set $\Sigma := u^{-1}\{M\}$, which is relatively closed in Ω since u is continuous. Consider $\xi \in \Sigma$ and take $B(\xi, r) \Subset \Omega$ for some $r > 0$. By the mean value property

$$M - u(\xi) = \frac{n}{\omega_{n-1} r^n} \int_{B(\xi,r)} (M-u)\ dx.$$

Since $u \leq M$ in Ω and $u(\xi) = M$, this implies that $u \equiv M$ in $B(\xi, r)$, and hence Σ is both relatively closed and open in open in Ω. Therefore either $\Sigma = \emptyset$ or $\Sigma = \Omega$, as Ω is connected. □

Exercise 1.18 (Uniqueness of Dirichlet boundary value problems)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ be given. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the* Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \textit{in } \Omega \\ u = g & \textit{on } \partial\Omega. \end{cases}$$

Theorem 1.19 (Harnack Inequality)**.** *Suppose $u \geq 0$ is harmonic in Ω. Then for any domain $\Omega' \Subset \Omega$ there exist a constant $C(\Omega, \Omega')$ such that*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

PROOF. We present two proofs.

Method 1. Consider $B(\xi, 4r) \Subset \Omega$ and $x, y \in B(\xi, r)$. By the mean value property

$$u(x) = \frac{n}{\omega_{n-1} r^n} \int\limits_{B(x,r)} u \, dz$$
$$\leq \frac{n}{\omega_{n-1} r^n} \int\limits_{B(\xi,2r)} u \, dz \leq \frac{n}{\omega_{n-1} r^n} \int\limits_{B(y,3r)} u \, dx = 3^n u(y).$$

It follows that

$$\sup_{B(\xi,r)} u \leq 3^n \inf_{B(\xi,r)} u.$$

Suppose now $\Omega' \Subset \Omega$. We choose $r > 0$ such that $4r < d(\Omega', \partial\Omega)$. Since $\overline{\Omega'}$ is compact, we can cover Ω by finitely many, say $k \in \mathbb{N}$, balls B_i of radius r and such that $B_i \cap B_{i+1} \neq \emptyset$. Therefore for any $x, y \in \Omega'$

$$u(x) \leq 3^{nk} u(y).$$

Method 2. By maximum principle $u > 0$ in Ω or $u \equiv const$. Consider $B(\xi, 2r) \Subset \Omega$. Applying corollary 1.12 to u in $B(z, r)$ for $z \in B(\xi, r)$, we have for some positive constant C depending only on n.

$$|\nabla u(z)| \leq \frac{C}{r} \, u(z) \quad \text{which implies} \quad \sup_{B(\xi,r)} |\nabla \log u| \leq \frac{C}{r}.$$

For any $x, y \in B(\xi, r)$ we then have

$$\log \frac{u(x)}{u(y)} = \int_0^1 [\log u((1-t)x + ty)]' dt$$
$$\leq |x - y| \int_0^1 |\nabla \log u((1-t)x + ty)| dt \leq \frac{C}{r} |x - y| \leq C.$$

So

$$u(x) \leq e^C u(y).$$

This yields the desired result as in the first method. □

Exercise* 1.20 (Harnack convergence theorem). *Let $(u_m)_{m \geq 1}$ be a monotone sequence of harmonic functions in Ω. Suppose there exists a point $\xi \in \Omega$ such that $(u_m(\xi))_{m \geq 1}$ is bounded. Then the sequence $(u_m)_{m \geq 1}$ converges uniformly on compact subsets of Ω to a harmonic function.*

DEFINITION 1.21. *Let $u \in C^2(\Omega)$. We say*

$$u \text{ is subharmonic in } \Omega \text{ if } -\Delta u \leq 0 \text{ in } \Omega,$$
$$u \text{ is superharmonic in } \Omega \text{ if } -\Delta u \geq 0 \text{ in } \Omega.$$

Exercise 1.22. *If $u \in C^2(\Omega)$ is subharmonic in Ω, then for any $B(\xi, r) \Subset \Omega$.*

$$u(\xi) \leq \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{\partial B(\xi,r)} u(x)\; d\sigma(x) \quad \text{and} \quad u(\xi) \leq \frac{n}{\omega_{n-1} r^{n}} \int\limits_{B(\xi,r)} u(x)\; dx.$$

Exercise 1.23. *Let u be harmonic in Ω and suppose $\psi : u(\Omega) \to \mathbb{R}$ is convex. Then show that $\psi \circ u$ is subharmonic in Ω.*

Exercise 1.24 (A Comparison result)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f \in C(\Omega)$. Suppose $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ solves the Poisson's equation:*

$$-\Delta u = f \;\; in \; \Omega.$$

If $u_1 \leq u_2$ on $\partial\Omega$, then

$$u_1(x) \leq u_2(x) \;\; \text{for all } x \in \Omega.$$

3. Fundamental Solutions

The Laplace's equation is invariant under rotations so it seems natural to look for radial solutions of the form $v(r) = u(x)$ where $r = |x|$.
One obtains

$$\Delta u = v'' + \frac{n-1}{r} v'.$$

This implies that radial harmonic functions in $\mathbb{R}^n$ satisfies the ODE:

$$v'' + \frac{n-1}{r} v' = 0.$$

Hence for $r > 0$

$$v(r) = \begin{cases} c_1 + c_2 \log r & \text{for } n = 2, \\ c_3 + c_4 r^{2-n} & \text{for } n \geq 3, \end{cases}$$

for some constants c_i, $1 \leq i \leq 4$. Note that v is singular at $r = 0$. We fix the constants so that

$$-\int\limits_{|x|=r} \frac{\partial v}{\partial \nu}\, d\sigma = 1 \qquad \text{for any } r > 0.$$

So we set

$$\Gamma(x) = \Gamma(|x|) := \begin{cases} -\dfrac{1}{2\pi} \log|x| & \text{for } n = 2 \\ \dfrac{1}{\omega_{n-1}(n-2)} \dfrac{1}{|x|^{n-2}} & \text{for } n \geq 3. \end{cases} \tag{1.2}$$

We have thus obtained a *singular solution* $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^n)$ to

$$-\Delta\Gamma = 0 \text{ in } \mathbb{R}^n \setminus \{0\},$$

satisfying $\displaystyle\int\limits_{|x|=r} \frac{\partial v}{\partial \nu}\, d\sigma = -1$ for any $r > 0$.

Exercise 1.25. *Show that for $n = 1$, $\Gamma(x) = -|x|/2$.*

Exercise 1.26. *Let $n \geq 2$. Show that $\Gamma, |\nabla\Gamma| \in L^1_{\text{loc}}(\mathbb{R}^n)$.*

Theorem-Definition 1.27 (Fundamental Solution). *We call the function Γ defined in (1.2) the fundamental solution of the Laplace equation.*

Similarly, fixing $x \in \mathbb{R}^n$ we consider

$$\Gamma(x,y) := \Gamma(x-y)$$

which then satisfies

$$-\Delta_y \Gamma(x,\cdot) = 0 \text{ in } \mathbb{R}^n \setminus \{x\} \text{ and } \int\limits_{|y-x|=r} \frac{\partial v}{\partial \nu}\, d\sigma(y) = -1 \quad \text{for any } r > 0.$$

Theorem 1.28 (Green's representation formula). *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then for any $\xi \in \Omega$ we have*

$$u(\xi) = -\int\limits_{\Omega} \Gamma(\xi,y)\Delta u(y)\, dy + \int\limits_{\partial\Omega} \left(\Gamma(\xi,y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(\xi,y) \right) d\sigma(y).$$

Proof. Applying the Gauss-Green formula to u and $\Gamma(\xi,\cdot)$ in the domain $\Omega \setminus B(\xi,\varepsilon)$ for $\varepsilon > 0$ small, we obtain

$$\int\limits_{\Omega\setminus B(\xi,\varepsilon)} (\Gamma\Delta u - u\Delta\Gamma)\, dy = \int\limits_{\partial\Omega} \left(\Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu} \right) d\sigma(y) - \int\limits_{\partial B(\xi,\varepsilon)} \left(\Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu} \right) d\sigma(y)$$

Since Γ is harmonic in $\Omega \setminus B(\xi,\varepsilon)$ we have

$$\int\limits_{\Omega\setminus B(\xi,\varepsilon)} \Gamma\Delta u\, dy = \int\limits_{\partial\Omega} \left(\Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu} \right) d\sigma(y) - \int\limits_{\partial B(\xi,\varepsilon)} \left(\Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu} \right) d\sigma(y)$$

We now let $\varepsilon \to 0$ and evaluate the above integrals. Since $u \in C^2(\Omega)$ we have as $\varepsilon \to 0$

$$\int\limits_{B(\xi,\varepsilon)} \Gamma\Delta u\, dy = \mathrm{O}\left(\int\limits_{B(\xi,\varepsilon)} \Gamma(\xi,y)\, dy \right) = \mathrm{o}(1),$$
$$\int\limits_{\partial B(\xi,\varepsilon)} \Gamma(\xi,y)\frac{\partial u}{\partial \nu}\, d\sigma(y) = \mathrm{O}\left(\varepsilon^{n-1}\Gamma(\varepsilon)\right) = \mathrm{o}(1).$$

And by the mean value property we have

$$-\int\limits_{\partial B(\xi,\varepsilon)} u(y)\frac{\partial \Gamma}{\partial \nu_y}(\xi,y)\, d\sigma(y) = \frac{1}{\omega_{n-1}\varepsilon^{n-1}} \int\limits_{\partial B(\xi,\varepsilon)} u(y)\, d\sigma(y) = u(\xi).$$

Combining and letting $\varepsilon \to 0$ we obtain

$$\int\limits_{\Omega} \Gamma(\xi,y)\Delta u(y)\, dy = \int\limits_{\partial\Omega} \left(\Gamma(\xi,y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(\xi,y) \right) d\sigma(y) - u(\xi)$$

□

Exercise* 1.29. *Suppose $u \in C\left(\overline{B(0,2r)}\right)$ is harmonic in $B(0,2r)$. Show that for some $C > 0$ depending only on n*

$$\|u\|_{L^\infty(B(0,r))} \le C\|u\|_{L^2(B(0,2r))} \quad \text{and} \quad \|\nabla u\|_{L^\infty(B(0,r))} \le C\|u\|_{L^\infty(B(0,2r))}.$$

Remark 1.30. *Taking $\varphi \in C_c^\infty(\Omega)$ we obtain by the Green's representation formula.*

$$\varphi(\xi) = -\int\limits_\Omega \Gamma(\xi, y)\Delta\varphi(y)\ dy.$$

This can be written symbolically as

$$-\Delta_y \Gamma(\xi, y) := \delta_\xi$$

where δ_ξ is the Dirac delta, a distribution, which acts on $C_c^\infty(\Omega)$ by $\delta_\xi[\varphi] = \varphi(\xi)$. $\Delta\Gamma(\xi, \cdot)$ can be defined as a distribution which acts on $C_c^\infty(\Omega)$ by

$$\Delta\Gamma(\xi, \cdot)[\varphi] := \int\limits_\Omega \Gamma(\xi, y)\Delta\varphi(y)\ dy.$$

We thus have

Theorem 1.31 (Solving the Poisson's equation)**.** *Suppose $f \in C_c^\infty(\mathbb{R}^n)$ and let $u := \Gamma * f$. Then $u \in C^\infty(\mathbb{R}^n)$ solves the Poisson's equation*

$$-\Delta u = f \ \text{in } \mathbb{R}^n.$$

4. Green's Function: Existence techniques I

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Then for any $\xi \in \Omega$ we have by the Green's representation formula

$$u(\xi) = -\int\limits_\Omega \Gamma(\xi, y)\Delta u(y)\ dy + \int\limits_{\partial\Omega} \left(\Gamma(\xi, y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(\xi, y)\right)\ d\sigma(y).$$

Then if one knows Δu, one can reconstruct u from its value on the boundary $u\ |_{\partial\Omega}$ and the value of its normal derivative on the boundary $\frac{\partial u}{\partial \nu}\ |_{\partial\Omega}$. Note that the normal derivative cannot be prescribed completely arbitrarily as by the Integration by parts formula

$$\int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu}\ d\sigma = \int\limits_\Omega \Delta u\ dx$$

We intend to eliminate the term $\dfrac{\partial u}{\partial \nu}$ by adjusting Γ. We fix $x \in \Omega$ and consider

$$\gamma(x, y) = \Gamma(x, y) - \Phi(x, y)$$

where $\Phi(x, \cdot) \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a harmonic corrector term satisfying the boundary problem

$$\begin{cases} -\Delta_y \Phi(x, y) = 0 & \text{in } \Omega \\ \quad \Phi(x, y) = \Gamma(x, y) & \text{on } \partial\Omega. \end{cases}$$

then by the Gauss-Green formula we can write

$$\int\limits_\Omega \Phi(x, y)\Delta u(y)\ dy = \int\limits_{\partial\Omega} \left(\Gamma(x, y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial \Phi}{\partial \nu_y}(x, y)\right)\ d\sigma(y).$$

This leads us to the notion of the Green's function.

Definition 1.32 (Green's function). *Let Ω be a domain in $\mathbb{R}^n$. A function $G : \overline{\Omega} \times \overline{\Omega} \setminus \{(x, x) : x \in \overline{\Omega}\} \longrightarrow \mathbb{R}$ satisfying for each fixed $x \in \Omega$*

- *$G(x, y) - \Gamma(x, y)$ is harmonic in Ω,*

- *$G(x, y) \equiv 0$ on in $\partial\Omega$,*

is called the Green's function of the Laplacian in Ω. Further for any $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ one can write the following representation formula

$$u(x) = -\int_{\Omega} G(x,y)\Delta u(y)\ dy - \int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu_y}(x,y)\ d\sigma(y). \tag{1.3}$$

Note 1.33. *By the maximum principle, the Green's function is unique if it exists, and symbolically for each fixed $x \in \Omega$*

$$\begin{cases} -\Delta_x G(x,y) = \delta_x & \text{in } \Omega \\ G(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the *Dirichlet boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where f and g are given functions, f defined on Ω and g defined on the boundary $\partial\Omega$. Using the Green's representation formula (1.3) we can invert the problem and express u in terms of f and g. So we have a formula for the solution of the Dirichlet boundary value problem, provided we can construct the Green's function.

Exercise* 1.34 (Bounds on the Green Funtions)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$. Show that for $x, y \in \Omega$ with $x \neq y$*

$$0 < G(x,y) < \frac{1}{\omega_{n-1}(n-2)|x-y|^{n-2}} \qquad \text{for } n \geq 3,$$

$$0 < G(x,y) < -\frac{1}{2\pi}\log|x-y| + \frac{1}{2\pi}\log diam(\Omega) \qquad \text{for } n = 2.$$

Proposition 1.35. *The Green's function $G(x,y)$ is symmetric in $\Omega \times \Omega$, that is, $G(x,y) = G(y,x)$ for $x \neq y \in \Omega$.*

Proof. Pick $\xi_1 \neq \xi_2 \in \Omega$. Then applying the Gauss-Green formula to $G_1 := G(\xi_1, \cdot)$ and $G_2 := G(\xi_2, \cdot)$ in $\Omega \setminus (B(\xi_1, \varepsilon) \cup B(\xi_2, \varepsilon))$ for ε small we get

$$\int_{\Omega\setminus(B(\xi_1,\varepsilon)\cup B(\xi_2,\varepsilon))} (G_1\Delta G_2 - G_2\Delta G_1)\ dy = \int_{\partial\Omega}\left(G_1\frac{\partial G_2}{\partial\nu} - G_2\frac{\partial G_1}{\partial\nu}\right)\ d\sigma(y)$$

$$-\int_{\partial B(\xi_1,\varepsilon)}\left(G_1\frac{\partial G_2}{\partial\nu} - G_2\frac{\partial G_1}{\partial\nu}\right)\ d\sigma(y) - \int_{\partial B(\xi_2,\varepsilon)}\left(G_1\frac{\partial G_2}{\partial\nu} - G_2\frac{\partial G_1}{\partial\nu}\right)\ d\sigma(y)$$

Since G_i is harmonic in $\Omega \setminus \{\xi\}$ and $G_i \equiv 0$ on $\partial\Omega$, $i = 1, 2$, we obtain

$$\int_{\partial B(\xi_1,\varepsilon)}\left(G_1\frac{\partial G_2}{\partial\nu} - G_2\frac{\partial G_1}{\partial\nu}\right)\ d\sigma(y) + \int_{\partial B(\xi_2,\varepsilon)}\left(G_1\frac{\partial G_2}{\partial\nu} - G_2\frac{\partial G_1}{\partial\nu}\right)\ d\sigma(y) = 0.$$

One has, as $\varepsilon \to 0$

$$\int_{\partial B(\xi_i,\varepsilon)} G_i\frac{\partial G_j}{\partial\nu}\ d\sigma = \mathrm{o}(\varepsilon) \quad \text{and} \quad \int_{\partial B(\xi_i,\varepsilon)} G_j\frac{\partial G_i}{\partial\nu}\ d\sigma = G_j(\xi_i) + \mathrm{o}(\varepsilon),$$

where $1 \leq i \neq j \leq 2$. Therefore letting $\varepsilon \to 0$ we obtain $G_1(\xi_2) = G_2(\xi_1)$ which gives $G(\xi_1, \xi_2) = G(\xi_2, \xi_1)$. □

The next result yields an expression for Green's functions in balls.

Proposition 1.36. *The Green's function for the ball $B(0,R)$ is given by*

$$G(x,y) = \frac{1}{(n-2)\omega_{n-1}}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right) \qquad \text{for } n \geq 3,$$

$$G(x,y) = -\frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right) \qquad \text{for } n = 2.$$

And for $x = 0$

$$G(0,y) = \frac{1}{(n-2)\omega_{n-1}}\left(|y|^{2-n} - |R|^{2-n}\right) \qquad \text{for } n \geq 3,$$

$$G(0,y) = -\frac{1}{2\pi}\left(\log|y| - \log|R|\ \right) \qquad \text{for } n = 2.$$

Proof. We start by defining reflection with respect to the sphere $\partial B(0,R)$.

Definition 1.37. *For $x \in \mathbb{R}^n \setminus \{0\}$ we let*

$$\tilde{x} = \frac{R^2}{|x|^2}x$$

The map $x \longmapsto \tilde{x}$ is a conformal map (preserves angles) and is called the inversion across the sphere $\partial B(0,R)$.

One has for $y \in \partial B(0,R)$

$$|\tilde{x}-y|^2 = |\tilde{x}|^2 - 2(\tilde{x},y) + |y|^2 = \frac{R^2}{|x|^2}\left(|x|^2 - 2(x,y) + R^2\right),$$
$$= \frac{R^2}{|x|^2}|x-y|^2.$$

The function $\Gamma(\tilde{x},y)$ has pole at $\tilde{x}$ and so is harmonic in the ball $B(0,R)$. Therefore in order to have vanishing boundary values we take

$$G(x,y) = \Gamma(|x-y|) - \Gamma\left(\frac{|x|}{R}|\tilde{x}-y|\right).$$

For $n \geq 3$ we have

$$G(x,y) = \frac{1}{(n-2)\omega_{n-1}}\left(|x-y|^{2-n} - \frac{R^{n-2}}{|x|^{n-2}}|\tilde{x}-y|^{2-n}\right).$$

Exercise 1.38. *The case $n = 2$ goes similarly.*

□

Exercise 1.39. *Suppose G is the Green's function in $B(0,R)$. Then there holds*

$$\frac{\partial G}{\partial \nu_y}(x,y) = \frac{|x|^2 - R^2}{\omega_{n-1}R|x-y|^n} \qquad \text{for any } x \in B(0,R) \text{ and } y \in \partial B(0,R).$$

Definition 1.40 (Poisson kernel). *The Poisson kernel for the ball $B(0,r)$ is defined as:*

$$K(x,y) := \frac{\partial G}{\partial \nu_y}(x,y)$$
$$= \frac{|x|^2 - R^2}{\omega_{n-1}R|x-y|^n} \qquad \text{for } x \in B(0,R) \text{ and } y \in \partial B(0,R).$$

Theorem 1.41 (Poisson Integral Formula)**.** *Let $g \in C(\partial B(0,R))$ and consider the function u defined by*

$$u(x) := \frac{R^2 - |x|^2}{\omega_{n-1} R} \int\limits_{\partial B(0,R)} \frac{1}{|x-y|^n} g(y)\ d\sigma(y) \quad \text{for } x \in B(0,R).$$

Then $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ and satisfies the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } B(0,R) \\ u = g & \text{on } \partial B(0,R). \end{cases}$$

Proof. Suppose u solves the above boundary value problem, then $u \in C^\infty(\Omega)$ and by the Green's representation formula (1.3) we have for $x \in B(0,R)$

$$\begin{aligned} u(x) &= -\int\limits_{\partial B(0,R)} u(y) \frac{\partial G}{\partial \nu_y}(x,y)\ d\sigma(y) = -\int\limits_{\partial B(0,R)} u(y) K(x,y)\ d\sigma(y) \\ &= \frac{R^2 - |x|^2}{\omega_{n-1} R} \int\limits_{\partial B(0,R)} \frac{1}{|x-y|^n} u(y)\ d\sigma(y). \end{aligned}$$

Note that taking $u \equiv 1$ gives that

$$\int\limits_{\partial B(0,R)} K(x,y)\ d\sigma(y) = -1 \ \text{ for all } x \in B(0,R).$$

We only need to prove the continuity of u up to the boundary $\partial B(0,R)$. Let $\xi \in \partial B(0,R)$ and let $x \in B(0,R)$, then

$$\begin{aligned} u(x) - u(\xi) &= -\int\limits_{\partial B(0,R)} u(y) K(x,y)\ d\sigma(y) + \int\limits_{\partial B(0,R)} u(\xi) K(x,y)\ d\sigma(y) \\ &= \int\limits_{\partial B(0,R)} \left(g(\xi) - g(y)\right) K(x,y)\ d\sigma(y). \end{aligned}$$

Then for $\delta > 0$ small as $x \to \xi$

$$
\begin{aligned}
|u(x) - u(\xi)| &\le -\int_{\partial B(0,R)} |g(\xi) - g(y)|\, K(x,y)\ d\sigma(y) \\
&\le -\int_{\partial B(0,R)\cap|y-\xi|<\delta} |g(\xi) - g(y)|\, K(x,y)\ d\sigma(y) \\
&\quad -\int_{\partial B(0,R)\cap|y-\xi|\ge\delta} |g(\xi) - g(y)|\, K(x,y)\ d\sigma(y) \\
&\le -\int_{\partial B(0,R)\cap|y-\xi|<\delta} |g(\xi) - g(y)|\, K(x,y)\ d\sigma(y) \\
&\quad -2\sup_{\partial B(0,R)} |g| \int_{\partial B(0,R)\cap|y-\xi|\ge\delta} \frac{|x|^2 - R^2}{\omega_{n-1} R |x-y|^n}\ d\sigma(y) \\
&\le -\int_{\partial B(0,R)\cap|y-\xi|<\delta} |g(\xi) - g(y)|\, K(x,y)\ d\sigma(y) \\
&\quad -2\sup_{\partial B(0,R)} |g| \frac{2^n}{\omega_{n-1} R \delta^n} \int_{\partial B(0,R)\cap|y-\xi|\ge\delta} (|x|^2 - R^2)\ d\sigma(y)
\end{aligned}
$$

Taking δ sufficiently small and x close to $\xi \in \partial B(0,R)$ one obtains that $\lim_{x\to\xi} u(x) = u(\xi)$. □

Exercise 1.42 (Harnack's Inequality). *Let u be a non-negative harmonic function in $B(\xi, R)$. Then for $r = |x - \xi| < R$*

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r} u(\xi) \le u(x) \le \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r} u(\xi).$$

Exercise 1.43. *Using the above estimate prove the* Liouville's theorem*, that is, a non-negative harmonic function in R^n is constant.*

Exercise 1.44 (Green's in the upper half space). *Find the Green's function for the Laplace operator in the upper half space $\mathbb{R}^n_+ = \{x : x_n > 0\}$ and then derive a formal integral representation for a solution of the Dirichlet problem:*

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \{x : x_n = 0\}. \end{cases}$$

Exercise 1.45 (Schwartz reflection principle). *Suppose u is harmonic in $B_+(0,R) := B(0,R) \cap \mathbb{R}^n_+$, and $u \in C^2(\overline{B_+(0,R)})$ with $u \equiv 0$ on $B(0,R) \cap \{x_n = 0\}$. Set*

$$\tilde{u}(x) = \begin{cases} u(x) & \text{when } x_n \ge 0 \\ -u(x_1, \ldots, -x_n) & \text{when } x_n < 0 \end{cases} \quad \text{for } x \in B(0,R)$$

Show that $\tilde{u}$ is harmonic in $B(0,R)$.

The fundamental solution of the Laplace operator Δ has an isolated singularity and is harmonic everywhere else. It turns out that the isolated singularity of a harmonic function can be removed if the singularity is "better" than that of the fundamental solution.

Proposition 1.46 (Singularity removal theorem)**.** *Suppose u is harmonic in $B(0,R)\setminus\{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log|x|) & \textit{for } n=2 \\ o(|x|^{2-n}) & \textit{for } n\geq 3 \end{cases} \quad \textit{as } |x|\to 0.$$

Then u can be defined at $x=0$ so that it is C^2 and is harmonic in $B(0,R)$.

Proof. Let v solve

$$\begin{cases} -\Delta v = 0 & \text{in } B(0,R/2) \\ v = u & \text{on } \partial B(0,R/2). \end{cases}$$

The existence of v is guaranteed by Theorem 1.41, and by the maximum principle

$$\sup_{B(0,R)} |v| \leq \sup_{\partial B(0,R/2)} |u| \text{ in } B(0,R/2).$$

Set $w := u - v$. We will show that $w\equiv 0$ in $B(0,R/2)$. Let $r << 1$. Now w is harmonic in $B(0,R/2)\setminus B(0,r)$ with $w\equiv 0$ on $\partial B(0,R/2)$ and

$$\sup_{\partial B(0,r)} |w| \leq 2 \sup_{\partial B(0,r)} |u|.$$

Which then implies as $r\to 0$

$$\sup_{\partial B(0,r)} |x|^{n-2}|w(x)| = o(1) \text{ for } n\geq 3,$$
$$\sup_{\partial B(0,r)} \log|x||w(x)| = o(1) \text{ for } n=2.$$

which we write

$$\sup_{z\in\partial B(0,r)} \frac{|w(z)|}{\Gamma(z)} = o(1) \quad \text{as } r\to 0.$$

Using the maximum principle we obtain

$$\frac{|w(x)|}{\Gamma(x)} \leq \sup_{z\in\partial B(0,r)} \frac{|w(z)|}{\Gamma(z)} \quad \text{for } x\in B(0,R/2)\setminus B(0,r),$$

So for each fixed $x\in B(0,R/2)\setminus B(0,r)$

$$|w(x)| \leq \sup_{z\in\partial B(0,r)} \frac{|w(z)|}{\Gamma(z)}\,\Gamma(x).$$

Passing to the limits as $r\to 0$ implies $w\equiv 0$ in $B(0,R/2)\setminus\{0\}$. □

Remark 1.47 (Just continuity is not enough!!)**.** *For any $u\in C^2$ the Laplacian Δu is continuous. Conversely, we can ask whether u is C^2 when Δu is continuous. This is not true in general!*

Exercise 1.48. *Let* $0 < R < 1$ *and consider the ball* $B(0,R) \subset \mathbb{R}^2$*. Let*

$$f = \begin{cases} \dfrac{x_1^2 - x_2^2}{2|x|^2}\left[\dfrac{4}{(-\log|x|)^{1/2}} + \dfrac{1}{2(-\log|x|)^{3/2}}\right] & \text{if } |x| \neq 0 \\ 0 & \text{if } |x| = 0. \end{cases}$$

Then $f \in C(\overline{B(0,R)})$*. Consider*

$$u = (x_1^2 - x_1^2)(-\log|x|)^{1/2} \in C(\overline{B(0,R)}) \cap C^\infty(\overline{B(0,R)} \setminus \{0\}).$$

We have $-\Delta u = f$ *in* $B(0,R) \setminus \{0\}$*, but note* $u \notin C^2(B(0,R))$ *as* $\lim_{|x|\to 0} |\partial_{x_1 x_1} u| = +\infty$*.*

Show that with f *defined as above the equation* $-\Delta u = f$ *has no* C^2*-solutions in* $B(0,R)$*.*

CHAPTER 2

Maximum Principle for Elliptic PDEs

The goal of this chapter is to obtain maximum principle type results for general second order linear elliptic operators, extending the case of the Laplacian. We obtain *apriori bounds* and then establish the existence of solutions.

DEFINITION 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Consider the partial differential operator*

$$\mathcal{L}u := -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

for $u \in C^2(\Omega) \cap C(\overline{\Omega})$. We will assume the following conditions on the coefficients:

(i) *Symmetry: $a_{ij}(x) = a_{ji}(x)$ for all $1 \le i, j \le n$ and $x \in \Omega$.*

(ii) *Ellipticity: There exists a constant $\Lambda > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \Lambda|\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

that is, the matrix $\big(a_{ij}(x)\big)_{1\le i,j<n}$ is positive definite for all $x \in \Omega$ with its smallest eigenvalue always greater than or equal to Λ. In this case, we say that $\mathcal{L}$ **is an uniformly elliptic operator** *in Ω.*

(iii) *Boundedness: a_{ij}, b_i, $c \in C(\overline{\Omega})$ and hence for some constant C*

$$\big|a_{ij}(x)\big|, \big|b_i(x)\big|, |c(x)| \le C \quad \text{for all } 1 \le i, j \le n \text{ and } x \in \Omega.$$

In this chapter we consider the equation $\mathcal{L}u = f$ in Ω with suitable boundary conditions.

1. Maximum Principles

Ω will always denote a bounded domain in $\mathbb{R}^n$.

Theorem 2.2 (Weak Maximum Principle)**.** *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{L}u \le 0$ in Ω (a subsolution) with $c(x) \ge 0$ in Ω. Then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+,$$

where $u^+(x) := \max\{u(x), 0\}$.

PROOF. We break the proof into multiple steps based on our assumptions on the operator $\mathcal{L}$, progressively improving and completing the proof of the theorem in the last step.

Step 1: If $\mathcal{L}u = -\Delta u < 0$ in Ω.

Proof of Step 1: Suppose $\sup_{\Omega} u^+ = u(\xi)$ for some point $\xi \in \Omega$. Then ξ is a point of maximum, so $\nabla u(\xi) = 0$ and the Hessian matrix $D^2 u(\xi)$ is nonpositive or negative semidefinite. By the definition of the Laplacian $\Delta u(\xi) = trace(D^2u(\xi)) \leq 0$, a contradiction since $-\Delta u < 0$ in Ω. Therefore u attains its nonnegative maximum only on $\partial\Omega$. □

Step 2: If $\mathcal{L}u < 0$ in Ω with $c \geq 0$.
Proof of Step 2: Suppose $\sup_\Omega u^+ = u(\xi)$ for some point $\xi \in \Omega$. Then again, ξ is a point of maximum, $\nabla u(\xi) = 0$ and the Hessian matrix $D^2u(\xi)$ is negative semidefinite. By the ellipticity condition the coefficient matrix $A := \big(a_{ij}(\xi)\big)_{1\leq i,j<n}$ is positive definite, and hence the product matrix $A \times D^2u(\xi)$ is negative semidefinite. We obtain $\sum_{i,j=1}^{n} a_{ij}(\xi)\partial^2_{ij}u(\xi) = trace\big(A \times D^2u(\xi)\big) \leq 0$ and then $\mathcal{L}u \geq 0$, a contradiction. So u attains its nonnegative maximum only on $\partial\Omega$ □

Step 3: If $\mathcal{L}u \leq 0$ in Ω with $c \geq 0$.
Proof of Step 3: For $\varepsilon > 0$ consider $v_\varepsilon(x) := u(x)+\varepsilon e^{\alpha x_1}$, with α to be determined later. We get

$$\mathcal{L}v_\varepsilon(x) = \mathcal{L}u(x) - \varepsilon e^{\alpha x_1}\left(a_{11}(x)\alpha^2 - b_1(x)\alpha - c(x)\right)$$

Since b_1 and x are bounded and $a_{11}(x) \geq \Lambda > 0$ for all $x \in \Omega$, by choosing $\alpha > 0$ large enough we obtain

$$a_{11}(x)\alpha^2 - b_1(x)\alpha - c(x) > 0 \text{ for all } x \in \Omega$$

which implies $\mathcal{L}v_\varepsilon < 0$ in Ω for all $\varepsilon > 0$. By Step 2, v attains its nonnegative maximum only on $\partial\Omega$, that is

$$\sup_\Omega v_\varepsilon \leq \sup_{\partial\Omega} v_\varepsilon^+$$

Then for all $\varepsilon > 0$.

$$\sup_\Omega u \leq \sup_\Omega v_\varepsilon \leq \sup_{\partial\Omega} v_\varepsilon^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon \sup_{\partial\Omega} e^{\alpha x_1}$$

Letting $\varepsilon \to 0$, we obtain the proof of the weak maximum principle.

□

Exercise 2.3. *If $c \equiv 0$, then the requirement of non-negativity of u in Theorem 2.2 can be removed.*

Exercise 2.4 (Uniqueness of Dirichlet boundary value problems)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ be given. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the* Dirichlet boundary value problem:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

when $c(x) \geq 0$.

Remark 2.5. *The boundedness of Ω is essential to get the above uniqueness. Also, the non-negativity assumption of c cannot be removed in general.*

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Then $u(x, y) = \sin(x)\sin(y)$ also solves the boundary value problem:

$$\begin{cases} -\Delta u - 2u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 2.6 (Strong Maximum Principle)**.** *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\mathcal{L}u \le 0$ in Ω with $c(x) \ge 0$ in Ω. Then the nonnegative maximum of u over $\overline{\Omega}$ can be attained only on $\partial\Omega$ unless u is constant.*

Proof. For the proof of the strong maximum principle we will use the Hopf boundary point lemma.

Lemma 2.7 (Hopf Lemma)**.** *Consider the ball $B(0,R)$ and let $\xi \in \partial B(0,R)$. Suppose $u \in C^2(B(0,R)) \cap C\left(\overline{B(0,R)}\right)$ satisfies $\mathcal{L}u \le 0$ in $B(0,R)$ with $c(x) \ge 0$ in $B(0,R)$. Assume*

$$u(x) < u(\xi) \quad \text{for all } x \in \overline{B(0,R)} \setminus \{\xi\} \quad \text{and } u(\xi) \ge 0$$

Then there holds

$$\liminf_{t \to 0^+} \frac{u(\xi - t\nu) - u(\xi)}{t} < 0,$$

where $\nu(\xi)$ is the outward normal vector to $\partial B(0,R)$ at ξ.

If $u \in C^1\left(\overline{B(0,R)}\right)$, then

$$\frac{\partial u}{\partial \nu}(\xi) > 0.$$

Proof. We perturb u and write for all $\varepsilon > 0$ $v_\varepsilon(x) := u(x) + \varepsilon h(x)$ for some nonnegative function h. We want h such that $\frac{\partial h}{\partial \nu}(\xi) < 0$ and $\mathcal{L}v_\varepsilon < 0$, and we choose $\varepsilon > 0$ appropriately so that v_ε attains its nonnegative maximum only at the point ξ.

Denote $\Sigma = B(0,R) \setminus B\left(0, \frac{1}{2R}\right)$ and consider $h(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}$ with α to be determined. We have

$$\begin{aligned}\mathcal{L}h = &- e^{-\alpha|x|^2}\Big[4\alpha^2 \sum_{i,j=1}^n a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^n a_{ii}(x) + 2\alpha \sum_{i=1}^n b_i(x)x_i - c(x)\Big] \\ &- \ c(x)e^{-\alpha R^2} \\ &\le -e^{-\alpha|x|^2}\Big[4\alpha^2 \sum_{i,j=1}^n a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^n (a_{ii}(x) - b_i(x)x_i) - c(x)\Big].\end{aligned}$$

By the ellipticity condition we have

$$\sum_{i,j=1}^n a_{ij}(x)x_i x_j \ge \Lambda |x|^2 \ge \Lambda \left(\frac{1}{2R}\right)^2 > 0 \ \text{ in } \Sigma.$$

So for α large enough we have $\mathcal{L}h < 0$ in Σ and therefore $\mathcal{L}v_\varepsilon = \mathcal{L}u + \varepsilon \mathcal{L}h < 0$ in Σ for all $\varepsilon > 0$. By Step 2 of the weak maximum principle theorem, v_ε cannot attain its nonnegative maximum inside Σ. Next we prove that for $\varepsilon > 0$ sufficiently small, v_ε attains its nonnegative maximum at ξ. Consider v_ε on the boundary $\partial\Sigma$

- For $x \in \partial B\left(0, \frac{1}{2R}\right)$, since $u(x) < u(\xi)$, we have $u(x) < u(\xi) - \delta$ for some $\delta > 0$. Take $\varepsilon > 0$ small such that $\varepsilon h < \delta$ on $\partial B\left(0, \frac{1}{2R}\right)$. Hence for $\varepsilon > 0$ small we have $v_\varepsilon(x) < u(\xi)$ for $x \in \partial B\left(0, \frac{1}{2R}\right)$.
- On $\partial B(0,R)$, $h(x) \equiv 0$ and $u(x) < u(\xi)$ for $x \ne \xi$. Hence $v_\varepsilon(x) < u(\xi) = v_\varepsilon(\xi)$ on $\partial B(0,R) \setminus \{\xi\}$.

Therefore we conclude for $\varepsilon > 0$ sufficiently small,

$$\frac{v_\varepsilon(\xi) - v_\varepsilon(\xi - t\nu)}{t} \geq 0 \quad \text{for any small } t > 0.$$

Hence by letting $t \to 0$ we obtain, or $\varepsilon > 0$ sufficiently small

$$\liminf_{t \to 0^+} \frac{u(\xi) - u(\xi - t\nu)}{t} \geq -\varepsilon \frac{\partial h}{\partial \gamma}(\xi).$$

By definition of h we have

$$\frac{\partial h}{\partial \nu}(\xi) < 0,$$

which completes the proof of the Hopf lemma. □

Proof of the Strong Maximum Principle: Let $M := \max_{\overline{\Omega}} u^+$, and set $\Sigma := \{x \in \Omega : u(x) = M\}$, which is relatively closed in Ω since u is continuous. We will show that either $\Sigma = \Omega$ or $\Sigma = \emptyset$.

We proceed by contradiction and suppose $\Sigma \neq \emptyset$. If $\Sigma \neq \Omega$, we take the largest ball $B(\xi_0, r) \subset \Omega \setminus \Sigma = \{x \in \Omega : u(x) < M\}$ and so $u(\xi) = M$ for some $\xi \in \partial B(\xi_0, r)$.Then $\nabla u(\xi) = 0$ as ξ is an interior maximum point of u.

We then have $\mathcal{L}u \leq 0$ in $B(\xi_0, r)$, and without loss of generality we can assume that $u(x) < u(\xi)$ for all $x \in \overline{B(\xi_0, r)} \setminus \{\xi\}$, since we can construct a tangent ball $\widetilde{B}$ to $B(\xi_0, r)$ at ξ such that $\widetilde{B} \subset B(\xi_0, r)$ (draw!). By the Hopf Lemma we obtain that $\dfrac{\partial u}{\partial \nu}(\xi) > 0$, which is a contradiction. □

Exercise 2.8 (Comparison Principle)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{L}u \leq 0$ in Ω with $c(x) \geq 0$ in Ω. If $u \leq 0$ on $\partial\Omega$, then show that $u \leq 0$ in Ω. In fact, show that either $u < 0$ in Ω or $u \equiv 0$ in Ω.*

Definition 2.9. *An open set Ω satisfies the interior sphere property at $\xi \in \partial\Omega$ if there is an open ball $B(x, r) \subset \Omega$ such that $\xi \in \partial B(x, r)$.*

The interior sphere condition is satisfied by open sets with C^2-boundary, but it need not be satisfied by open sets with a C^1-boundary. (Why?)

Exercise 2.10. *Let Ω staisfy the interior sphere property and suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{L}u \leq 0$ in Ω with $c(x) \geq 0$ in Ω. Assume that u attains its nonnegative maximum at $\xi \in \overline{\Omega}$. Then $\xi \in \partial\Omega$ and for any outward pointing vector γ at ξ to $\partial\Omega$*

$$\frac{\partial u}{\partial \gamma}(\xi) > 0,$$

or $u \equiv const.$ in Ω.

Exercise* 2.11. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the interior sphere property. Consider the following boundary value problem :*

$$(*) \qquad \begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \alpha(x) u = g & \text{on } \partial\Omega. \end{cases}$$

where $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$ are given. Suppose $c(x) \geq 0$ in Ω and $\alpha \in C(\partial\Omega)$ with $\alpha \geq 0$ on $\partial\Omega$.

(1) *If $c \not\equiv 0$ or $\alpha \not\equiv 0$, then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to $(*)$.*
(2) *If $c \equiv 0$ or $\alpha \equiv 0$, then $C^2(\Omega) \cap C(\overline{\Omega})$ solutions to $(*)$ differ by additive constants.*

The following theorem generalizes the comparison principle with no restriction on the sign of c.

Theorem 2.12 (Comparison Principle of Serrin)**.** *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{L}u \geq 0$ in Ω in Ω. If $u \geq 0$ in Ω then show that either $u > 0$ in Ω or $u \equiv 0$ in Ω.*

Proof. We present two proofs
Method 1. Suppose $u(\xi) = 0$ for some point $\xi \in \Omega$. We will show that $u \equiv 0$ in Ω.
We write $c(x) = c^+(x) - c^-(x)$, where

$$\underbrace{c^+(x) := \max\{c(x), 0\}}_{\text{positive part of } c}, \qquad \underbrace{c^-(x) := \max\{-c(x), 0\}}_{\text{negative part of } c}, .$$

Then u satisfies

$$-\sum_{i,j}^{n} a_{ij}\partial_{ij}^2 u + \sum_{i=1}^{n} b_i \partial_i u + c^+ u \geq c^- u \geq 0,$$

and we obtain by the strong maximum principle $u \equiv 0$.
Method 2. Set $v = ue^{-\alpha x_1}$ for some $\alpha > 0$ to be determined. Since $\mathcal{L}u \geq 0$ we obtain

$$-\sum_{i,j}^{n} a_{ij}\partial_{ij}^2 v - \sum_{i=1}^{n} \left[\alpha\left(a_{1i} + a_{i1}\right) - b_i\right]\partial_i v - \left(a_{11}\alpha^2 - b_1\alpha - c\right)v \geq 0,$$

Choosing $\alpha > 0$ large such that $a_{11}\alpha^2 - b_1\alpha - c > 0$ implies

$$-\sum_{i,j}^{n} a_{ij}\partial_{ij}^2 v - \sum_{i=1}^{n} \left[\alpha\left(a_{1i} + a_{i1}\right) - b_i\right]\partial_i v \geq 0,$$

and then by the strong maximum principle either $v > 0$ or $v \equiv 0$ in Ω.

□

Exercise* 2.13. *Prove the general maximum principle for $\mathcal{L}$ with no restriction on the sign of c.*

Theorem 2.14. *Suppose there exists a $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\varphi > 0$ in $\overline{\Omega}$ and $\mathcal{L}\varphi \geq 0$ in Ω. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{L}u \leq 0$ in Ω, then $\dfrac{u}{\varphi}$ cannot assume its nonnegative maximum in Ω unless $\dfrac{u}{\varphi} \equiv const.$*

2. Apriori Estimates

$$\mathcal{L} := -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} + \sum_{i}^{n} b_i(x)\partial_i + c(x)$$

$\mathcal{L}$ is an uniformly elliptic operator in Ω with ellipticity constant Λ with continuous coefficients and we set Θ such that

$$\sum_{i,j=1}^{n} \max_{\Omega} |a_{ij}| + \sum_{i=1}^{n} \max_{\Omega} |b_i| \leq \Theta.$$

Proposition 2.15. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f \in C(\Omega)$, $g \in C(\partial\Omega)$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the boundary value problem:*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with $c(x) \geq 0$. Then the following estimate holds

$$\sup_{\Omega} \big|u\big| \leq C \max_{\Omega} |f| + \max_{\partial\Omega} |g|$$

where $C > 0$ is a constant depending only on Λ, Θ and $\big|\Omega\big|$.

PROOF. We construct an auxiliary function φ such that

$$\begin{aligned} \mathcal{L}\big(\varphi \pm u\big) = \mathcal{L}\varphi \pm f \geq 0 \ \text{ or } \ \mathcal{L}\varphi \geq \mp f \ \text{ in } \ \Omega, \\ \varphi \pm u = \varphi \pm g \geq 0 \ \text{ or } \ \varphi \geq \mp g \ \text{ on } \ \partial\Omega. \end{aligned}$$

Denote

$$F := \max_{\Omega} |f| \ \text{ and } \ G := \max_{\partial\Omega} |g|$$

We want the function φ to satisfy

$$\begin{cases} \mathcal{L}\varphi \geq F \ \text{ in } \ \Omega, \\ \varphi \geq G \ \text{ on } \ \partial\Omega. \end{cases}$$

Without loss of generality, we assume $\Omega \subset B(0,R)$. Set $\varphi(x) = G + \big(e^{\alpha R^2} - e^{\alpha |x|^2}\big)F$. We get

$$\begin{aligned} \mathcal{L}\varphi =& \Big(4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x) x_i x_j + 2\alpha \sum_{i=1}^{n} \big[a_{ii}(x) - b_i x_i\big]\Big) e^{\alpha |x|^2} F \\ &+ c(x)\varphi \\ \geq& \underbrace{\Big(4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x) x_i x_j + 2\alpha \sum_{i=1}^{n} \big[a_{ii}(x) - b_i x_i\big] \Big)}_{\geq 1 \text{ for } \alpha \text{ large}} e^{\alpha |x|^2} F. \\ \geq& F. \end{aligned}$$

Thus for α large we get we obtain

$$\begin{cases} \mathcal{L}\varphi \geq F \ \text{ in } \ \Omega, \\ \varphi \geq G \ \text{ on } \ \partial\Omega. \end{cases}$$

By the comparison principle we conclude $-\varphi \leq u \leq \varphi$ and in particular

$$\sup_{\Omega} \big|u\big| \leq \big(e^{\alpha R^2} - 1\big)F + G,$$

where $\alpha > 0$ is a constant depending only on Λ and Θ.

□

Proposition 2.16. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f \in C(\Omega)$, $g \in C(\partial\Omega)$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the boundary value problem:*

$$(*) \qquad \begin{cases} \mathcal{L}u = f & \textit{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \alpha(x)u = g & \textit{on } \partial\Omega. \end{cases}$$

with $c(x) \geq 0$ and $\alpha \in C(\partial\Omega)$ with $\alpha \geq \alpha_0 > 0$ on $\partial\Omega$. Then the following estimate holds

$$\sup_{\Omega} |u| \leq C\Big(\max_{\Omega} |f| + \max_{\partial\Omega} |g| \Big)$$

where $C > 0$ is a constant depending only on Λ, Θ, α_0 and diam(Ω).

Exercise 2.17. *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the boundary value problem:*

$$\begin{cases} -\Delta u = u - u^3 & \textit{in } \Omega \\ \quad u = 0 & \textit{on } \partial\Omega, \end{cases}$$

Show that $\sup_{\Omega} |u| \leq 1$. Can the solution take the values ± 1?

CHAPTER 3

The Heat Equation

We first introduce the heat equation and derive its fundamental solution using Fourier transforms. Next we study the initial value problem. Using the fundamental solution we discuss the regularity of solutions. We also discuss the maximum principle for the heat equation and its applications.

Consider the heat equation in $\mathbb{R}^n$:

$$u_t - \Delta u = 0,$$

where $u = u(x,t)$, with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The heat equation models the temperature of a body conducting heat.

1. The Fourier Transform

Definition 3.1. *The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is the space of functions $f \in C^\infty(\mathbb{R}^n;\mathbb{C})$ such that:*

$$\sup_{x\in\mathbb{R}^n}\Big[(1+|x|^2)^{\frac{m}{2}}\partial^\alpha f(x)\Big] < +\infty, \quad \textit{for all multi-index } \alpha \in \mathbb{Z}_+^n \textit{ and } m \in \mathbb{Z}_+,$$

here $\mathbb{Z}_+$ denotes the set of nonnegative integers.

Exercise 3.2 (Schwartz functions are rapidly decreasing.)**.** *The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ consists of smooth functions which together with all its derivatives decay faster than any polynomial, that is,*

$$\lim_{|x|\to+\infty}\big(|x|^m\partial^\alpha f(x)\big) = 0, \quad \textit{for all multi-index } \alpha \in \mathbb{Z}_+^n \textit{ and } m \in \mathbb{Z}_+.$$

Exercise 3.3. *Show that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.*

Exercise 3.4. *Show that*

(1) $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$.

(2) $\dfrac{1}{(1+|x|^2)^2} \notin \mathcal{S}(\mathbb{R}^n)$.

Exercise 3.5. *Properties of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$*

- $\mathcal{S}(\mathbb{R}^n)$ *is a vector space.*
- $\mathcal{S}(\mathbb{R}^n)$ *is an algebra.*
- $\mathcal{S}(\mathbb{R}^n)$ *is closed under multiplication by polynomials.*
- $\mathcal{S}(\mathbb{R}^n)$ *is closed under differentiation.*
- $\mathcal{S}(\mathbb{R}^n)$ *is closed under translations and multiplication by $e^{ix\cdot\xi}$.*
- $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

Definition 3.6 (The Fourier transform). *The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is defined as:*

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad \text{for } \xi \in \mathbb{R}^n.$$

Exercise 3.7. *Fourier transform is well defined in $\mathcal{S}(\mathbb{R}^n)$ and satisfies the following properties.*

(P1) *Linearity: The Fourier transform is linear operator*

$$\big(c_1 f_1 + c_2 f_2\big)^{\wedge} = c_1 \widehat{f}_1 + c_2 \widehat{f}_2,$$

for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ and $c_1, c_1 \in \mathbb{R}$.

(P2) *Differentiation: For any $f \in \mathcal{S}(\mathbb{R}^n)$, we have by integration by parts*

$$\widehat{\partial_j f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \partial_j f(x)\, dx = i\xi_j \widehat{f}(\xi) \quad \text{for all } 1 \le j \le n.$$

Hence for any multi-index $\alpha \in \mathbb{Z}_+^n$

$$\widehat{\partial^\alpha f}(\xi) == (i\xi)^\alpha \widehat{f}(\xi).$$

For a polynomial $P(\xi)$, $\xi \in \mathbb{R}^n$ given by: $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$, we define the differential operator $P(\partial)$ by

$$P(\partial) f := \sum_{|\alpha| \le m} a_\alpha \partial^\alpha f.$$

Then

$$\widehat{P(\partial) f}(\xi) = P(i\xi) \hat{f}(\xi).$$

(P3) *Multiplication: For $f \in \mathcal{S}(\mathbb{R}^n)$ we have $\widehat{f} \in C^\infty(\mathbb{R}^n)$.*

$$\partial_j \widehat{f}(\xi) = -\frac{i}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} x_j f(x)\, dx = -i\, \widehat{x_j f}(\xi) \quad \text{for all } 1 \le j \le n,$$

and hence for any multi-index $\alpha \in \mathbb{Z}_+^n$

$$\partial_\alpha \widehat{f}(\xi) = (-i)^{|\alpha|} \widehat{x^\alpha f}(\xi).$$

As a consequence $\left|\xi^\alpha \partial^\beta \widehat{f}(\xi)\right|$ is bounded for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$, and we have $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

(P4) *Translation and dilation: Let $f \in \mathcal{S}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. We have*

$$f\,\widehat{(\cdot - x_0)}(\xi) = e^{-\xi\cdot x_0} \widehat{f}(\xi) \quad \text{and} \quad \widehat{f\,(\lambda\cdot)}(\xi) = \lambda^{-n} \widehat{f}\left(\frac{\xi}{\lambda}\right).$$

(P5) *Convolution: For $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, the convolution $f_1 * f_2 \in \mathcal{S}(\mathbb{R}^n)$, and we have*

$$\widehat{f_1 * f_2}(\xi) = (2\pi)^{n/2} \widehat{f}_1(\xi) \widehat{f}_2(\xi).$$

Example 3.8 (The Fourier Transform of a Gaussian is a Gaussian)**.** *Consider* $\mathcal{G}(x) := e^{-|x|^2}$ *in* $\mathbb{R}^n$*. We have for* $\xi \in \mathbb{R}^n$

$$\begin{aligned}
\widehat{\mathcal{G}}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-|x|^2}\ dx \\
&= \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-\left(|x|^2 + ix\cdot\xi\right)}\ dx \\
&= \frac{e^{-\frac{|\xi|^2}{4}}}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-\left|x + i\frac{\xi}{2}\right|^2}\ dx \\
&= \frac{e^{-\frac{|\xi|^2}{4}}}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-|x|^2}\ dx \\
&= \frac{e^{-\frac{|\xi|^2}{4}}}{(2\pi)^{n/2}} \prod_{i=i}^{n} \underbrace{\int\limits_{-\infty}^{+\infty} e^{-x_i^2}\ dx_i}_{=\sqrt{\pi}} \\
&= 2^{-\frac{n}{2}} e^{-\frac{|\xi|^2}{4}}.
\end{aligned}$$

And for any $\lambda > 0$*,* $\widehat{\mathcal{G}(\lambda\cdot)}(\xi) = (2\lambda^2)^{-n/2} e^{-\frac{|\xi|^2}{4\lambda^2}}$*.*

The Fourier transform can be used to solve certain classes of PDE. Let P be a polynomial in $\mathbb{R}^n$ of degree m. Consider the following linear partial differential equation of degree m

$$P(\partial)u := f \quad \text{in } \mathbb{R}^n.$$

By applying Fourier transforms, we have

$$P(i\xi)\widehat{u}(\xi) = \widehat{f}(\xi).$$

Then the solution u has the Fourier Transform given by

$$\widehat{u}(\xi) = \frac{\widehat{f}(\xi)}{P(i\xi)}.$$

This motivates to inverse Fourier Transform.

Theorem 3.9. *Suppose* $f \in \mathcal{S}$*. Then*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)\ d\xi \qquad \text{for } x \in \mathbb{R}^n$$

Proof. First, by the definition of the Fourier transform

$$\frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{f}(\xi)\ d\xi = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-iy\cdot\xi} f(y)\ dy d\xi$$

We cannot simply change the order of integrations since there is no absolute convergence. Instead, we introduce a limiting process to have the absolute convergence.

Then

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{f}(\xi)\ d\xi = \frac{1}{(2\pi)^n}\lim_{t\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}e^{-t|\xi|^2}f(y)\ dyd\xi$$

We write

$$K_t(x-y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}e^{-t|\xi|^2}\ d\xi$$

This is the inverse Fourier transform of the Gaussian Function $(2\pi)^{-n/2}e^{-t|\xi|^2}$. We have

$$K_t(x-y) = \frac{1}{(4\pi t)^{n/2}}e^{-|x-y|^2/4t}.$$

The kernel K_t satisfies

(1) $K_t > 0$ in $\mathbb{R}^n$.

(2) $\displaystyle\int_{\mathbb{R}^n} K_t(x)\ dx = 1$

(3) $\lim_{t\to 0} K_t(x) = 0$ uniformly in $\mathbb{R}^n \setminus \{|x| \geq \delta\}$, for all $\delta > 0$.

Exercise 3.10. *Then*

$$\begin{aligned}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{f}(\xi)\ d\xi &= \lim_{t\to 0}\int_{\mathbb{R}^n} K_t(x-y)f(y)\ dy\\ &= f(x).\end{aligned}$$

As a result, the Fourier transform is an automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. □

Suppose u, v are complex valued functions. We consider the their L^2-product

$$\big(u, v\big)_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} u\overline{v}\ dx.$$

The Fourier transform is a linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Now since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can complete the space $\mathcal{S}(\mathbb{R}^n)$ under $\|\cdot\|_{L^2(\mathbb{R}^n)}$ to get $L^2(\mathbb{R}^n)$. Hence, the Fourier transform can be extended to a bounded linear operator on $L^2(\mathbb{R}^n)$.

Theorem 3.11 (Parseval's Identity)**.** *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\big(u, v\big)_{L^2(\mathbb{R}^n)} = \big(\widehat{u}, \widehat{v}\big)_{L^2(\mathbb{R}^n)}.$$

In particular for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)},$$

so the Fourier transform is an isometry of $L^2(\mathbb{R}^n)$.

Exercise 3.12. *Prove the Parseval's Identity.*

Exercise* 3.13. *Show that the Fourier transform operator is a bounded linear operator from $L^1(\mathbb{R}^n)$ to the set of bounded continuous functions in $\mathbb{R}^n$.*

2. The Fundamental Solution and the Heat Kernel

The heat equation $u_t - \Delta u = 0$ is invariant under spatial rotations, space and time translations, and multiplication by constants. Note that the heat equation is not invariant under the change $t \mapsto -t$. This indicates that the heat equation describes an irreversible process and distinguishes between past and future. The heat equation is invariant under the dilations $(x, t) \mapsto (\lambda x, \lambda^2 t)$, for all $\lambda > 0$ which makes the quotient $|x|^2/t$ invariant.

Consider the initial value problem or the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

We assume that $u \in C^2\big(\mathbb{R}^n \times (0, +\infty)\big) \cap C\big(\mathbb{R}^n \times [0, +\infty)\big)$. Using Fourier transforms we will derive a formal solution. In the following, we will take the Fourier transform of u with respect to the spatial variable $x \in \mathbb{R}^n$ and we write

$$\widehat{u}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x, t)\ dx, \qquad \text{for } \xi \in \mathbb{R}^n,\ t \geq 0$$

We obtain

$$\begin{cases} \widehat{u}_t + |\xi|^2 \widehat{u} = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ \widehat{u}(\cdot, 0) = \widehat{u}_0 & \text{in } \mathbb{R}^n. \end{cases}$$

The solution of this initial-value problem ODE is given by

$$\widehat{u}(\xi, t) = \widehat{u_0}(\xi) e^{-t|\xi|^2}.$$

We write

$$\mathcal{K}(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x)\cdot\xi} e^{-t|\xi|^2}\ d\xi \qquad \text{for } x \in \mathbb{R}^n,\ t > 0.$$

This is the inverse Fourier transform of the Gaussian Function $(2\pi)^{-n/2} e^{-t|\xi|^2}$. We have

$$\widehat{u}(\xi, t) = (2n)^{n/2} \widehat{u_0}(\xi) \widehat{\mathcal{K}}(\xi, t).$$

Then

$$\mathcal{K}(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \qquad \text{for } x \in \mathbb{R}^n,\ t > 0$$

which is called the fundamental solution of the heat equation. We have obtained

$$u(x, t) = \int_{\mathbb{R}^n} \mathcal{K}(x - y, t) u_0(y)\ dy \qquad \text{for } x \in \mathbb{R}^n,\ t > 0.$$

$\mathcal{K}$ is the heat kernel of $\mathbb{R}^n$ and satisfies the following properties:

(1) $\mathcal{K}(x, t)$ is smooth for any $x \in \mathbb{R}^n$ and $t > 0$.
(2) $\mathcal{K}(x, t) > 0$ for any $x \in \mathbb{R}^n$ and $t > 0$.
(3) $(\partial_t - \Delta_x)\mathcal{K}(x, t) = 0$ for any $x \in \mathbb{R}^n$ and $t > 0$.
(4) $\displaystyle\int_{\mathbb{R}^n} \mathcal{K}(x, t)\ dx = 1$ for any $t > 0$.

(5) For any $\delta > 0$,

$$\lim_{t\to 0} \int\limits_{|x|>\delta} \mathcal{K}(x,t)\ dx = 0.$$

We now solve the initial-value problems of the heat equation.

Theorem 3.14 (Solution of the initial value problem)**.** *Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Consider u defined by*

$$u(x,t) := \int\limits_{\mathbb{R}^n} \mathcal{K}(x-y,t)g(y)\ dy \qquad \text{for } x \in \mathbb{R}^n,\ t > 0,$$

where $\mathcal{K}$ is the heat kernel of $\mathbb{R}^n$. Then $u \in C^\infty(\mathbb{R}^n \times (0,+\infty)) \cap C\big(\mathbb{R}^n \times [0,+\infty)\big)$ ans satisfies the initial value problem:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(\cdot,0) = g & \text{in } \mathbb{R}^n. \end{cases}$$

Proof. The heat kernel $\mathcal{K}$ is smooth for all $x \in \mathbb{R}^n, t > 0$, and from the exponential decay of $\mathcal{K}$ it follows that for all $m \in \mathbb{Z}_+$

$$\frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} |x-y|^m e^{-|x-y|^2/4t} |g(y)|\ dy < +\infty \qquad \text{for all } x \in \mathbb{R}^n,\ t > 0.$$

This shows that

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)\ dy\ \in C^\infty(\mathbb{R}^n \times (0,+\infty)).$$

Differentiating we obtain

$$\partial_t u(x,t) - \Delta_x u(x,t) = \int\limits_{\mathbb{R}^n} \big[(\partial_t - \Delta_x)\mathcal{K}(x-y,t)\big] g(y)\ dy = 0 \ \text{ for } x \in \mathbb{R}^n,\ t > 0.$$

We now show

$$\lim_{(x,t)\to(x_0,t)} u(x,t) = g(x_0).$$

Fix $x_0 \in \mathbb{R}^n$, we have

$$u(x,t) - g(x_0) = \int\limits_{\mathbb{R}^n} \mathcal{K}(x-y,t)\big(g(y) - g(x_0)\big)\ dy,$$

$$\big|u(x,t) - g(x_0)\big| \le \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-|x-y|^2/4t} \big|g(y) - g(x_0)\big|\ dy$$

$$\begin{aligned} \big|u(x,t) - g(x_0)\big| \le& \frac{1}{(4\pi t)^{n/2}} \int\limits_{B(x_0,\delta)} e^{-|x-y|^2/4t} \big|g(y) - g(x_0)\big|\ dy \\ &+ \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-|x-y|^2/4t} \big|g(y) - g(x_0)\big|\ dy, \end{aligned}$$

for some $\delta > 0$. Using the continuity of g, for any $\varepsilon > 0$ we choose $\delta > 0$ such that $\big|g(y) - g(x_0)\big| < \varepsilon$ for $|y - x_0| < \delta$. Then

$$\big|u(x,t) - g(x_0)\big| \le \varepsilon \int_{\mathbb{R}^n} \mathcal{K}(x-y,t)\ dy + 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \mathcal{K}(x-y,t)\ dy.$$

Since $|x - y| \ge \delta/2$ for $|y - x_0| \ge \delta$ and $|x - x_0| < \delta/2$, we have

$$\big|u(x,t) - g(x_0)\big| \le \varepsilon + 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(x,\delta/2)} \mathcal{K}(x-y,t)\ dy.$$

Now for any $\delta > 0$,

$$\lim_{t\to 0} \int_{|x|>\delta} \mathcal{K}(x,t)\ dx = 0.$$

Therefore for some $t_0 > 0$ depending only on ε, δ and $\|g\|_{L^\infty}$, we have

$$\big|u(x,t) - g(x_0)\big| \le 2\varepsilon \ \text{ for } |x - x_0| < \delta/2,\ 0 < t < t_0.$$

This show the continuity at $t = 0$. □

Remark 3.15. *Symbolically we can write as*

$$\begin{cases} \partial_t \mathcal{K} - \Delta \mathcal{K} = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(\cdot,0) = \delta_0 & \text{in } \mathbb{R}^n, \end{cases}$$

where δ_0 is the Dirac delta.

Remark 3.16.

- Heat equation has infinite speed of propagation: *For a solution $u(x,t)$ of the heat equation, the value $u(x,t) > 0$ for all $(x.t) \in \mathbb{R}^n \times (0,+\infty)$, if the initial nonnegative value is positive somewhere. Thus the values of the initial data near a point affects the value of the solution $u(x,t)$ of the heat equation for all points and all positive time.*
- The heat equation regularizes or smoothens out its initial values*: Note that the solution $u(x,t)$ of the heat equation is smooth for all $t > 0$, even if the initial value is only bounded.*

Exercise 3.17 (Decay estimates)**.** *Suppose the initial data g in Theorem 3.14 has a compact support. Then*

$$|u(x,t)| \le \frac{1}{(4\pi t)^{n/2}} e^{-\frac{dist(x,K)}{4t}} \|g\|_{L^1}.$$

Theorem 3.18 (Solution of the Nonhomogeneous Problem)**.** *Let $f \in C^1(\mathbb{R}^n \times [0,+\infty)) \cap L^\infty(\mathbb{R}^n \times (0,+\infty))$ and consider the Cauchy Problem:*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(\cdot,0) = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Then

$$u(x,t) := \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\ dyds \qquad \text{for } x \in \mathbb{R}^n,\ t > 0.$$

PROOF. For all $s > 0$, let $\widetilde{u}(x,t;s)$ solve the initial value problem

$$\begin{cases} \partial_t \widetilde{u}(x,t;s) - \Delta \widetilde{u}(x,t;s) = 0 & \text{in } \mathbb{R}^n \times (s,+\infty), \\ \widetilde{u}(x,s;s) = f(x,s) & \text{in } \mathbb{R}^n \times \{t = s\}. \end{cases}$$

Then

$$\widetilde{u}(x,t;s) = \int_{\mathbb{R}^n} \mathcal{K}(x-y,t-s) f(y,s)\ dyds \quad \text{for } x \in \mathbb{R}^n,\ t > s.$$

Consider (Duhamel's Principle)

$$u(x,t) = \int_0^t \widetilde{u}(x,t;s) ds = \int_0^t \int_{\mathbb{R}^n} \mathcal{K}(x-y,t-s) f(y,s)\ dyds \quad \text{for } x \in \mathbb{R}^n,\ t > 0.$$

Then we have $u \in C^{2,1}\big(\mathbb{R}^n \times (0,+\infty)\big) \cap C\big(\mathbb{R}^n \times [0,+\infty)\big)$ and it solves the nonhomogeneous boundary value problem, since

$$\begin{aligned} \partial_t u(x,t) =& \widetilde{u}(x,t;t) + \int_0^t \partial_t \widetilde{u}(x,t;s) ds = f(x,t) + \int_0^t \Delta_x \widetilde{u}(x,t;s) ds \\ =& f(x,t) + \Delta_x u(x,t). \end{aligned}$$

□

Corollary 3.19. *Combining the previous two theorems we obtain that*

$$u(x,t) := \int_{\mathbb{R}^n} \mathcal{K}(x-y,t) g(y)\ dy + \int_0^t \int_{\mathbb{R}^n} \mathcal{K}(x-y,t-s) f(y,s)\ dyds,$$

solves the evolution problem:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g & \text{in } \mathbb{R}^n \times \{0\}. \end{cases}$$

The boundedness assumption on the initial condition u_0 in Theorem 3.14 can be relaxed. The heat kernel $\mathcal{K}$ has an exponential decay in space variables, with a large decay rate for small time. This suggests in the convolution, a fixed exponential growth from initial values can be offset by the fast exponential decay in the fundamental solution at least for a short time period.

Theorem 3.20 (Short time existence)**.** *Let $u_0 \in C(\mathbb{R}^n)$ satisfy the growth condition:*

$$\big|u_0(x)\big| \le M e^{\alpha |x|^2} \quad \textit{for all } x \in \mathbb{R}^n,$$

for some constants $M, \alpha \ge 0$. Consider

$$u(x,t) := \int_{\mathbb{R}^n} \mathcal{K}(x-y,t) u_0(y)\ dy \quad \textit{for } x \in \mathbb{R}^n,\ t > 0,$$

where $\mathcal{K}$ is the heat kernel of $\mathbb{R}^n$. Then $u \in C^\infty(\mathbb{R}^n \times (0,T)) \cap C\big(\mathbb{R}^n \times [0,T)\big)$ and satisfies the initial value problem:

$$\begin{cases} \partial_t u - \Delta u = 0 & \textit{in } \mathbb{R}^n \times (0,T] \\ u(\cdot,0) = u_0 & \textit{in } \mathbb{R}^n, \end{cases}$$

with $T < \dfrac{1}{4\alpha}$. *Moreover*

$$\big|u(x,t)\big| \le \widetilde{M} e^{\widetilde{\alpha}|x|^2} \quad \textit{for all } x \in \mathbb{R}^n, \ \textit{and } t \in (0,T],$$

for some constants $\widetilde{M} > 0, \widetilde{\alpha} \ge 0$ *depending only on* M, α *and* T.

PROOF. Recall $\mathcal{K}(x-y,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$. We choose $T > 0$ such that $\alpha - \frac{1}{4t} < 0$ for all $t \in (0,T]$, that is, $T < \frac{1}{4\alpha}$. Then for all $x \in \mathbb{R}^n$ and $t \in (0,T]$

$$\begin{aligned}
\big|u(x,t)\big| &\le \int_{\mathbb{R}^n} \mathcal{K}(x-y,t)|u_0(y)|\, dy \le \frac{M}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\left(\alpha|y|^2 - \frac{1}{4t}|x-y|^2\right)}\, dy \\
&\le \frac{M}{(4\pi t)^{n/2}} e^{\left(\frac{\alpha}{1-4\alpha t}\right)|x|^2} \underbrace{\int_{\mathbb{R}^n} e^{-\left(\frac{1-4\alpha t}{4t}\right)\left|y - \frac{1}{1-4\alpha t}x\right|^2}\, dy}_{=\frac{(4\pi t)^{n/2}}{(1-4\alpha t)^{n/2}}} \\
&\le \frac{M}{(1-4\alpha t)^{n/2}} e^{\left(\frac{\alpha}{1-4\alpha t}\right)|x|^2},
\end{aligned}$$

and therefore we have

$$\big|u(x,t)\big| \le \widetilde{M} e^{\widetilde{\alpha}|x|^2} \quad \text{for all } x \in \mathbb{R}^n, \ \text{and } t \in (0,T],$$

for some constants $\widetilde{M} > 0, \widetilde{\alpha} \ge 0$ depending only on M, α and T. Then next,

Exercise 3.21. *Proceeding as in Theorem 3.14 we obtain* $u \in C^\infty(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T))$.

□

DEFINITION 3.22. *Consider a smooth* (*or* C^2) *domain* $\Omega \subset \mathbb{R}^n$. *We will consider the parabolic cylinder* $\Omega_T := \Omega \times (0,T]$ *for* $T > 0$. *The parabolic boundary is defined as*

$$\partial_P \Omega_T := \big\{\Omega \times \{t = 0\}\big\} \cup \big\{\partial\Omega \times [0,T]\big\}$$

The parabolic boundary consists of the bottom and the sides of $\Omega \times [0,T]$.

DEFINITION 3.23. *We denote the collection of functions which are* C^2 *in the* x*-variable and* C^1 *in the* t*-variable as:*

$$C^{2,1}\big(\Omega_T\big) := \big\{u : \Omega_T \to \mathbb{R} \ \ s.t \ \ u, D_x u, D_x^2 u, \partial_t u \in C\big(\Omega_T\big)\big\}.$$

Proposition 3.24 (Representation Formula)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded smooth domain and assume that* $u \in C^{2,1}\big(\Omega_T\big) \cap C\big(\overline{\Omega_T}\big)$. *We have for* $(x,t) \in \Omega_T$

$$\begin{aligned}
u(x,t) = &\int_0^t \int_\Omega \mathcal{K}(x-y,t-s)\big(\partial_s u_s(y,s) - \Delta u(y,s)\big)\, dyds + \int_\Omega \mathcal{K}(x-y,t)u(y,0)\, dy \\
&+ \int_0^t \int_{\partial\Omega} \Big[\mathcal{K}(x-y,t-s)\frac{\partial u}{\partial \nu_y}(y,s) - u(y,t)\frac{\partial \mathcal{K}}{\partial \nu_y}(x-y,t-s)\Big]\, d\sigma(y)ds,
\end{aligned}$$

where

$$\mathcal{K}(x-y,t-s) := \frac{1}{(4\pi(t-s))^{n/2}} e^{-|x|^2/4(t-s)} \qquad \text{for } x \in \mathbb{R}^n,\ t>0$$

Proof. Note that $\big(\partial_s + \Delta_y \mathcal{K}\big)(x-y,t-s+\varepsilon) = 0$. We obtain by integration by parts formula for $\varepsilon > 0$

$$\int_0^t \int_\Omega \mathcal{K}(x-y,t-s+\varepsilon)\big(\partial_s u - \Delta_y u\big)\ dyds =$$

$$\int_\Omega \int_0^t \mathcal{K}(x-y,t-s+\varepsilon)\partial_s u(y,s)dsdy - \int_0^t \int_\Omega \mathcal{K}(x-y,t-s+\varepsilon)\Delta_y u(y,s)\ dyds$$

$$= \int_\Omega \Big[\mathcal{K}(x-y,\varepsilon)u(y,t) - \mathcal{K}(x-y,t+\varepsilon)u(y,0) - \int_0^t \partial_s \mathcal{K}(x-y,t-s+\varepsilon)\ u(y,t)ds\Big]dy$$

$$-\int_0^t \Big(\int_\Omega \Delta_y \mathcal{K}(x-y,t-s+\varepsilon)u(y,s)\ dy\Big)ds -$$

$$\int_0^t \int_{\partial\Omega} \Big(\mathcal{K}(x-y,t-s+\varepsilon)\frac{\partial u}{\partial \nu}(y,s) - u(y,s)\frac{\partial \mathcal{K}}{\partial \nu}(x-y,t-s+\varepsilon)\Big)\ d\sigma(y)ds$$

$$= \int_\Omega \mathcal{K}(x-y,\varepsilon)u(y,s)\ dx - \int_\Omega \mathcal{K}(x-y,t+\varepsilon)u(y,s)\ dx$$

$$-\int_0^t \int_{\partial\Omega} \Big(\mathcal{K}(x-y,t-s+\varepsilon)\frac{\partial u}{\partial \nu}(y,s) - u(y,s)\frac{\partial \mathcal{K}}{\partial \nu}(x-y,t-s+\varepsilon)\Big)\ d\sigma(y)ds$$

Letting $\varepsilon \to 0$ gives us the representation formula (why?). □

Theorem 3.25 (Cauchy Problem for the Heat Equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let $f \in C\big(\overline{\Omega} \times [0,+\infty)\big)$, $g \in C\big(\Omega\big)$ and $\varphi \in C\big(\partial\Omega \times (0,+\infty)\big)$. Then the initial value problem:*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,+\infty) \\ \quad u = \varphi & \text{in } \partial\Omega \times (0,+\infty), \\ \quad u = g & \text{in } \Omega \times \{0\}, \end{cases}$$

has a unique solution $u \in C^{2,1}\big(\Omega_T\big) \cap C\big(\overline{\Omega_T}\big)$ given by:

$$u(x,t) = \int_0^t \int_\Omega \mathcal{K}_\Omega(x-y,t-s)f(y,s)\ dydt + \int_\Omega \mathcal{K}_\Omega(x-y,t)g(y)\ dy$$

$$-\int_0^t \int_{\partial\Omega} \frac{\partial \mathcal{K}_\Omega}{\partial \nu_y}(x-y,t-s)\varphi(y,s)\ d\sigma(y)ds.$$

Here $\mathcal{K}_\Omega$ is the Heat Kernel of Ω.

3. Maximum Principle

We discuss the maximum principle for the heat equation and its applications. Here $\Omega \subset \mathbb{R}^n$ will be a bounded smooth domain.

Theorem 3.26. *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $\partial_t u - \Delta u \le 0$ in Ω_T. Then*

$$\max_{\overline{\Omega_T}} u = \max_{\partial_P \Omega_T} u,$$

that is, the maximum of u is attained on its parabolic boundary $\partial_P \Omega_T$.

Proof. **Step 1:** If $\partial_t u - \Delta u < 0$ in Ω_T..
Proof of Step 1: Suppose $\max_{\Omega_T} u = u(\xi_0, t_0)$ for some point $(\xi_0, t_0) \in \Omega_T$. Then $\nabla_x u(\xi_0, t_0) = 0$ and also the Hessian matrix $D_x^2 u(\xi, t_0)$ is negative semidefinite, so $\Delta u(\xi) = \mathrm{trace}(D_x^2 u(\xi)) \le 0$. Moreover, $\partial_t u(\xi_0, t_0) \ge 0$. We then have $\partial_t u - \Delta u \ge 0$ in Ω_T, a contradiction and therefore u must attain its maximum on $\partial_P \Omega_T$. □

Step 2: If $\partial_t u - \Delta u \le 0$ in Ω_T. Consider the auxiliary function

$$v_\varepsilon(x,t) = u(x,t) - \varepsilon t.$$

Then $\partial_t v - \Delta v = \partial_t u - \Delta u - \varepsilon < 0$ in Ω_T. Hence

$$\max_{\overline{\Omega_T}} u + \varepsilon T \le \max_{\overline{\Omega_T}} v_\varepsilon = \max_{\partial_P \Omega_T} v_\varepsilon \le \max_{\partial_P \Omega_T} u + \varepsilon T,$$

Letting $\varepsilon \to 0$, we get the desired result. □

Exercise 3.27 (Weak maximum principle)**.** *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy $\partial_t u - \Delta u + c(x,t)u \le 0$ in Ω_T, with $c(x,t) \ge 0$ in Ω_T. Then*

$$\max_{\overline{\Omega_T}} u = \max_{\partial_P \Omega_T} \{u, 0\},$$

that is, the nonnegative maximum of u is attained on its parabolic boundary $\partial_P \Omega_T$.

We can remove the condition on the sign of c.

Theorem 3.28. *Suppose $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies $\partial_t u - \Delta u + c(x,t)u \le 0$ in Ω_T, where $c(x,t) \ge c_0$ in Ω_T for some constant $c_0 > 0$. If $u \le 0$ in $\partial_P \Omega_T$ then $u \le 0$ in Ω_T.*

Proof. Take $v(x,t) := e^{-c_0 t} u(x,t)$, then

$$\partial_t v - \Delta v + (c + c_0)v = e^{-c_0 t}\left(\partial_t u - \Delta u + c(x,t)u\right) \le 0$$

Then by the weak maximum principle for the heat equation, since $c + c_0 \le 0$ in Ω_T we have

$$\max_{\Omega_T} v = \max_{\partial_P \Omega_T} \{0, v\} \le \max_{\partial_P \Omega_T} \{0, e^{-c_0 t} u\} \le 0.$$

Hence $u(x,t) \le 0$ in for all $(x,t) \in \Omega_T$. □

Corollary 3.29 (Comparison Principle)**.** *Suppose $u, v \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ satisfy*

$$\partial_t u - \Delta u + c(x,t)u \le \partial_t v - \Delta v + c(x,t)v \quad \text{in } \Omega_T.$$

If $u \le v$ in $\partial_P \Omega_T$ then $u \le v$ in Ω_T.

Solutions are not unique without further conditions on u, like the boundedness assumption or the exponential growth assumption.

Let $\alpha > 0$ and set

$$\mathcal{B}(x,t) := \frac{1}{(1-4\alpha t)^{n/2}} e^{\frac{\alpha}{1-4\alpha t}|x|^2} \quad \text{for } x \in \mathbb{R}^n,\ t \in \left(0, \frac{1}{4\alpha}\right).$$

Then

$$\begin{cases} \partial_t \mathcal{B} - \Delta \mathcal{B} = 0 & \text{in } \mathbb{R}^n \times \left(0, \frac{1}{4\alpha}\right) \\ \mathcal{B}(\cdot, 0) = e^{\alpha|x|^2} & \text{in } \mathbb{R}^n. \end{cases}$$

Theorem 3.30. *Suppose $u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C\big(\mathbb{R}^n \times [0,T]\big)$ and satisfies the initial value problem:*

$$\begin{cases} \partial_t u - \Delta u \le 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) = g & \text{in } \mathbb{R}^n, \end{cases}$$

and the growth condition:

$$\big|u(x,t)\big| \le M e^{\alpha|x|^2} \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T],$$

for some constants $M, \alpha \ge 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u \le \sup_{\mathbb{R}^n} g.$$

Proof. We divide the interval $[0,T]$ into subintervals of length $\dfrac{1}{4a}$ with $a > \alpha$. It then suffices to prove the theorem in $[0,(4a)^{-1}]$ by repeatedly applying the result on $[0,(4a)^{-1}]$, $[(4a)^{-1}, 2(4a)^{-1}] \ldots$

Consider for $\delta > 0$ and $R > 0$

$$V_{\delta,R}(x,t) := \delta \frac{M e^{(\alpha - a)R^2}}{(1-4at)^{n/2}} e^{\frac{a}{1-4at}|x|^2} \quad \text{for } (x,t) \in \mathbb{R}^n \times \left(0, (4a)^{-1}\right).$$

Then $\partial_t V_{\delta,R} - \Delta V_{\delta,R} = 0$ in $\mathbb{R}^n \times \big(0,(4a)^{-1}\big)$ and so in particular

$$\partial_t(u - V_{\delta,R}) - \Delta(u - V_{\delta,R}) \le 0 \text{ in } B(0,R) \times \big(0,(4a)^{-1}\big)$$

Also, for R large

$$u - V_{\delta,R} \le \sup_{|x|=R} |g| \text{ in } \partial B(0,R) \times \big[0,(4a)^{-1}\big] \quad \text{for } R \text{ large},$$

$$\text{and} \quad u - V_{\delta,R} \le g \text{ in } B(0,R) \times \{t = 0\}.$$

Then by the maximum principle we have

$$\sup_{B(0,R)\times[0,(4a)^{-1}]} \Big[u(x,t) - V_{\delta,R}(x,t)\Big] \le \sup_{|x|\le R} g \le \sup_{\mathbb{R}^n} g.$$

Letting $\delta \to 0$ gives us for all $R > 0$

$$\sup_{B(0,R)\times(0,(4a)^{-1})} u(x,t) \le \sup_{\mathbb{R}^n} g,$$

and we obtain the theorem. □

Corollary 3.31 (Uniqueness)**.** *Suppose $u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C\big(\mathbb{R}^n \times [0,T]\big)$ and satisfies the Cauchy problem:*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

and the growth condition:

$$\big|u(x,t)\big| \le M e^{\alpha|x|^2} \quad \textit{for all } (x,t) \in \mathbb{R}^n \times [0,T],$$

for some constants $M, \alpha \ge 0$. Then $u \equiv 0$ in $\mathbb{R}^n \times [0,T]$.

Remark 3.32. *There are counterexample for uniqueness if the above growth condition does not hold. Both $v \equiv 0$ and*

$$\tilde{v}(x,t) = \sum_{j=0}^{+\infty} \frac{h^{(j)}(t)}{(2j)!} x^{2j} \qquad \textit{where} \;\; h(t) = \begin{cases} e^{-\frac{1}{t^2}} & \textit{for } t > 0 \\ 0 & \textit{for } t \le 0, \end{cases}$$

satisfy

$$\begin{cases} \partial_t u - \Delta u = 0 & \textit{in } \mathbb{R} \times (0,+\infty) \\ u(\cdot,0) = 0 & \textit{in } \mathbb{R}. \end{cases}$$

4. Regularity

We discuss regularity of solutions of the heat equation with the help of the fundamental solution. For any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, we define

$$Q_r(x,t) := B(x,r) \times \big(t - r^2, t\big].$$

Theorem 3.33. *Let $u \in C^{2,1}(Q_R(\xi,\tau))$ solves the heat equation $\partial_t u - \Delta u = 0$ in $Q_r(\xi,\tau)$. Then $u \in C^\infty(Q_R(\xi,\tau))$ and we have the gradient estimate*

$$\big|\nabla_x u(\xi,\tau)\big| \le \frac{C}{R} \sup_{Q_R(\xi,\tau)} |u|$$

for some positive constant $C > 0$ depending only on n.

Proof. Consider the heat kernel

$$\mathcal{K}(x-y,t-s) := \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x|^2}{4(t-s)}} \qquad \text{for } x \in \mathbb{R}^n, \; t > 0$$

We choose a cut-off function $\eta \in C_c^\infty\big(Q_{\frac{3}{4}R}(\xi,\tau)\big)$ such that $\eta \equiv 1$ in $Q_{\frac{1}{2}R}(\xi,\tau)$.

Exercise 3.34. *We have*

$$\eta(x,t)u(x,t) = \int_{\tau-R^2}^{t} \int_{B(\xi,R)} u(y,s)\big[\partial_s + \Delta_y\big]\big(\eta(y,s)\mathcal{K}(x-y,t-s)\big)\; dyds$$

We denote $\widetilde{\mathcal{K}}(y,s) := \mathcal{K}(x-y,t-s)$. Then for $(x,t) \in Q_{\frac{1}{4}R}(\xi,\tau)$

$$\begin{aligned}
u(x,t) &= \int_{\tau-R^2}^{t} \int_{B(\xi,R)} u(y,s)\big[\partial_s + \Delta_y\big]\big(\eta(y,s)\widetilde{\mathcal{K}}(y,s)\big)\; dyds \\
&= \int_{\tau-R^2}^{t} \int_{B(\xi,R)} u(y,s)\Big(\big[\partial_s\eta(y,s) + \Delta_y\eta(y,s)\big]\widetilde{\mathcal{K}}(y,s) + \big[\underbrace{\partial_s\widetilde{\mathcal{K}}(y,s) + \Delta_y\widetilde{\mathcal{K}}(y,s)}_{=0}\big]\eta(y,s) \\
&\quad + 2\nabla_y\eta(y,s)\cdot\nabla_y\widetilde{\mathcal{K}}(y,s)\Big)\; dyds \\
&= \int_{B(\xi,R)\times(\tau-R^2,t)} u(y,s)\Big(\big[\partial_s\eta(y,s) + \Delta_y\eta(y,s)\big]\widetilde{\mathcal{K}}(y,s) + 2\nabla_y\eta(y,s)\cdot\nabla_y\widetilde{\mathcal{K}}(y,s)\Big)\; dyds
\end{aligned}$$

Note that each term in the integrand involves a derivative of η, which vanishes in $Q_{\frac{1}{2}R}(\xi,\tau)$, since $\eta \equiv 1$ in $Q_{\frac{1}{2}R}(\xi,\tau)$. Then letting

$$\mathcal{D} := B\left(\xi, (3R/4)\right) \times \left(\tau - (3R/4)^2, t\right] \setminus B\left(\xi, (R/2)\right) \times \left(\tau - (R/2)^2, t\right].$$

We then obtain for $(x,t) \in Q_{\frac{1}{4}R}(\xi,\tau)$

$$u(x,t) = \int_{\mathcal{D}} u(y,s)\Big(\big[\partial_s \eta(y,s) + \Delta_y \eta(y,s)\big]\widetilde{\mathcal{K}}(y,s) + 2\nabla_y \eta(y,s)\cdot\nabla_y \widetilde{\mathcal{K}}(y,s)\Big)\ dyds.$$

Note that there are no singularity in the integral domain since the distance between any $(y,s) \in \mathcal{D}$ and any $(x,t) \in Q_{\frac{1}{4}R}(\xi,\tau)$ has a positive lower bound. This proves that $u \in C^\infty\big(Q_{\frac{1}{4}R}(\xi,\tau)\big)$.

Next we have for $(x,t) \in Q_{\frac{1}{4}R}(\xi,\tau)$

$$\begin{aligned}\nabla_x u(x,t) = \int_{\mathcal{D}} u(y,s)\Big(\big[\partial_s \eta(y,s) + \Delta_y \eta(y,s)\big]\nabla_x \widetilde{\mathcal{K}}(y,s) + \\ 2\nabla_x\big(\nabla_y \eta(y,s)\cdot\nabla_y \widetilde{\mathcal{K}}(y,s)\big)\Big)\ dyds.\end{aligned}$$

We can choose the cut-off function η such that for some positive constant $C > 0$ depending only on n (why?)

$$\left|\nabla_y \varphi\right| \le \frac{C}{R} \qquad \left|\partial_s \varphi\right| + \left|\nabla_y^2 \varphi\right| \le \frac{C}{R^2}$$

We also have the following bounds on the heat kernel for some positive constant $C > 0$ depending only on n (why?)

$$\left|\nabla_x \widetilde{\mathcal{K}}\right| \le C\frac{|x-y|}{(t-s)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t-s)}}, \quad \left|\nabla_x \widetilde{\mathcal{K}}\right| \le C\frac{|x-y|^2 + (t-s)}{(t-s)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(t-s)}}.$$

This gives us that for some positive constant $C > 0$ depending only on n we have for $(x,t) \in Q_{\frac{1}{4}R}(\xi,\tau)$

$$\begin{aligned}\left|\nabla_x u(x,t)\right| &\le C\int_{\mathcal{D}}\left(\frac{|x-y|}{(t-s)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left(\left|\partial_s\varphi\right| + \left|\nabla_y^2\varphi\right|\right) + \right.\\ &\left.\frac{|x-y|^2 + (t-s)}{(t-s)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t-s)}}\left|\nabla_y\varphi\right|\right)|u(y,s)|\ dyds.\\ &\le C\int_{\mathcal{D}}\left(\frac{1}{R(t-s)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t-s)}} + \frac{R}{(t-s)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(t-s)}}\right)|u(y,s)|\ dyds.\end{aligned}$$

Exercise 3.35. *We have*

$$\int_{\mathcal{D}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{\frac{n}{2}+j}}\ dyds \le \frac{C}{R^{2(j-1)}} \qquad \textit{for } j = 1, 2.$$

This gives us the bounded

$$\left|\nabla_x u(x,t)\right| \leq \frac{C}{R} \sup_{Q_R(\xi,\tau)} |u| \quad \text{for } (x,t) \in Q_{\frac{1}{4}R}(\xi,\tau).$$

And this completes the proof of the theorem. □

Corollary 3.36. *We have the following estimate on the derivatives*

$$\left|\partial_t^k \nabla_x^m u(\xi,\tau)\right| \leq \left(\frac{c}{r}\right)^{m+2k} n^k e^{m+2k-1}(m+2k)! \sup_{Q_r(\xi,\tau)} |u|$$

for some positive constant $c > 0$ depending only on n.

Proof. For x-derivatives, proceeding as in the case of the Laplacian, we can show for any multi-index α with $|\alpha| = m$

$$\left|\partial_x^\alpha u(\xi,\tau)\right| \leq \left(\frac{c}{r}\right)^m n^k e^{m-1}(m)! \sup_{Q_r(\xi,\tau)} |u|$$

For t-derivatives, we use $\partial_t = \Delta u$ and hence $\partial_t^k u = \Delta^k u$ for any nonnegative integer k. Hence

$$\left|\partial_t^k \nabla_x^m u(\xi,\tau)\right| \leq n^k \max_{|\beta|=m+2k} \left|\partial_x^\beta u(\xi,\tau)\right|.$$

□

CHAPTER 4

Distributions and Weak Derivatives

Weak derivatives and distribution theory leads to view linear differential equations as linear functionals acting on a space of test functions. Using this perspective, given suitable estimates, one can obtain simple and general existence results for weak solutions of linear PDEs by functional analytic tools.

Our presentation will be rather vague and we will omit proofs. The references are the books of Rudin [19], Hormander [12], Leoni [15], Kesavan [14]. Also see the wonderful texts by Strichartz [21] and Taylor [22].

1. Weak Derivatives

Ω will throughout denote a domain in $\mathbb{R}^n$. Motivated by the integration by parts formula, we define the notion of *weak derivatives*.

DEFINITION 4.1 (Weak Derivatives). *A function $f \in L^1_{\text{loc}}(\Omega)$ is said to be weakly differentiable with respect to x_i if there exists a function, denoted by, $g_i \in L^1_{\text{loc}}(\Omega)$ such that :*

$$\int_\Omega f \partial_i \varphi \, dx = -\int_\Omega g_i \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

The function g_i is called the i-th weak partial derivative of f, and is denoted by $\partial_i f$. We write:

$$\int_\Omega f \partial_i \varphi \, dx = -\int_\Omega \partial_i f \ \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Higher-order weak derivatives are defined similarly.

DEFINITION 4.2. *Let α be a multi-index. A function $f \in L^1_{\text{loc}}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{\text{loc}}(\Omega)$ of order $|\alpha|$, if*

$$\int_\Omega \partial^\alpha f \ \varphi \, dx = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Note that the weak derivative of a function, if it exists, is unique almost everywhere in Ω. If f has sufficient regularity so that $\partial^\alpha f$ exits in the usual sense, then $\partial^\alpha f$ is also the α-th weak derivative of f. The existence of a weak derivative however does not imply, in general, the existence of a pointwise derivative almost everywhere.

Example 4.3. *Consider $f(x) = x_+$, that is,*

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then for any $\varphi \in C_c^\infty(\Omega)$, by integration by parts we obtain

$$\int_{-\infty}^{+\infty} f\varphi' \, dx = \int_{0}^{+\infty} x\varphi' \, dx = -\int_{0}^{+\infty} \varphi \, dx = -\int_{-\infty}^{+\infty} \chi_{[0,+\infty)}\varphi \, dx,$$

where χ_S denotes the characteristic function of the set S. Therefore the given f is weakly differentiable, with

$$f' = \chi_{[0,+\infty)}, \text{ a step-function given by } \chi_{[0,+\infty)}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Example 4.4. *The step-function*

$$\chi_{[0,+\infty)}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is not weakly differentiable. To see this, note for any $\varphi \in C_c^\infty(\Omega)$

$$\int_{-\infty}^{+\infty} \chi_{[0,+\infty)}\varphi' \, dx = \int_{0}^{+\infty} \varphi' \, dx = -\varphi(0).$$

Then the weak derivative $f' \in L^1_{\text{loc}}(\Omega)$, if it exists, would have to satisfy

$$\int_{-\infty}^{+\infty} f'\varphi \, dx = -\varphi(0) \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

By considering test functions with $\varphi(0) = 0$, it follows that $f' \equiv 0$ almost everywhere, but this gives a contradiction for test functions with $\varphi(0) \neq 0$.

In this example, the pointwise derivative of the function is zero everywhere except at 0, where the function is discontinuous, but the function is not weakly differentiable.

Exercise 4.5. *If $f : (a, b) \to \mathbb{R}$ is weakly differentiable and $f' = 0$, then $f \equiv const$.*

Exercise* 4.6. *Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as:*

$$f(x) := \frac{1}{|x|^p}$$

Show that f is weakly differentiable when $p < n - 1$ and

$$\nabla f(x) = -p\frac{x}{|x|^{p+2}}.$$

That is, f is weakly differentiable provided that the pointwise derivative, which is defined almost everywhere, is locally integrable.

We collect some results, without proofs, from real analysis which is useful for understanding weak derivatives in the simplest context of functions of a single variable. Proofs can be found in Folland [8] or Evans-Gariepy[6].

Definition 4.7. *A function $f : [a, b] \to \mathbb{R}$ is Lipschitz, if for some constant C we have*

$$\big|f(x) - f(y)\big| \leq C\big|x - y\big| \qquad \text{for all } x, y \in [a, b].$$

Note the Lipschitz constant can be characterized as: $\sup\limits_{[a,b]} \dfrac{\big|f(x) - f(y)\big|}{\big|x - y\big|}$.

Theorem 4.8 (Rademacher's theorem in 1-D)**.** *A locally Lipschitz in $\mathbb{R}$ is almost everywhere differentiable.*

Theorem 4.9. *If $f : [a, b] \to \mathbb{R}$ is Lipschitz, then the pointwise derivative f' exists almost everywhere in (a, b) and $f' \in L^\infty(a, b)$.*

Theorem 4.10. *Suppose $f \in L^1_{\text{loc}}(a, b)$. Then f is Lipschitz in $[a, b]$ if and only if f is weakly differentiable in (a, b) and $f' \in L^\infty(a, b)$. Moreover, the Lipschitz constant of f is equal to the sup-norm of f'.*

Definition 4.11 (Absolutely continuous functions). *A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^{m} \big|f(b_i) - f(a_i)\big| < \varepsilon,$$

for any finite collection $\big\{[a_i, b_i] : 1 \le i \le m\big\}$ of non-overlapping subintervals of $[a, b]$ with

$$\sum_{i=1}^{m} |b_i - a_i| < \delta.$$

Exercise 4.12. *A Lipschitz function is absolutely continuous.*

Example 4.13 (Fundamental Theorem of Calculus)**.** *Let $f \in L^1(a, b)$ and set*

$$F(x) := \int_a^x f \; dt.$$

Then F is absolutely continuous in $[a, b]$ and $F' = g$ almost everywhere in (a, b).

Absolutely continuous functions are precisely the ones for which the fundamental theorem of calculus holds. The following result gives an explicit characterization of weakly differentiable functions of a single variable.

Theorem 4.14. *A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if The pointwise derivative f' exists almost everywhere in (a, b), $f' \in L^1(a, b)$ and*

$$f(x) = f(a) + \int_a^x f' \; dt.$$

Theorem 4.15. *$f \in L^1_{\text{loc}}(a, b)$ is absolutely continuous if and only if f is weakly differentiable in (a, b) and $f' \in L^1(a, b)$.*

2. Distributions

We weaken the notion of derivative and consider distributions, which in contrast to functions, are always differentiable. The space of distributions may be thought of as the smallest extension of the space of continuous functions that is closed under differentiation. The theory of distributions given by Laurent Schwartz at the end of the 1940's and has had a major influence in the development of the theory of PDEs henceforth.

Consider a domain $\Omega \subset \mathbb{R}^n$. We construct a topology on the space $C_c^\infty(\Omega)$. First we see that the space $C^\infty(K)$ is a Fréchet Space, where $K \subset \Omega$ is compact.

Theorem-Definition 4.16. *Let $K \subset \Omega$ be a compact set. We set:*

$$\mathcal{D}_K(\Omega) := \{\varphi \in C_c^\infty(\Omega) : supp(\varphi) \subset K\}.$$

For $l \in \mathbb{N}$, we define the norm $\|\cdot\|_{K,l}$ on $\mathcal{D}_K(\Omega)$ by

$$\|\varphi\|_{K,l} := \sup\left\{\left|\partial^\alpha \varphi(x)\right| : x \in K,\ \alpha \text{ is multi-index with } |\alpha| \leq l\right\}.$$

The family of norms $\|\cdot\|_{K,l}$ turns $\mathcal{D}_K(\Omega)$ into a locally convex topological vector space and a local base for the topology τ_K is given by all sets of the form

$$\left\{\varphi \in \mathcal{D}_K(\Omega) : \|\varphi\|_{K,l} < \frac{1}{l}\right\}, \qquad \text{where } l \in \mathbb{N}.$$

We define:

$$d_K(\varphi, \psi) := \max_{i\in\mathbb{N}} \frac{1}{2^i} \frac{\|\varphi - \psi\|_{K,i}}{1 + \|\varphi - \psi\|_{K,i}} \qquad \text{for } \varphi, \psi \in \mathcal{D}_K(\Omega).$$

Then

(a) *d_K is a metric on $\mathcal{D}_K(\Omega)$.*
(b) *The space $\big(\mathcal{D}_K(\Omega), d_K\big)$ is an infinite dimensional, locally convex, complete metric space, that is, a Fréchet Space.*
(c) *The topology τ_K is determined by the metric d_K.*

The space $C_c^\infty(\Omega)$ is given the inductive limit topology which makes it a complete Topological Vector Space.

Theorem-Definition 4.17 (Inductive limit topology)**.** *Consider a sequence of compact sets $\big(K_m\big)$ in Ω such that $K_m \subset int(K_{m+1})$ for all $m \in \mathbb{N}$, with $K_0 = \emptyset$ and $\Omega = \bigcup_{m\in\mathbb{N}} K_m$. Given two sequences $\mathbf{l} := (l_m) \in Z_+$ and $\mathbf{a} := (\varepsilon_m) \in (0,1)$, with $l_m \to +\infty$ and $a_m \to 0$ as $m \to +\infty$, we let:*

$$p_{\mathbf{l},\mathbf{a}}(\varphi) := \sup_{m\geq 1} \sup_{x\in\Omega\setminus K_m} \frac{1}{a_m} \sum_{|\alpha|\leq l_m} \left|\partial^\alpha \varphi(x)\right| \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

The family of semi-norms $p_{\mathbf{l},\mathbf{a}}$ generates a complete topology on $\mathcal{D}(\Omega)$, called the inductive limit topology. Further, for any compact set $K \subset \Omega$ the Fréchet space topology coincides with the relative topology of $\mathcal{D}_K(\Omega)$ as a subset $\mathcal{D}(\Omega)$.

It is difficult to work directly with the inductive limit topology, however, one can characterize convergence in this topology sequentially.

Theorem-Definition 4.18 (The space of Test Functions)**.** *The space $\mathcal{D}(\Omega)$ consists of functions $f \in C_c^\infty(\Omega)$. $\mathcal{D}(\Omega)$ is a complete space and a sequence of functions $(\varphi_m)_m \in C_c^\infty(\Omega)$ converges to $\varphi \in C_c^\infty(\Omega)$ in $\mathcal{D}(\Omega)$ if*

(i) *There exists $\Omega' \Subset \Omega$ such that $supp(\varphi_m) \subset \Omega'$ for all $m \in \mathbb{N}$.*
(ii) *$\partial^\alpha \varphi_m \longrightarrow \partial^\alpha \varphi$ as $m \to +\infty$, uniformly on Ω for every multi-index α.*

Let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ be a linear functional and let write $T(\varphi) = \langle T, \varphi\rangle$ and so:

$$\langle T, c_1\varphi_1 + c_2\varphi_2\rangle = c_1 \langle T, \varphi_1\rangle + c_2 \langle T, \varphi_2\rangle \quad \text{for all} \quad \varphi_1, \varphi_2 \in \mathcal{D}(\Omega) \quad \text{and} \quad c_1, c_2 \in \mathbb{R}.$$

Definition 4.19. *The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the space of distributions on Ω. $\mathcal{D}'(\Omega)$ is given the weak-star topology.*

The space $\mathcal{D}(\Omega)$ is not metrizable. Despite this, one can still prove that linear functionals on $\mathcal{D}(\Omega)$ are continuous if and only if they are sequentially continuous.

Theorem-Definition 4.20 (Distributions). *Let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ be a linear functional and we write $T(\varphi) = \langle T, \varphi\rangle$. The following are equivalent:*

(i) *T is continuous.*
(ii) *T is bounded.*
(iii) *If $\varphi_m \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$ with respect to τ, then*
$$\lim_{m\to+\infty} T(\varphi_m) = T(\varphi).$$
(iv) *The restriction of T to $\mathcal{D}(K)$ is continuous for every compact set $K \subset \Omega$.*
(v) **For every compact set $K \subset \Omega$, there exists an integer $i_K \in \mathbb{Z}_+$ and a constant $C_K > 0$ such that**
$$\big|\langle T, \varphi\rangle\big| \le C_K \sum_{|\alpha|\le i_K} \sup\big|\partial^\alpha \varphi\big| \qquad \textbf{for all } \varphi \in \mathcal{D}_K(\Omega).$$

Thus, the space of distributions is the dual of the space of test functions.

Definition 4.21 (The order of a Distribution). *We define the order of a distribution $T \in \mathcal{D}'(\Omega)$ as the smallest integer i_0 such that $\big|\langle T, \varphi\rangle\big| \le C_K\|\varphi\|_{K,i_0}$ for all $\varphi \in \mathcal{D}_K(\Omega)$ and for all compact sets $K \subset \Omega$. If no such integer exists, then the distribution T is said to have infinite order.*

Exercise 4.22. *Let $\psi \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. Then $\psi T \in \mathcal{D}'(\Omega)$, where :*
$$\langle \psi T, \varphi\rangle := \langle T, \psi\varphi\rangle \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$
Also, prove the Leibnitz formula
$$\partial^\alpha\big(\psi T\big) = \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}\partial^\beta\psi\partial^{\alpha-\beta}T.$$

Example 4.23.

(i) *The Dirac delta at $\xi \in \Omega$ is the distribution $\delta_\xi \in \mathcal{D}'(\Omega)$ defined by*
$$\langle \delta_\xi, \varphi\rangle = \varphi(\xi) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$
$\delta_\xi \in \mathcal{D}'(\Omega)$ has order zero.

(ii) *Let μ be a Radon measure on Ω, then*
$$\langle T_\mu, \varphi\rangle = \int_\Omega \varphi\, d\mu \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$
defines a distribution $T_\mu \in \mathcal{D}'(\Omega)$ of order zero.

(iii) *Let $f \in L^1_{\mathrm{loc}}(\Omega)$, then*
$$\langle T_f, \varphi\rangle = \int_\Omega f\varphi\, dx \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$
defines a distribution $T_f \in \mathcal{D}'(\Omega)$ of order zero.

Exercise 4.24. *Consider*
$$\Psi_m(x) = \begin{cases} m & \text{for } 0 < x < \dfrac{1}{m}, \\ 0 & \text{otherwise.} \end{cases}$$
Show that $\Psi_m \to \delta_0$ in $\mathcal{D}'(\Omega)$.

Exercise* 4.25. *Fix* $\xi_0 \in \Omega$. *Then* $\langle T, \varphi \rangle := \partial^\alpha \varphi(\xi_0)$, *defines a distribution of order* $|\alpha|$.

Remark 4.26. *Distributions of order zero may be identified with measures.*

Theorem-Definition 4.27 (Derivative of a distribution)**.** *Let* $T \in \mathcal{D}'(\Omega)$. *For a multiindex* α, *the derivative* $\partial^\alpha T \in \mathcal{D}'(\Omega)$ *of order* $|\alpha|$ *is defined by:*

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Exercise 4.28. *Show that* $\partial^\alpha T$ *defined above is indeed a distribution.*

Example 4.29. *If* $f \in L^1_{\text{loc}}(\Omega)$, *and let* α *be a multi-index. The* α*-th weak derivative of* f *is the distribution* $\partial^\alpha T_f$ *given by:*

$$\langle \partial^\alpha T_f, \varphi \rangle = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Example 4.30. *If* $f \in L^1_{\text{loc}}(\Omega)$. *The distributional Laplacian of* f *is given by:*

$$\langle T_{\Delta f}, \varphi \rangle = \int_\Omega f \Delta \varphi \, dx = \langle T_f, \Delta \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We say $f \in L^1_{\text{loc}}(\Omega)$ *is subharmonic if*

$$\langle T_{-\Delta u}, \varphi \rangle = -\int_\Omega u \Delta \varphi \, dx \leq 0 \quad \text{for all } \varphi \geq 0 \in \mathcal{D}(\Omega).$$

Lemma 4.31. *Let* $\Phi \in C^\infty\big(\Omega_1 \times \Omega_2\big)$ *where* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *are open, and suppose* $\Phi(x, y) = 0$ *for* $x \notin K$, *for some compact* $K \subset \Omega_1$. *Let* $T \in \mathcal{D}'(\Omega)$ *be a distribution, and consider*

$$\psi(y) := \langle T, \Phi(\cdot, y) \rangle .$$

Then $\psi \in C^\infty(\Omega_2)$ *and* $\partial^\alpha \psi(y) = \left\langle T, \partial_y^\alpha \Phi(\cdot, y) \right\rangle$.

Proof. By Taylor's expansion

$$\Phi(x, y + h) = \Phi(x, y) + \sum_{i=1}^n h_i \partial_{y_i} \Phi(x, y) + \mathrm{O}\left(|h|^2\right) \quad \text{as } h \to 0.$$

Hence

$$\langle T, \Phi(\cdot, y + h) \rangle = \langle T, \Phi(\cdot, y) \rangle + \sum_{i=1}^n h_i \langle T, \Phi(\cdot, \partial_{y_i} y) \rangle + \mathrm{O}\left(|h|^2\right) \quad \text{as } h \to 0.$$

This shows that $\psi(y) := \langle T, \Phi(\cdot, y) \rangle$ is differentiable and

$$\partial_i \psi(y) = \langle T, \partial_{y_i} \Phi(\cdot, y), \rangle \quad 1 \leq i \leq n.$$

Successive differentiation completes the lemma. □

Exercise* 4.32. *Consider a sequence of distributions* $(T_m)_m \in \mathcal{D}'(\Omega)$ *such that the limit*

$$\langle T, \varphi \rangle := \lim_{m \to +\infty} \langle T_m, \varphi \rangle \quad \text{exists for all } \varphi \in \mathcal{D}(\Omega).$$

Show that $T \in \mathcal{D}'(\Omega)$ *and for any multi-index* α

$$\langle \partial^\alpha T, \varphi \rangle = \lim_{m \to +\infty} \langle \partial^\alpha T_m, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Theorem 4.33 (Support of a Distribution). *Consider a distribution $T \in \mathcal{D}'(\Omega)$. Let $\Omega' \subset \Omega$ be an open set. We say that $T = 0$ in Ω' if $\langle T, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega')$. Let U be the union of all open sets Ω' where $T = 0$. The support of T is the complement of U in Ω, that is, the support of T is the complement in Ω of the largest open set where it vanishes. The support of T will be written as $supp(T)$.*

Exercise 4.34. *Let $T \in \mathcal{D}'(\Omega)$. Show that $\partial^\alpha T = 0$ in $\Omega \setminus supp(T)$, for every multi-index α.*

Theorem-Definition 4.35 (Distributions with compact support). *Let $T \in \mathcal{D}'(\Omega)$ with compact support. Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ in a neighbourhood of $supp(T)$. We define*

$$\langle T, \phi\rangle := \langle T, \eta\phi\rangle \qquad \textit{for all} \;\; \psi \in C^\infty(\mathbb{R}^n).$$

The space of distributions with compact support in $\mathcal{D}'(\Omega)$ is identical to the dual of $C^\infty(\Omega)$ with the topology induced by the semi-norms $\|\phi\|_{K,k} = \sum_{|\alpha|\le k} \sup_K \left|\partial^\alpha \phi\right|$, where K ranges over all compact subsets of Ω and k over all nonnegative integers. We write $\mathcal{E}(\Omega)$ for the space $C^\infty(\Omega)$ equipped with this topology, and the space of distributions with compact support in Ω as $\mathcal{E}'(\Omega)$.

Exercise* 4.36. *Let $T \in \mathcal{D}'(\Omega)$ with compact support. Then T has finite order.*

Using partitions of unity, one can obtain a representation for every distribution.

Theorem 4.37. *Let $T \in \mathcal{D}'(\Omega)$. Then for every multi-index α there exists $g_\alpha \in C(\Omega)$ such that*

(i) *Any compact $K \subset \Omega$ intersects the support of finitely many g_α's.*

(ii)

$$\langle T, \varphi\rangle = \sum_\alpha \langle \partial^\alpha T_{g_\alpha}, \varphi\rangle = \sum_\alpha (-1)^\alpha \int_\Omega g_\alpha \partial^\alpha \varphi \quad \textit{for all } \varphi \in \mathcal{D}(\Omega).$$

While distribution theory provides an effective general framework for the analysis of linear PDEs, it is less useful for nonlinear PDEs because one cannot define a product of distributions that extends the usual product of smooth functions in an unambiguous way.

3. Convolution of Distributions

Fix $\phi \in \mathcal{D}(\mathbb{R}^n)$. We approximate $\check{\psi} * \phi$ with the Riemann sum

$$u_h(x) := h^m \sum_{y\in\mathbb{Z}^n} \check{\psi}(x - hy)\phi(hy), \qquad \text{for } x \in \mathbb{R}^n,$$

where $h > 0$. Then $u_h(x) \to$ For every multi-index α

$$\partial_x^\alpha u_h(x) := h^m \sum_{y\in\mathbb{Z}^n} \partial_x^\alpha \check{\psi}(x - hy)\phi(hy), \qquad \text{for } x \in \mathbb{R}^n.$$

We first define the convolution of a distribution T and a function $\phi \in C_c^\infty(\mathbb{R}^n)$.

Theorem-Definition 4.38. *The convolution $T * \phi$ of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ and a function $\phi \in \mathcal{D}(\mathbb{R}^n)$ is defined by*

$$(T * \phi)(x) := \langle T, \phi(x - \cdot)\rangle,$$

where the right hand side denotes T acting on $\phi(x-y)$ as a function of y. The convolution $T*\phi$ satisfies

(a) $T*\phi \in C^\infty(\mathbb{R}^n)$.
(b) $supp(T*\phi) \subset supp(T) + supp(\phi)$.
(c) *For any multi-index α*

$$\partial^\alpha(T*\phi) = (\partial^\alpha T)*\phi = T*\partial^\alpha\phi.$$

(d) $(T*\psi)*\phi = T*(\psi*\phi)$, *for all* $\psi,\phi \in \mathcal{D}(\mathbb{R}^n)$.

Proof. From Lemma 4.31 it follows that $T*\phi \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(T*\phi) = T*\partial^\alpha\phi$. From the definitions we obtain that $((\partial^\alpha T)*\phi)(x) = \langle \partial^\alpha T, \phi(x-\cdot)\rangle = (-1)^{2|\alpha|}\langle T, \partial^\alpha\phi(x-\cdot)\rangle = (T*\partial^\alpha\phi)(x)$.

We have $(T*\phi)(x) = 0$ if $x-y \notin supp(\phi)$ for $y \in supp(T)$, which implies that $supp(T*\phi) \subset supp(T) + supp(\phi)$.

Associativity is a consequence of the result that, for $\phi \in C^j_c(R^n)$ and $\psi \in C_c(\mathbb{R}^n)$, the Riemann sum

$$\sum_{y\in\mathbb{Z}^n} \phi(x-hy)h^n\psi(hy) \longrightarrow \phi*\psi(x) \ \text{ in } C^j_c(\mathbb{R}^n) \ \text{ as } h\to 0.$$

See Hormander's book [12]. □

Theorem 4.39. *Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and let $(\eta_\varepsilon)_{\varepsilon>0}$, be a family of standard mollifiers, that is,*

$$\eta_\varepsilon \in C^\infty_c(\mathbb{R}^n) \ \text{ with } \ supp(\eta_\varepsilon) \subset B(0,\varepsilon),\ \eta_\varepsilon > 0 \text{ and } \int\limits_{B(0,\varepsilon)} \eta_\varepsilon = 1.$$

Then the sequence $(T*\eta_\varepsilon)_\varepsilon$ converges to T in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon\to 0$.

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n} (T*\eta_\varepsilon)(x)\varphi(x)\,dx = \langle T,\varphi\rangle \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^n).$$

Proof. We denote $\check{\phi}(x) = \phi(-x)$. We note that for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we can write $\langle T,\varphi\rangle = (T*\check{\varphi})(0)$. This gives

$$\langle T*\eta_\varepsilon,\varphi\rangle = ((T*\eta_\varepsilon)*\check{\varphi})(0) = (T*(\eta_\varepsilon*\check{\varphi}))(0) = \langle T, \check{\eta}_\varepsilon*\varphi\rangle$$

We have $\check{\eta}_\varepsilon*\varphi \to \varphi$ as $supp(\check{\eta}_\varepsilon) \to \{0\}$. Therefore it follows that for all $\varphi\in\mathcal{D}(\mathbb{R}^n)$

$$\langle T*\eta_\varepsilon,\varphi\rangle \longrightarrow \langle T,\varphi\rangle \quad \text{ as } \varepsilon\to 0.$$

□

Remark 4.40. *Given $T\in\mathcal{D}'(\mathbb{R}^n)$, there exists a sequence $(T_m)\in\mathcal{D}(\mathbb{R}^n)$ such that $T_m \longrightarrow T$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Exercise 4.41. *Show that $(T*\phi_m)\to 0$ in $C^\infty(\mathbb{R}^n)$ if $\phi_m\to 0$ in $\mathcal{D}(\mathbb{R}^n)$.*

Exercise 4.42. *We define the translation operator τ_ξ as:*

$$(\tau_\xi\phi)(x) := \phi(x-\xi), \ \text{ where } \xi\in\mathbb{R}^n.$$

Show that:

- $(\tau_\xi\phi)(x) = \delta_\xi*\phi$, *where δ_ξ denotes the Dirac measure at $\xi\in\mathbb{R}^n$. In particular $\delta_0*\phi = \phi$*

- *For any $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have*

$$T * (\tau_\xi \phi) = \tau_\xi (T * \phi).$$

Proposition 4.43. *Let $\Psi : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ be a linear map such that*

- *Ψ is continuous in the sense that: $\Psi(\phi_m) \to 0$ in $C(\mathbb{R}^n)$ if $\phi_m \to 0$ in $\mathcal{D}(\mathbb{R}^n)$.*
- *Further, Ψ commutes with translations, that is, $\tau_\xi(\Psi(\phi)) = \Psi(\tau_\xi \phi)$, for all $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$.*

Then there exists a unique $T_\Psi \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\Psi(\varphi) = T_\Psi * \varphi \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Definition 4.44. *The convolution of two distributions T_1 and T_2, one of which has a compact support, is the unique distribution T such that:*

$$T * \varphi = T_1 * (T_2 * \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Theorem 4.45. *Consider the convolution of two distributions T_1 and T_2, one of which has compact support. We have*

- *$T_1 * T_2 = T_2 * T_1$.*
- *$T * \delta_0 = T$ for all $T \in \mathcal{D}'(\mathbb{R}^n)$.*

Proposition 4.46. *Differentiation can be interpreted as a convolution.*

- *$\partial^\alpha T = \partial^\alpha \delta_0 * T$ for all $T \in \mathcal{D}'(\mathbb{R}^n)$.*
- *Consider two distributions T_1 and T_2, one of which has compact support. We have for all multi-index α*

$$\partial^\alpha (T_1 * T_2) = \partial^\alpha T_1 * T_2 = T_1 * \partial^\alpha T_2.$$

Exercise 4.47. *Consider a partial differential operator with constant coefficients: $\mathcal{P} = \sum a_\alpha \partial^\alpha$ (here the sum is finite). Consider two distributions T_1 and T_2, one of which has compact support. We have*

- *$\mathcal{P}T = (\mathcal{P}\delta_0) * T$, for all $T \in \mathcal{D}'(\mathbb{R}^n)$.*
- *Consider two distributions T_1 and T_2, one of which has compact support. We have*

$$\mathcal{P}(T_1 * T_2) = \mathcal{P}T_1 * T_2 = T_1 * \mathcal{P}T_2.$$

4. Tempered Distributions and Fourier Transforms

We recall

Definition 4.48. *Let $\phi : \mathbb{R}^n \to \mathbb{C}$. We define for multi-indices α, β*

$$\|\phi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \phi \right|.$$

The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is the space of functions $f \in C^\infty(\mathbb{R}^n; \mathbb{C})$ such that $\|f\|_{\alpha,\beta} < +\infty$, for all multi-indices α, β.

$\mathcal{S}(\mathbb{R}^n)$ consists of smooth functions which together with all its derivatives decay faster than any polynomial.

Theorem-Definition 4.49. *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space with the topology induced by the family of semi-norms $\|\cdot\|_{\alpha,\beta}$.*

A linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is continuous if and only if there exists a constant $C > 0$ and $m, l \in Z_+$ such that

$$\big|\langle T, \phi\rangle\big| \leq C\|\phi\|_{m,l} \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\|\phi\|_{m,l} := \sum_{|\alpha|=m, |\beta|=l} \|\phi\|_{\alpha,\beta}.$$

Definition 4.50. *The dual space of $\mathcal{S}(\mathbb{R}^n)$ is called the space of tempered distributions and is denoted as $\mathcal{S}'(\mathbb{R}^n)$.*

Proposition 4.51. *$\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and take a cut-off function $\eta \in C^\infty(\mathbb{R}^n)$

$$\eta(x) = \begin{cases} 1 & \text{in } B(0,1), \\ 0 & \text{in } \mathbb{R}^n \setminus B(0,2). \end{cases}$$

Let $\phi_\varepsilon(x) = \eta(\varepsilon x)\phi(x)$ for $x \in \mathbb{R}^n$, for all $\varepsilon > 0$. Then $\phi_\varepsilon \in C_c^\infty\left(B(0, 2/\varepsilon)\right)$ and we have

$$|\phi(x) - \phi_\varepsilon(x)| = (1 - \eta(\varepsilon x))|\phi(x)| = 0 \quad \text{in } B(0, 1/\varepsilon).$$

Therefore $\phi_\varepsilon \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ as $\varepsilon \to 0$. □

Theorem 4.52. *For all tempered distributions $T \in \mathcal{S}'(\mathbb{R}^n)$ its restriction to $\mathcal{D}(\mathbb{R}^n)$ is a distribution.*

Proof. By definition of continuity, there exists a constant $C > 0$ and $m, l \in Z_+$ such that $\big|\langle T, \phi\rangle\big| \leq C\|\phi\|_{m,l}$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Given a compact set K, we choose $R > 0$ such that $K \subset B(0, R)$. If $\psi \in \mathcal{D}_K(\mathbb{R}^n)$, then $\sup\limits_{x\in\mathbb{R}^n} \big|x^\alpha \partial^\beta \psi\big| \leq (1 + R^m)\big|\partial^\beta \psi\big|$, which implies

$$\big|\langle T, \psi\rangle\big| \leq (1 + R^m)\|\psi\|_{K,l} \quad \text{for all } \psi \in \mathcal{D}_K(\mathbb{R}^n).$$

Therefore $T \in \mathcal{D}'(\mathbb{R}^n)$. □

Theorem 4.53. *$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.*

Proof. $\mathcal{S}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$. For $p = 1$ we have

$$\int\limits_{\mathbb{R}^n} \|\phi\|\ dx = \int\limits_{\mathbb{R}^n} \frac{1 + |x|^{n+1}}{1 + |x|^{n+1}} \|\phi\|\ dx \leq C\|\phi\|_{n+1,0} \int\limits_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+1}}\ dx < +\infty.$$

For $1 < p < +\infty$ we can write

$$\int\limits_{\mathbb{R}^n} \|\phi\|^p\ dx \leq \|\phi\|^{p-1} \int\limits_{\mathbb{R}^n} \|\phi\|\ dx \leq C\|\phi\|^p_{n+1,0}.$$

This shows $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. Given $f \in L^p(\mathbb{R}^n)$ consider the linear functional on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\phi \mapsto \int\limits_{\mathbb{R}^n} f\phi\ dx \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

By Hölder's inequality

$$\left| \int_{\mathbb{R}^n} f\phi \, dx \right| \le \|f\|_{L^p} \|\phi\|_{L^{p'}} \le C\|f\|_{L^p} \|\phi\|_{n+1,0},$$

that is,

$$\langle T_f, \phi \rangle := \int_{\mathbb{R}^n} f\phi \, dx \le C\|f\|_{L^p} \|\phi\|_{n+1,0}.$$

This implies $T_f \in \mathcal{S}'(\mathbb{R}^n)$ and hence $f \in L^p(\mathbb{R}^n) \mapsto T_f \in \mathcal{S}'(\mathbb{R}^n)$ is an embedding. $\square$

Exercise 4.54. *For a polynomial* $\mathcal{P} : \mathbb{R}^n \to \mathbb{C}$, *we define*

$$\langle T_{\mathcal{P}}, \phi \rangle := \int_{\mathbb{R}^n} \mathcal{P}\phi \, dx \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^n).$$

Show that $T_{\mathcal{P}} : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ *is continuous and hence* $T_{\mathcal{P}}$ *is a tempered distribution.*

Theorem-Definition 4.55. *Let* $T \in \mathcal{S}'(\mathbb{R}^n)$. *For a multiindex* α, *the derivative* $\partial^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$ *of order* $|\alpha|$ *is defined by:*

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Exercise* 4.56. *Show that if* $T \in \mathcal{D}'(\mathbb{R}^n)$ *has compact support, then* $T \in \mathcal{S}'(\mathbb{R}^n)$.

Exercise 4.57. *Let* $T \in \mathcal{S}'(\mathbb{R}^n)$. *Show that* $\phi T \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\mathcal{P}T \in \mathcal{S}'(\mathbb{R}^n)$ *for any* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and any polynomial* $\mathcal{P}$.

Remark 4.58. *We have*

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n),$$
$$\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

We define the Fourier transform of tempered distributions.

DEFINITION 4.59. *For* $T \in \mathcal{S}'(\mathbb{R}^n)$, *the Fourier transform* $\hat{T}$ *is defined by*

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

The inverse Fourier transform $\hat{T}$ *is defined by*

$$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n),$$

where $\check{\phi}(x) = \phi(-x)$.

Remark 4.60. *The Fourier transformation is an isomorphism of* $\mathcal{S}'(\mathbb{R}^n)$ *(with the weak topology). The Fourier inversion formula*

$$\check{T} = \widehat{\widehat{T}} \text{ holds for all } T \in \mathcal{S}(\mathbb{R}^n).$$

Example 4.61.

$$\widehat{1} = (2\pi)^{n/2} \delta_0.$$

This holds since for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ *by the Fourier inversion formula for functions in* $\mathcal{S}(\mathbb{R}^n)$

$$\langle \widehat{1}, \phi \rangle = \langle 1, \widehat{\phi} \rangle = \int_{\mathbb{R}^n} \widehat{\phi}(\xi) \, d\xi = \int_{\mathbb{R}^n} e^{i0\cdot\xi} \widehat{\phi}(\xi) \, d\xi = (2\pi)^{n/2} \phi(0) \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Exercise 4.62. *Calculate* $\widehat{\delta_0}$.

CHAPTER 5

Sobolev Spaces

Sobolev spaces consist of functions whose weak derivatives belong to L^p. These spaces provide a useful setting for the analysis of PDEs. We refer to the books: Adams [1], Evans [5], Gilbarg-Trudinger [9] and Zimmer [23].

Definition 5.1. *Let $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$. Let Ω be a domain in $\mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ consists of locally integrable functions in Ω such that the function and all its weak derivatives of order less than or equal to k, belongs to $L^p(\Omega)$.*

$$W^{k,p}(\Omega) := \left\{ u \in L^1_{\text{loc}}(\Omega) \ : \ \partial^\alpha u \in L^p(\Omega) \ \ \textit{for all } |\alpha| \leq k \right\}$$

For $p = 2$ we write

$$H^k(\Omega) := W^{k,p}(\Omega).$$

We identify functions that are equal almost everywhere.

Theorem 5.2 (see Adams [1])**.** *The Sobolev space $W^{k,p}(\Omega)$ is a is a Banach space equipped with the norm:*

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega \left|\partial^\alpha u\right|^p \right)^{1/p} \qquad \textit{for } \ 1 \leq p < +\infty.$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha| \leq k} \sup_\Omega \left|\partial^\alpha u\right| \qquad \textit{for } \ p = +\infty.$$

Exercise 5.3. *Let $1 \leq p < +\infty$. Show that the following are equivalent Sobolev norms*

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \left(\int_\Omega \left|\partial^\alpha u\right|^p \right)^{1/p},$$

$$\|u\|_{W^{k,p}(\Omega)} := \max_{|\alpha| \leq k} \left(\int_\Omega \left|\partial^\alpha u\right|^p \right)^{1/p}.$$

Exercise 5.4. *Show that the space $H^k(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v \rangle = \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha u \cdot \partial^\alpha v \ dx$$

Several important properties of Sobolev spaces can be obtained by regarding $W^{k,p}(\Omega)$ as a closed subspace of disjoint copies of Ω.

Theorem 5.5 (see Adams [1])**.** *The Sobolev space $W^{k,p}(\Omega)$ is separable for $1 \leq p < +\infty$, and is uniformly convex and reflexive for $1 < p < +\infty$.*

1. Approximation of Sobolev functions

Theorem 5.6. *$C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and $1 \le p < +\infty$.*

PROOF. Let $u \in W^{k,p}(\mathbb{R}^n)$ and let $(\eta_\varepsilon)_{\varepsilon>0}$, be a family of standard mollifiers, that is,

$$\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n) \text{ with } supp(\eta_\varepsilon) \subset B(0,\varepsilon),\ \eta_\varepsilon > 0 \text{ and } \int_{B(0,\varepsilon)} \eta_\varepsilon = 1.$$

Then

Exercise 5.7. $\eta_\varepsilon * u \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$.

We have for all multi-index α such that $|\alpha| \le k$

$$\partial^\alpha\big(\eta_\varepsilon * u\big) = \eta_\varepsilon * \partial^\alpha u \longrightarrow u \text{ in } L^p \text{ as } \varepsilon \to 0.$$

It follows that $(\eta_\varepsilon * u \to u$ in $W^{k,p}(\mathbb{R}^n)$ as $\varepsilon \to 0$ and therefore $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Now let $u \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\phi(x) = \begin{cases} 1 & \text{in } B(0,1), \\ 0 & \text{in } \mathbb{R}^n \setminus B(0,2). \end{cases}$$

Set $\phi_R(x) = \phi\left(\dfrac{x}{R}\right)$ and $u_R = \phi_R u$. Then, by the Leibnitz rule

$$\partial^\alpha u_R = \phi_R \partial^\alpha u + \frac{1}{R} F_R,$$

where F_R is uniformly bounded in L^p for all R. Hence, by the dominated convergence theorem

$$\partial^\alpha u_R \longrightarrow \partial^\alpha u \quad \text{in } L^p \text{ as } R \to +\infty.$$

So $u_R \longrightarrow \partial^\alpha u$ in $W^{k,p}$ as $R \to +\infty$. This completes the proof that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$. □

Theorem 5.8 (Meyers-Serrin). *$C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for any domain Ω in $\mathbb{R}^n$, $k \in \mathbb{N}$ and $1 \le p < +\infty$.*

For a proof see Adams [1]. We could then define $W^{k,p}(\Omega)$ equivalently as the completion of $C^\infty(\Omega)$ with respect to the Sobolev norm, that is, o the space of smooth functions in Ω whose derivatives of order less than or equal to k belong to $L^p(\Omega)$. If Ω is a proper subset of $\mathbb{R}^n$ then $C_c^\infty(\mathbb{R}^n)$ is not dense in $W^{k,p}(\mathbb{R}^n)$. Instead the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is the space of functions in $W^{k,p}(\Omega)$ that *vanish on the boundary* $\partial\Omega$, which will be denoted by $W_0^{k,p}(\Omega)$. For a bounded domain Ω, in general, the space $C^k(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$.

By the Meyers-Serrin theorem, Sobolev functions can be approximated by functions with infinite degree if smoothness in the interior of Ω. If smoothness up to the boundary is required, some regularity of the boundary $\partial\Omega$ is required.

Theorem 5.9. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$. Then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for any $k \in \mathbb{N}$ and $1 \le p < +\infty$.*

More generally, it suffices that Ω satisfies a segment condition. See Adams [1].

DEFINITION 5.10. *Let Ω be a domain in $\mathbb{R}^n$. We define*

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}, \quad \textit{with respect to the } W^{k,p} \textit{ norm }.$$

DEFINITION 5.11. *$f : \Omega \to \mathbb{R}$ is said to Lipschitz if*

$$\big|f(x) - f(y)\big| \leq C|x - y| \qquad \textit{for all} \;\; x, y \in \Omega,$$

for some constant $C > 0$.

Proposition 5.12. *Let Ω be a domain in $\mathbb{R}^n$ and let $u \in L^1_{loc}(\Omega)$. Then u is weakly differentiable on Ω if and only if, up to changing u on a set of measure zero, we have*

(a) *u is absolutely continuous on almost every segment parallel to the axis of coordinates contained in Ω.*
(b) *The first order partial derivatives of u, which exists almost everywhere by (a), are in $L^1_{loc}(\Omega)$.*

A simple critera for belonging to the first order Sobolev spaces in differentiation is given by the following result.

Proposition 5.13. *Let Ω be a bounded domain in $\mathbb{R}^n$, any Lipschitz function $u : \Omega \to \mathbb{R}$ belongs to all the Sobolev spaces $W^{1,p}(\Omega)$ for all $p \geq 1$.*

Proposition 5.14. *Let Ω be a domain in $\mathbb{R}^n$ and let $u \in L^1_{loc}(\Omega)$.*

Theorem 5.15. *Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous function and let $u \in W^{1,p}(\Omega)$ for some $p \geq 1$. If $h \circ u \in L^p(\mathbb{R}^n)$, then $h \circ u \in W^1, p(\mathbb{R}^n)$ and for almost all $x \in \mathbb{R}^n$*

$$\big|\nabla(h \circ u)(x)\big| = \big|h'(u(x))\big|\big|\nabla u(x)\big|.$$

Consequently. if $u \in W^{1,p}(\Omega)$ then $|u| \in W^{1,p}(\Omega)$ and $\big|\nabla |u|\big| = \big|\nabla u\big|$ almost everywhere in $\mathbb{R}^n$.

See Ziemer [23] for proofs.

2. Sobolev Embedding I

We study Sobolev embeddings and inequalities, which are some of the most important tools in the analysis of partial differential equations.

DEFINITION 5.16. *We say that a Banach space $\big(X, \|\cdot\|_X\big)$ is continuously embedded, or embedded for short, in the Banach space $\big(Y, \|\cdot\|_Y\big)$ if there is a one-to-one, bounded linear map $\iota : X \to Y$.*

The boundedness of ι means $\|\iota u\|_Y \leq C\|u\|_X$, for some constant $C > 0$. We can think of ι as identifying elements of the smaller space X with elements of the larger space Y: if X is a subset of Y, then ι is the inclusion map. We write an embedding as $X \hookrightarrow Y$, or as $X \subset Y$.

The general idea of Sobolev's embedding is the following: if we have information on the regularity of the derivatives of a function (as they are in L^p), we should be able to get information about the function itself (as it is in L^q for some $q > p$). Some regularity on the derivatives should imply more regularity on the function itself. We are looking for an estimate of the form

$$\|u\|_{L^q} \leq C\|\nabla u\|_{L^p} \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n), \tag{5.1}$$

for some constant $C = C(p, q, n)$.

We use a rescaling argument to get the value of the exponent q such that the inequality (5.1) holds. Let $\lambda > 0$ and we rescale the function $u \in C_c^\infty(\mathbb{R}^n)$

$$u_\lambda(x) := u(\lambda x) \quad \text{for } x \in \mathbb{R}^n.$$

Then $\nabla u_\lambda(x) = \lambda \nabla u(\lambda x)$, and we get by a change of variables $x \mapsto \lambda x$

$$\int_{\mathbb{R}^n} |u_\lambda|^q \, dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u|^q \, dx, \qquad \int_{\mathbb{R}^n} |\nabla u_\lambda|^p \, dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |\nabla u|^p \, dx.$$

If (5.1) is true then for all $\lambda > 0$

$$\|u_\lambda\|_{L^q} \le C \|\nabla u_\lambda\|_{L^p}$$

from which we get $\dfrac{1}{\lambda^{n/q}} \|u\|_{L^q} \le C \dfrac{\lambda}{\lambda^{n/p}} \|\nabla u\|_{L^p}$, that is,

$$\|u\|_{L^q} \le \lambda^{\left(\frac{np}{n-p} - q\right)\left(\frac{n-p}{pq}\right)} C \|\nabla u\|_{L^p}.$$

We want the above inequality to be independent of λ so that (5.1) holds for all $u \in C_c^\infty(\mathbb{R}^n)$, otherwise letting $\lambda \to 0$ or $\lambda \to +\infty$ we can violate the inequality (5.1). So the only exponent q for which an inequality of the type (5.1) could hold for all functions $u \in C_c^\infty(\mathbb{R}^n)$ is:

$$q = \frac{np}{n-p} := p^*$$

p^* is called the conjugate Sobolev exponent of p.

Thus, an inequality of the form (5.1) is is possible only if $q = p^*$; we will show that (5.1) is true when $q = p^*$. This result was obtained by Sobolev (1938), who used potential-theoretic methods. The proof we give is due to Nirenberg (1959). The inequality is usually called the *Gagliardo-Nirenberg inequality* or *Sobolev inequality* (or Gagliardo-Nirenberg-Sobolev inequality ...)

To prove (5.1), we first start with the case $p = 1$ and then using the Hölder inequality we extend it for all $p \ge 1$.

Proposition 5.17. *We have for all $u \in C_c^\infty(\mathbb{R}^n)$*

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \le \frac{1}{2} \int_{R^n} |\nabla u| \, dx.$$

Proof. Take $u \in C_c^\infty(\mathbb{R}^n)$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \le j \le n$ we can write by using integration by parts

$$u(x) = \int_{-\infty}^{x_j} \partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n) \, ds = \int_{x_j}^{+\infty} \partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n) \, ds,$$

and thus

$$|u(x)| \le \frac{1}{2} \le \int_{-\infty}^{+\infty} \left| \partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n) \right| \, ds.$$

We then obtain

$$|u(x)|^n \leq \frac{1}{2^n} \prod_{j=1}^{n} \int_{-\infty}^{+\infty} \left|\partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\right| \, ds,$$

$$|u(x)|^{\frac{n}{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\prod_{j=1}^{n} \int_{-\infty}^{+\infty} \left|\partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\right| \, ds \right)^{\frac{1}{n-1}}.$$

Integrating we have

(5.2)

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \, dx \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} \int_{-\infty}^{+\infty} \left|\partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\right| \, ds \right)^{\frac{1}{n-1}} dx.$$

We next show by induction on n that

$$\int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} \int_{-\infty}^{+\infty} \left|\partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\right| \, ds \right)^{\frac{1}{n-1}} dx \leq \left(\prod_{j=1}^{n} \int_{\mathbb{R}^n} |\partial_j u| \, dx \right)^{\frac{1}{n-1}}.$$

Claim: Let $\{F_j : 1 \leq j \leq n\}$ be a family of nonnegative functions in $C_c^\infty(\mathbb{R}^n)$. We have

(5.3)

$$\int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} \int_{-\infty}^{+\infty} F_j(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n) \, ds \right)^{\frac{1}{n-1}} dx \leq \left(\prod_{j=1}^{n} \int_{\mathbb{R}^n} F_j(x) \, dx \right)^{\frac{1}{n-1}}.$$

Proof of Claim: We say that $\mathcal{H}_n$ holds if (5.3) holds for any family of nonnegative functions $\{F_j : 1 \leq j \leq n\}$ in $C_c^\infty(\mathbb{R}^n)$. For $n = 2$ we want to show that:

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\int_{-\infty}^{+\infty} F_1(x_1, x_2) \, dx_1 \right) \left(\int_{-\infty}^{+\infty} F_2(x_1, x_2) \, dx_2 \right) dx \\ &\qquad \leq \left(\int_{\mathbb{R}^2} F_1(x_1, x_2) \, dx_1 \right) \left(\int_{\mathbb{R}^2} F_2(x_1, x_2) \, dx_2 \right). \end{aligned}$$

Indeed for $n = 2$ we have by Fubini's theorem

$$\int_{\mathbb{R}^2} \left(\int_{-\infty}^{+\infty} F_1(x_1, x_2)\ dx_1 \right) \left(\int_{-\infty}^{+\infty} F_2(x_1, x_2)\ dx_2 \right) dx =$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} F_1(x_1, x_2)\ dx_1 \right) \left(\int_{-\infty}^{+\infty} F_2(x_1, x_2)\ dx_2 \right) dx_1 dx_2 =$$
$$\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_2(x_1, x_2)\ dx_2 dx_1 \right) \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_1(x_1, x_2)\ dx_1 dx_2 \right) =$$
$$\left(\int_{\mathbb{R}^2} F_1(x_1, x_2)\ dx \right) \left(\int_{\mathbb{R}^2} F_2(x_1, x_2)\ dx \right).$$

So then $\mathcal{H}_2$ holds for $n = 2$. Let $n \geq 3$ and suppose that $\mathcal{H}_{n-1}$ holds. We write $x \in \mathbb{R}^n$ as $x = (x_1, y)$ with $y \in \mathbb{R}^{n-1}$. Using the Hölder inequality, we have that

$$\int_{\mathbb{R}^{n-1}} \left(\prod_{j=1}^{n} \int_{-\infty}^{+\infty} F_j(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\ ds \right)^{\frac{1}{n-1}} dy =$$
$$\int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{+\infty} F_1(s, x_2, \ldots, x_n)\ ds \right)^{\frac{1}{n-1}} \left(\prod_{j=2}^{n} \int_{-\infty}^{+\infty} F_j(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\ ds \right)^{\frac{1}{n-1}} dy$$
$$\leq \left(\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{+\infty} F_1(s, x_2, \ldots, x_n)\ ds dy \right)^{\frac{1}{n-1}}$$
$$\times \left(\int_{\mathbb{R}^{n-1}} \left(\prod_{j=2}^{n} \int_{-\infty}^{+\infty} F_j(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\ ds \right)^{\frac{1}{n-2}} dy \right)^{\frac{n-2}{n-1}}.$$

Since $\mathcal{H}_{n-1}$ holds, we have

$$\int_{\mathbb{R}^{n-1}} \left(\prod_{j=2}^{n} \int_{-\infty}^{+\infty} F_j(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_n)\ ds \right)^{\frac{1}{n-2}} dy \leq \left(\prod_{j=2}^{n} \int_{\mathbb{R}^{n-1}} F_j(x_1, z)\ dz \right)^{\frac{1}{n-2}}.$$

We obtain that

$$\int\limits_{\mathbb{R}^{n-1}} \left(\prod_{j=1}^{n} \int\limits_{-\infty}^{+\infty} F_j(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n)\ ds \right)^{\frac{1}{n-1}} dy \leq$$

$$\left(\int\limits_{\mathbb{R}^{n-1}} \int\limits_{-\infty}^{+\infty} F_1(s,x_2,\ldots,x_n)\ dsdy \right)^{\frac{1}{n-1}} \left(\prod_{j=2}^{n} \int\limits_{\mathbb{R}^{n-1}} F_j(x_1,z)\ dz \right)^{\frac{1}{n-1}} \leq$$

$$\left(\int\limits_{\mathbb{R}^{n}} F_1(x)\ dx \right)^{\frac{1}{n-1}} \left(\prod_{j=2}^{n} \int\limits_{\mathbb{R}^{n-1}} F_j(x_1,z)\ dz \right)^{\frac{1}{n-1}}.$$

Integrating with respect to x_1 and using the Hölder inequalaties, we deduce that

$$\int\limits_{\mathbb{R}^{n}} \left(\prod_{j=1}^{n} \int\limits_{-\infty}^{+\infty} F_j(x_1,\ldots,x_{j-1},s,x_{j+1},\ldots,x_n)\ ds \right)^{\frac{1}{n-1}} dx \leq$$

$$\left(\int\limits_{\mathbb{R}^{n}} F_1(x)\ dx \right)^{\frac{1}{n-1}} \int\limits_{-\infty}^{+\infty} \left(\prod_{j=2}^{n} \int\limits_{\mathbb{R}^{n-1}} F_j(x_1,z)\ dz \right)^{\frac{1}{n-1}} dx_1.$$

$$\leq \prod_{j=1}^{n} \int\limits_{\mathbb{R}^{n}} F_j(x)\ dx.$$

So $\mathcal{H}_n$ holds if $\mathcal{H}_{n-1}$ holds and hence (5.3) is true. This proves the **claim**.

Coming back to (5.2), applying (5.3) with $F_i = |\partial_i u|$ we then obtain

$$\left(\int\limits_{\mathbb{R}^{n}} |u(x)|^{\frac{n}{n-1}}\ dx \right)^{\frac{n-1}{n}} \leq \frac{1}{2} \left(\prod_{j=1}^{n} \int\limits_{\mathbb{R}^{n}} |\partial_j u|\ dx \right)^{\frac{1}{n}} \leq \frac{1}{2} \int\limits_{\mathbb{R}^{n}} |\nabla u|\ dx.$$

This proves the proposition. □

Corollary 5.18. *Let $1 \leq p < n$. We have for all $u \in C_c^\infty(\mathbb{R}^n)$*

$$\left(\int\limits_{\mathbb{R}^{n}} |u|^{\frac{np}{n-p}}\ dx \right)^{\frac{n-p}{np}} \leq \frac{p(n-1)}{2(n-p)} \left(\int\limits_{\mathbb{R}^{n}} |\nabla u|^p\ dx \right)^{1/p}.$$

Proof. We have by Proposition 5.17

$$\int\limits_{\mathbb{R}^{n}} |u|^{\frac{np}{n-p}}\ dx = \int\limits_{\mathbb{R}^{n}} \left(|u|^{\frac{p(n-1)}{n-p}} \right)^{\frac{n}{n-1}} dx \leq \left(\frac{1}{2} \int\limits_{\mathbb{R}^{n}} \left| \nabla |u|^{\frac{p(n-1)}{n-p}} \right| dx \right)^{\frac{n}{n-1}}.$$

$$\leq \left(\frac{p(n-1)}{2(n-p)} \int\limits_{\mathbb{R}^{n}} \left| |u|^{\frac{p(n-1)}{n-p}-1} \right| |\nabla u|\ dx \right)^{\frac{n}{n-1}} = \left(\frac{p(n-1)}{2(n-p)} \int\limits_{\mathbb{R}^{n}} |u|^{\frac{n(p-1)}{n-p}} |\nabla u|\ dx \right)^{\frac{n}{n-1}}.$$

Applying the Hölder inequality this gives

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\, dx\right)^{\frac{n-1}{n}} \leq \frac{p(n-1)}{2(n-p)} \int_{\mathbb{R}^n} |u|^{\frac{n(p-1)}{n-p}} \left|\nabla u\right| \, dx$$

$$\leq \frac{p(n-1)}{2(n-p)} \left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} \left|\nabla u\right|^p \, dx\right)^{1/p}.$$

Therefore we obtain

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\, dx\right)^{\frac{n-p}{np}} \leq \frac{p(n-1)}{2(n-p)} \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{1/p},$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. □

Theorem 5.19. *Let $1 \leq p < n$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$, where p^* is the Sobolev conjugate exponent of p. And for all $u \in W^{1,p}(\mathbb{R}^n)$ we have*

$$\|u\|_{L^{p^*}} \leq C\|u\|_{W^{1,p}},$$

for some constant $C = C(n,p)$.

Proof. This follows from Corollary 5.18 by using the density of $C_c^\infty(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$. □

Corollary 5.20. *Let $1 \leq p < n$ and $p \leq q \leq p^*$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ and for all $u \in W^{1,p}(\mathbb{R}^n)$ we have*

$$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}},$$

for some constant $C = C(n,p,q)$.

Proof. Let $u \in W^{1,p}(\mathbb{R}^n)$ and we choose $0 \leq \theta \leq 1$ such that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}.$$

We have by Hölder inequality

$$\int_{\mathbb{R}^n} |u|^q \, dx = \int_{\mathbb{R}^n} |u|^{\theta q} |u|^{(1-\theta)q} \, dx \leq \left(\int_{\mathbb{R}^n} |u|^p \, dx\right)^{\frac{\theta q}{p}} \left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx\right)^{\frac{(1-\theta)q}{p^*}}$$

Therefore

$$\left(\int_{\mathbb{R}^n} |u|^q \, dx\right)^{1/q} \leq \left\|u\right\|_{L^p}^{\theta} \left\|u\right\|_{L^{p^*}}^{1-\theta}$$

and then by Young's inequality

$$\left(\int_{\mathbb{R}^n} |u|^q \, dx\right)^{1/q} \leq \left\|u\right\|_{L^p}^{\theta} \left\|u\right\|_{L^{p^*}}^{1-\theta} \leq \theta\left\|u\right\|_{L^p} + (1-\theta)\left\|u\right\|_{L^{p^*}}.$$

So we have obtained for some constant $C = C(n,p,q)$

$$\|u\|_{L^q} \le C\Big(\|u\|_{L^p} + \|u\|_{L^{p^*}}\Big) \quad \forall\, u \in W^{1,p}(\mathbb{R}^n).$$

By the Sobolev embedding Theorem 5.19 this gives us

$$\|u\|_{L^q} \le C\|u\|_{W^{1,p}} \quad \forall\, u \in W^{1,p}(\mathbb{R}^n).$$

□

Remark 5.21. *The constants in Proposition 5.17, Corollary 5.18 ... are not optimal.*
For $p = 1$, finding the best Sobolev constant is equivalent to the Isoperimetric Inequality which says that a sphere has minimal area among all surfaces enclosing a given volume.
For $1 < p < n$ the equality in 5.18 are achieved by Aubin-Talenti instantons which have the form: $\big(a + b|x|^{\frac{p}{p-1}}\big)^{\frac{p-n}{p}}$.

Remark 5.22. *More recently, a new elegant proof of the optimal Sobolev inequalities in $\mathbb{R}^n$ was given by Cordero-Erausquin-Nazaret-Villani using Optimal Transport theory.*

Theorem 5.23. *Let $1 \le p < n$. Then $W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{m,q}(\mathbb{R}^n)$ for all $0 \le m < k$ and q satisfying*

$$\frac{1}{q} = \frac{1}{p} - \frac{k-m}{n}.$$

Exercise* 5.24. *Prove the above embedding Theorem.*

Remark 5.25. *The Sobolev embedding in Theorem 5.19 does not hold in the limit case $p = n$ with $p^* = +\infty$.*
Consider f defined by

$$f(x) = \phi(x)\log\left(1 + \log\frac{1}{|x|}\right),$$

where $\phi \in C_c^\infty(\mathbb{R}^n)$ is the cut-off function

$$\phi(x) = \begin{cases} 1 & \text{in } B(0,1), \\ 0 & \text{in } \mathbb{R}^n \setminus B(0,2). \end{cases}$$

Then $|\nabla f| \in L^n(\mathbb{R}^n)$ and $f \in W^{1,n}(\mathbb{R}^n)$, but $f \notin L^\infty(\mathbb{R}^n)$.

3. Sobolev Embedding II

We show that if the weak derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$, then the function is continuous, and Hölder continuous. The following result is due to Morrey. The main idea is to estimate the difference $|u(x) - u(y)|$ in terms of $|\nabla u|$ by the mean value theorem, average the result over a ball $B(x,r)$ and estimate the result in terms of$\|\nabla u\|_{L^p}$ by Hölder inequality.

Theorem 5.26. *Let $n < p < +\infty$ and*

$$\alpha = 1 - \frac{n}{p}$$

Then for some constant $C = C(p,n)$ we have for all $u \in C_c^\infty(\mathbb{R}^n)$

$$[u]_\alpha \ \le C\|\nabla u\|_{L^p} \quad \text{and} \quad \|u\|_{L^\infty} \ \le C\|u\|_{W^{1,p}}$$

where $[\cdot]_\alpha$ is the Hölder seminorm:

$$[u]_{\alpha,\Omega} := \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

So for some constant $C = C(p,n)$ we have for all $u \in C_c^\infty(\mathbb{R}^n)$

$$\|u\|_{C^{0,\alpha}} \ \le C\|u\|_{W^{1,p}},$$

where $\|\cdot\|_{C^{0,\alpha}}$ is the Hölder norm:

$$\|u\|_{C^{0,\alpha}(\Omega)} := \sup_{x\in\Omega} |u(x)| \ + \sup_{x,y\in\Omega,\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Proof. We break the proof into steps.

Step 1: We have for any ball $B(\xi, r)$

$$\frac{n}{r^n} \int\limits_{B(\xi,r)} |u(x)-u(\xi)|\ dx \le \int\limits_{B(\xi,r)} \frac{|\nabla u(x)|}{|x-\xi|^{n-1}}\ dx.$$

Take $\nu \in \mathbb{R}^n$ with $|\nu| = 1$. We have for $s > 0$

$$\begin{aligned} u(\xi+t\nu)-u(\xi) \ &= \int_0^t \frac{d}{dt} u(\xi+s\nu)\ ds = \int_0^t \langle \nabla u(\xi+s\nu)\cdot\nu\rangle\ ds, \\ \big|u(\xi+t\nu)-u(\xi)\big| \ &\le \int_0^t \big|\nabla u(\xi+s\nu)\big|\ ds. \end{aligned}$$

Integrating over $\mathbb{S}^{n-1}$

$$\int\limits_{\mathbb{S}^n} \big|u(\xi+t\nu)-u(\xi)\big|\ d\sigma(\nu) \le \int\limits_{|\nu|=1} \int_0^t \big|\nabla u(\xi+s\nu)\big|\ ds\ d\sigma(\nu).$$

By change of coordinates using Polar-coordinates

$$\int\limits_{|\nu|=1} \int_0^t \big|\nabla u(\xi+s\nu)\big|\ ds\ d\sigma(\nu) = \int\limits_{|\nu|=1} \int_0^t s^{n-1} \frac{\big|\nabla u(\xi+s\nu)\big|}{s^{n-1}}\ ds\ d\sigma(\nu) = \int\limits_{B(\xi,t)} \frac{\big|\nabla u(y)\big|}{|y-\xi|^{n-1}} dy.$$

Thus

$$\int\limits_{\mathbb{S}^n} \big|u(\xi+t\nu)-u(\xi)\big|\ d\sigma(\nu) \le \int\limits_{B(\xi,t)} \frac{\big|\nabla u(y)\big|}{|y-\xi|^{n-1}} dy.$$

We then obtain

$$\int\limits_{B(\xi,r)} |u(y)-u(\xi)|\ dy = \int_0^r t^{n-1} \int\limits_{\mathbb{S}^n} \big|u(\xi+t\nu)-u(\xi)\big|\ d\sigma(\nu)\ dt \le \frac{r^n}{n} \int\limits_{B(\xi,t)} \frac{\big|\nabla u(x)\big|}{|x-\xi|^{n-1}} dx.$$

This completes the step.

Step 2: Let $x, y \in \mathbb{R}^n$, $r = |y - x|$. Then for $z \in B(x,r) \cap B(y,r) = \mathcal{B}$

$$|u(x) - u(y)| \leq |u(x) - u(z)| + |u(z) - u(y)|.$$

And then from Step 1

$$\begin{aligned}
|u(x) - u(y)| &\leq \frac{n}{\omega_{n-1} r^n} \left[\int_{\mathcal{B}} |u(x) - u(z)| \, dz + \int_{\mathcal{B}} |u(z) - u(y)| \, dz \right] \\
&\leq \frac{n}{\omega_{n-1} r^n} \int_{B(x,r)} |u(z) - u(x)| \, dz + \frac{n}{\omega_{n-1} r^n} \int_{B(x,r)} |u(z) - u(y)| \, dz \\
&\leq \frac{1}{\omega_{n-1}} \int_{B(x,r)} \frac{|\nabla u(z)|}{|z - x|^{n-1}} \, dz + \frac{1}{\omega_{n-1}} \int_{B(x,r)} \frac{|\nabla u(z)|}{|z - y|^{n-1}} \, dz.
\end{aligned}$$

Then by Hölder inequality

$$\begin{aligned}
&\leq \frac{1}{\omega_{n-1}} \left(\int_{B(x,r)} |\nabla u(z)|^p \, dz \right)^{1/p} \left(\int_{B(x,r)} \frac{1}{|z - x|^{(n-1)\frac{p}{p-1}}} \, dz \right)^{(p-1)/p} \\
&\quad + \frac{1}{\omega_{n-1}} \left(\int_{B(y,r)} |\nabla u(z)|^p \, dz \right)^{1/p} \left(\int_{B(y,r)} \frac{1}{|z - x|^{(n-1)\frac{p}{p-1}}} \, dz \right)^{(p-1)/p} \\
&\leq \frac{1}{\omega_{n-1}} \left(\int_0^r \frac{s^{n-1}}{s^{(n-1)\frac{p}{p-1}}} \, ds \right)^{(p-1)/p} \left[\left(\int_{B(x,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} \right] \\
&\quad + \frac{1}{\omega_{n-1}} \left(\int_0^r \frac{s^{n-1}}{s^{(n-1)\frac{p}{p-1}}} \, ds \right)^{(p-1)/p} \left(\int_{B(x,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} \\
&\leq \omega_{n-1}^{-1/p} \left(\int_0^r s^{\left[1 - \frac{p'}{p}(n-1)\right] - 1} \, ds \right)^{(p-1)/p} \left(\int_{B(x,2r)} |\nabla u(z)|^p \, dz \right)^{1/p} \\
&\leq \left(\frac{1}{\omega_{n-1}} \left(\frac{p-1}{p} \right) \left(\frac{p}{p-n} \right) \right)^{1/p} r^{\left(1 - \frac{n}{p}\right)} \left(\int_{B(x,2r)} |\nabla u(z)|^p \, dz \right)^{1/p}.
\end{aligned}$$

Therefore for a constant $C = C(p,n)$ we have

$$|u(x) - u(y)| \leq C \, |x - y|^{1 - n/p} \big\| \nabla u \big\|_{L^p(B(x,2r))}.$$

Using this estimate we then obtain

Exercise* 5.27.

$$\|u\|_{L^\infty} \leq C \|u\|_{W^{1,p}}.$$

Complete the proof of the above estimate.

□

Theorem 5.28. *Let $n < p \leq +\infty$ and $\alpha = 1 - n/p$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$, and there is a constant $C = C(n,p)$ such that*

$$\|u\|_{C^{0,\alpha}} \leq C \ \|u\|_{W^{1,p}} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^n).$$

Let Ω be bounded domain in $\mathbb{R}^n$, let $n < p \leq +\infty$ and $\alpha = 1 - n/p$. Then $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$. and there is a constant $C = C(n,p,\Omega)$ such that

$$\|u\|_{C^{0,\alpha}} \leq C \ \|u\|_{W^{1,p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

See Evans [5] section 5.4 for proofs.

4. Extension and Trace

The open upper half-space is defined as

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

A point $x \in \mathbb{R}^n$ will sometimes be written as $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

We first extend functions $u \in W^{1,p}(\mathbb{R}^n_+)$ to functions in $W^{1,p}(\mathbb{R}^n)$ without increasing their norm. An extension may be constructed by reflecting a function across the boundary $\partial\mathbb{R}^n_+$ in a way that preserves its differentiability. Such an extension map E is not, may not be unique. Then "flattening the boundary" the same can be done for a bounded domain Ω with C^1 boundary $\partial\Omega$.

Theorem 5.29. *Let $1 \leq p < +\infty$. There is a bounded linear map*

$$E : W^{1,p}(\mathbb{R}^n_+) \longrightarrow W^{1,p}(\mathbb{R}^n),$$

such that for all $u \in W^{1,p}(\mathbb{R}^n_+)$ we have

(a) *$Eu = u$ a.e. in $\mathbb{R}^n_+$,*
(b) *for some constant $C = C(n,p)$*

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n_+)}.$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1-boundary $\partial\Omega$. There is a bounded linear map

$$E : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n),$$

such that for all $u \in W^{1,p}(\Omega)$ we have

(a) *$Eu = u$ a.e. in Ω,*
(b) *For any open U such that $\Omega \Subset U$ we can have supp$(Eu) \subset U$.*
(c) *For some constant $C = C(n,p)$*

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

We refer to Evans [5] section 5.4 for a proof.

Let Ω be a smooth domain in $\mathbb{R}^n$. For $u \in C\left(\overline{\Omega}\right)$ the boundary values are defined pointwise as a continuous function as the restriction of u on $\partial\Omega$. In general, a Sobolev function is not equivalent pointwise a.e. to a continuous function and the boundary of a smooth domain has measure zero, so the boundary values cannot be defined pointwise. For example, we cannot make sense of the boundary values of an L^p-function as an L^p-function on the boundary. Nevertheless, we can define the boundary values of Sobolev functions by the so called trace map.

Functions in $C_c^\infty(\overline{\mathbb{R}^n_+})$ need not vanish on the boundary $\partial\mathbb{R}^n_+$. On the other hand functions in $C_c^\infty(\mathbb{R}^n)$ vanishes on the boundary $\partial\mathbb{R}^n_+$, and it is not true that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n_+)$. Roughly speaking, we can only approximate Sobolev functions that "vanish on the boundary" by functions in $C_c^\infty(\mathbb{R}^n_+)$.

DEFINITION 5.30. *The space $W_0^{k,p}(\mathbb{R}^n_+)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in $W^{k,p}(\mathbb{R}^n_+)$.*

We interpret $W_0^{k,p}(\mathbb{R}^n_+)$ as the space of functions that vanish on the boundary.

Theorem 5.31 (Trace theorem)**.** *Let $1 \le p < +\infty$. There is a bounded linear map*

$$T : W^{1,p}(\mathbb{R}^n_+) \longrightarrow L^p(\partial\mathbb{R}^n_+),$$

such that for any $u \in C_c^\infty(\overline{\mathbb{R}^n_+})$

$$Tu(x') = u(x', 0)$$

and

$$\|Tu\|_{L^p(\mathbb{R}^{n-1})} \le C\|u\|_{W^{1,p}(\mathbb{R}^n_+)}$$

for some constant $C = C(n,p)$. Furthermore, $u \in W_0^{1,p}(\mathbb{R}^n_+)$ if and only if $Tu = 0$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1-boundary $\partial\Omega$. There is a bounded linear map

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega),$$

such that for any $u \in C^\infty(\overline{\Omega})$

$$Tu = u \text{ on } \partial\Omega$$

and

$$\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

for some constant $C = C(n,p,\Omega)$. Furthermore, $u \in W_0^{1,p}(\Omega)$ if and only if $Tu = 0$.

PROOF. Let $u \in C_c^\infty(\overline{\mathbb{R}^n_+})$. We have

$$|u(x',0)|^p \le p \int_0^{+\infty} |u(x',t)|^{p-1}\ |\partial_n u(x',t)|\ dt.$$

Then Hölder inequality gives

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} |u(x',0)|^p\ dx' &\le p \int_{\mathbb{R}^{n-1}} \int_0^{+\infty} |u(x',t)|^{p-1}\ |\partial_n u(x',t)|\ dt\ dx', \\
&\le p \int_{\mathbb{R}^n} |u(x)|^{p-1}\ |\partial_n u(x)|\ dx, \\
&\le p \left(\int_{\mathbb{R}^n} |u(x)|^p\ dx \right)^{(p-1)/p} \left(\int_{\mathbb{R}^n} |\partial_n u(x)|^p\ dx \right)^{1/p}.
\end{aligned}$$

Thus

$$\big\|u(\cdot,0)\big\|_{L^p(\mathbb{R}^{n-1})} \leq p\ \|u\|^{p-1}_{L^p(\mathbb{R}^n)}\big\|\partial_n u\big\|_{L^p(\mathbb{R}^n)} \leq p\ \|u\|^p_{W^{1,p}(\mathbb{R}^n)}.$$

Therefore

$$T : C_c^\infty(\overline{\mathbb{R}^n_+}) \longrightarrow C_c^\infty(\mathbb{R}^{n-1}),$$

is a bounded map with respect to the $W^{1,p}(\mathbb{R}^n_+)$ and $L^p(\partial\mathbb{R}^n_+)$ norms, and thus extends by density and continuity to a map between these spaces. It follows that $Tu = 0$ if $u \in W_0^{1,p}(\mathbb{R}^n)$.

We refer to Evans [5] section 5.4 for a complete proof.

□

5. Compactness Results

Compactness results play a central role in the analysis of PDEs. In many situations it is much easier to obtain a solution to an approximation of the main equation. If one can show that the set of all solutions of the approximations is compact, then one can pass to limit to obtain a solution to the main equation in question.

A subspace S of a metric space X is said to be said to precompact if the closure of S in X is compact. Equivalently, S is precompact in X if if every bounded sequence in S has a subsequence that converges in X. The Arzelà-Ascoli theorem gives a basic criterion for compactness in function spaces: namely, a set of continuous functions on a compact metric space is precompact if and only if it is bounded and equicontinuous.

Definition 5.32. *We say that a Banach space $\big(X, \|\cdot\|_X\big)$ is compactly embedded in the Banach space $\big(Y, \|\cdot\|_Y\big)$ if the embedding $\iota : X \to Y$ is compact. That is, ι maps a bounded set in X to a precompact set in Y. Equivalently, for a bounded sequence $(x_m)_{m\in\mathbb{N}}$ in X, the sequence $(\iota(x)_m)_{m\in\mathbb{N}}$ has a convergent subsequence in Y.*

Theorem 5.33 (Arzelà-Ascoli)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. A subset S of $C(\overline{\Omega})$ (or $C^0(\overline{\Omega})$) equipped with the supremum norm $\|\cdot\|_\infty$ is precompact if and only if:*

(i) *S is bounded, that is, there exists a constant M such that*

$$\|u\|_\infty := \sup_{\overline{\Omega}} |u| \leq M \quad \text{for all } u \in S.$$

(ii) *S is equicontinuous, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \Omega$ with $|x - y| < \delta$ we have*

$$\big|u(x) - u(y)\big| < \varepsilon \quad \text{for all } u \in S.$$

Theorem 5.34 (Kolmogorov-Riesz)**.** *Let $1 \leq p < +\infty$. A subset S of $L^p(\mathbb{R}^n)$ is precompact if and only if:*

(i) *S is bounded, that is, there exists a constant M such that*

$$\|u\|_{L^p} \leq M \quad \text{for all } u \in S.$$

(ii) *S is tight, that is, for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\left(\int_{\mathbb{R}^n \setminus B(0,R)} |u|^p \, dx \right)^{1/p} < \varepsilon \quad \text{for all } u \in S.$$

(iii) *S is L^p-equicontinuous, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$ then*

$$\left(\int_{\mathbb{R}^n} \big|u(x+h) - u(x)\big|^p \, dx \right)^{1/p} < \varepsilon \quad \text{for all } u \in S.$$

Exercise 5.35. *Consider $\varphi_m = \chi_{(m,m+1)}$, $m \in \mathbb{N}$. Show that the sequence $(\varphi_m)_{m\in\mathbb{N}}$ is bounded and equicontinuous in $L^p(\mathbb{R}^n)$ for any $1 \le p < +\infty$, but it is not tight nor it is precompact.*

Theorem 5.36. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $1 \le p < n$ and let $1 \le q < p^* = \dfrac{np}{n-p}$. If S is a bounded in $W_0^{1.p}(\Omega)$, then S is precompact in $L^q(\Omega)$.*

Equivalently if $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence in $W_0^{1.p}(\Omega)$, that is

$$\|u_m\|_{W^{1,p}(\Omega)} \le M \quad \text{for all } m \in \mathbb{N},$$

for some constant M. Then there exists a subsequence (u_{m_i}) and $u_0 \in L^q(\Omega)$ such that

$$u_{m_i} \longrightarrow u_0 \quad \text{as } i \to +\infty \quad \text{in } L^q(\Omega).$$

Proof. By a density argument, we may assume that the functions in S are smooth. We may then extend the functions and their derivatives by zero outside Ω to obtain smooth functions on $\mathbb{R}^n$. We can then identify S as bounded subspace in $W_0^{1.p}(\mathbb{R}^n)$ with supports contained in Ω. We show that S is a compact subspace of $L^q(\mathbb{R}^n)$ for $1 \le q < p^*$. We use the Kolmogorov-Riesz theorem to show compactness.

$$\|u\|_{L^q(\mathbb{R}^n)} = \|u\|_{L^q(\Omega)} \le C\|u\|_{L^{p^*}(\Omega)} = C\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\nabla u\|_{L^p(\mathbb{R}^n)} \le M$$

so the set S is bounded. The set S is tight since the supports of all functions in S are contained in the compact set $\overline{\Omega}$.

We verify equicontinuity. We note that since u and so ∇u has support contained in the compact set $\overline{\Omega}$, we have

$$\|\nabla u\|_{L^1(\mathbb{R}^n)} \le \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Fixing $h \in \mathbb{R}^n$ we have

$$\big|f(x+h) - f(x)\big| = \left| \int_0^1 \big(h, \nabla u(x+th)\big) \, dt \right| \le |h| \int_0^1 |\nabla u(x+th)| \, dt.$$

Integrating with respect to x and using Fubini's theorem we get

$$\int_{\mathbb{R}^n} \big|f(x+h) - f(x)\big| \, dx \le C \, |h| \|\nabla u\|_{L^1(\mathbb{R}^n)} \le C \, |h| \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Thus we have

$$\big\|f(x+h)-f(x)\big\|_{L^1(\mathbb{R}^n)} \leq C\ |h|\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Using the interpolation inequality for any $1 \leq q < p^*$ we get that

$$\big\|f(x+h)-f(x)\big\|_{L^q(\mathbb{R}^n)} \leq \big\|f(x+h)-f(x)\big\|^{\theta}_{L^1(\mathbb{R}^n)}\big\|f(x+h)-f(x)\big\|^{1-\theta}_{L^{p^*}(\mathbb{R}^n)}.$$

Here $0 < \theta \leq 1$ is given that

$$\frac{1}{q} = \theta + \frac{1-\theta}{p^*},$$

The Sobolev embedding theorem implies that

$$\big\|f(x+h)-f(x)\big\|_{L^{p^*}(\mathbb{R}^n)} \leq C\ |h|^{\theta}\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

It follows that S is L^q-equicontinuous if the derivatives of functions in S are uniformly bounded in L^p, and the result follows.

□

Remark 5.37. *The assumptions that the domain Ω is bounded and that $q < p^*$ are necessary.*

Theorem 5.38. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $n < p < +\infty$. If $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence in $W_0^{1.p}(\Omega)$, that is*

$$\|u_m\|_{W^{1,p}(\Omega)} \leq M \quad \textit{for all } m \in \mathbb{N},$$

for some constant M. Then there exists a subsequence (u_{m_i}) and $u_0 \in C(\Omega)$ such that

$$u_{m_i} \longrightarrow u_0 \quad \textit{as } i \to +\infty \quad \textit{in } C(\Omega).$$

Exercise* 5.39. *Prove the above compactness result.*

CHAPTER 6

Appendix A

Parametrization of the Boundary

Let $\Omega \subset \mathbb{R}^n$ be an open connected set. $\partial\Omega$ will denote the boundary of Ω. Roughly speaking, $\partial\Omega$ is smooth if we can *flatten* it using a smooth map so that Ω looks like a half-space $\mathbb{R}^n_+$ near the boundary.

Definition 6.1. *Let $k \in \mathbb{N}$ and suppose that U, V are open subsets in $\mathbb{R}^n$ A map $\varphi : U \to V$ is a C^k-diffeomorphism if it one-to-one, onto, and both φ, φ^{-1} are C^k maps.*

We describe a *parametrization* around a point of the boundary $\partial\Omega$.

Theorem-Definition 6.2. *Let Ω be an open set in $\subset \mathbb{R}^n$ and let $\xi_0 \in \partial\Omega$. We say that the boundary $\partial\Omega$ is C^k, or that Ω is C^k, if for every point $\xi_0 \in \Omega$ there exists open sets $U, V \subset \mathbb{R}^n$ and a C^k-diffeomorphism $\varphi : U \to V$ such that:*

$$\varphi\left(U \cap \{x_n > 0\}\right) = V \cap \Omega \quad \text{and } \varphi\left(U \cap \{x_n = 0\}\right) = V \cap \partial\Omega.$$

We say that $\partial\Omega$ is smooth, or that Ω is smooth, if φ is a smooth diffeomorphism. Similarly we have the definition of an analytic boundary.

Remark 6.3. *The boundary $\partial\Omega$ of an bounded open set $\Omega \subset \mathbb{R}^n$ is an $(n-1)$ dimensional oriented manifold in $\mathbb{R}^n$, and as such, has a (outer) normal vector field which we denote by ν..*

Theorem-Definition 6.4 (Flattening the boundary)**.** *Suppose Ω is an open set in $\mathbb{R}^n$ with C^k-boundary and let $\xi_0 \in \partial\Omega$. Then there exists open sets $U, V \subset \mathbb{R}^n$, there exists an open interval $I \subset \mathbb{R}$, there exists an open set $U' \subset \mathbb{R}^{n-1}$, and there exists a C^k-diffeomorphism $\varphi : U \to V$ and $\varphi_0 \in C^k(U')$ such that up to a rotation of coordinates if necessary we have:*

(6.1)

- $0 \in U = U' \times I$ *and* $\xi_0 \in V$.
- $\varphi(0) = \xi_0$.
- $\varphi\left(U \cap \{x_n > 0\}\right) = V \cap \Omega$ *and* $\varphi\left(U \cap \{x_n = 0\}\right) = V \cap \partial\Omega$.
- $D_0\varphi = \mathbb{I}_{\mathbb{R}^n}$. *Here $D_x\varphi$ denotes the differential of φ at the point x and $\mathbb{I}_{\mathbb{R}^n}$ is the identity map on $\mathbb{R}^n$.*
- $D_0\varphi[e_n] = \nu_{\xi_0}$ *where ν_{ξ_0} denotes the outer unit normal vector to $\partial\Omega$ at the point p.*
- $\{D_0\varphi[e_1], \cdots, D_0\varphi[e_{n-1}]\}$ *forms an orthonormal basis of* $T_{\xi_0}\partial\Omega$.
- $\varphi(y, x_n) = \xi_0 + (y, x_n + \varphi_0(y))$ *for all* $(y, x_n) \in U' \times I = U$.
- $\varphi_0(0) = 0$ *and* $\nabla\varphi_0(0) = 0$.

Divergence Theorem: Integration by Parts Formula

Theorem 6.5. *Let Ω be an bounded open set in $\mathbb{R}^n$ with C^1-boundary $\partial\Omega$.*

(1) *Suppose* $u \in C^1(\overline{\Omega})$. *Then for all* $1 \le i \le n$

$$\int_{\Omega} \partial_{x_i} u \; dx = \int_{\partial\Omega} u\nu^i \; d\sigma$$

(2) *Suppose* $u, v \in C^1(\overline{\Omega})$. *Then for all* $1 \le i \le n$

$$\int_{\Omega} v \; \partial_{x_i} u \; dx = -\int_{\Omega} u \; \partial_{x_i} v \; dx + \int_{\partial\Omega} uv\nu^i \; d\sigma$$

Here $d\sigma$ *denotes the surface measure on the hypersurface* $\partial\Omega$.

Proposition 6.6 (Gauss-Green Formulas)**.** *Suppose* $u, v \in C^2(\overline{\Omega})$. *Then*

(1)

$$\int_{\Omega} \Delta u \; dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \; d\sigma.$$

(2)

$$\int_{\Omega} (\nabla u, \nabla v) \; dx = -\int_{\Omega} u\Delta v \; dx + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} \; d\sigma.$$

(3)

$$\int_{\Omega} (u\Delta v - v\Delta u) \, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma.$$

Change of Variable Formula

Theorem 6.7. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $\varphi : \Omega \to \mathbb{R}$ *be a* C^1 *map such that* $\varphi : \Omega \to \varphi(\Omega)$ *is a* C^1*- diffeomorphism. Consider* $f : \varphi(\Omega) \to \mathbb{R}$ *a nonnegative Lebesgue measurable function or an integrable function, then*

$$\int_{\varphi(\Omega)} f(x) \; dx = \int_{\Omega} f(\varphi(x))|D\varphi(x)| \; dx.$$

Polar Coordinates

Definition 6.8 (Polar Coordinates). *We define Polar Coordinates in* $\mathbb{R}^n \setminus \{0\}$ *by* $x = (r, \theta)$ *where* $r = |x| > 0$ *and* $\theta \in \mathbb{S}^{n-1}$. *In these coordinates, Lebesgue measure has the representation*

$$dx = r^{n-1} dr d\sigma(y)$$

where $d\sigma(y)$ *is the surface area measure on the unit sphere* $\mathbb{S}^{n-1}$.

Theorem 6.9. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be an integrable function*

(1) *For any point* $\xi_0 \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f \; dx = \int_0^{+\infty} \left(\int_{\partial B(\xi_0, r)} f \; d\sigma \right) dr = \int_0^{+\infty} \left(\int_{\mathbb{S}^{n-1}} f(\xi_0 + ry) \; d\sigma(y) \right) r^{n-1} \; dr.$$

(2) *And for all* $r > 0$

$$\frac{d}{dr}\left(\int\limits_{B(\xi_0,r)} f \; dx \right) = \int\limits_{\partial B(\xi_0,r)} f \; d\sigma.$$

CHAPTER 7

Appendix B

Convolutions

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. We define the convolution $f * g : \mathbb{R}^n \to \mathbb{R}$ by:

DEFINITION 7.1.

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\ dy$$

provided that the integral exists for almost every x in $\mathbb{R}^n$.

Exercise 7.2. *Show that, when defined*

(1) *The convolution product is commutative*

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\ dy = g * f(x)$$

(2) *The convolution product is associative*

$$f * (g * h) = (f * g) * h$$

(3) *If $f, g \in C^c(\mathbb{R}^n)$, then*

$$f * g \in C^c(\mathbb{R}^n) \text{ and } supp(f * g) \subset\ supp(f) +\ supp(g)$$

The following result, called Young's inequality, gives conditions for the convolution of L^p functions to exist and estimates its norm.

Theorem 7.3 (Young's Inequality)**.** *Let $1 \le p \le q \le r \le +\infty$ and suppose that*

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$$

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}.$$

Example 7.4 (Hölder Inequality)**.** *When $r = +\infty$*

Example 7.5 (Fubini Theorem)**.** *When $p = q = 1$ and $r = 1$*

Exercise 7.6. *Let $p = r$ and $q = 1$. The convolution product is then a bounded linear map on $L^p(\mathbb{R}^n)$.*

Mollifiers and Approximation by Smooth Functions

Definition 7.7 (Mollifiers). *Consider the function* $\eta \in C_c^{\infty}(\mathbb{R}^n)$ *with supp*(η)*=the ball* $B(0,1)$ *given by:*

$$\eta(x) = \begin{cases} \tau_n \exp\left[-1/\left(1-|x|\right)^2\right] & \textit{when } |x| < 1 \\ 0 & \textit{when } |x| \geq 1 \end{cases}$$

The constant τ_n *is a normalizing factor chosen in a such a way that*

$$\int_{\mathbb{R}^n} \eta \; dx = 1.$$

We rescale η and obtain functions having arbitrary small (or large) supports while preserving the integral(mass).

Exercise 7.8. *For* $\varepsilon > 0$ *we consider the function* η_ε *defined by:* $\eta_\varepsilon = \dfrac{1}{\varepsilon^n}\eta\left(\dfrac{x}{\varepsilon}\right)$. *Then* $\eta_\varepsilon \in C_c^{\infty}(\mathbb{R}^n)$, *is supported in the ball* $B(0,\varepsilon)$ *and* $\displaystyle\int_{\mathbb{R}^n} \eta_\varepsilon \; dx = 1$.

Definition 7.9. *Let* $f \in L^1_{\text{loc}}(\Omega)$. *The mollification or regularization of* u *is defined as:*

$$\eta_\varepsilon * f(\xi) = \int_{\mathbb{R}^n} \eta_\varepsilon(\xi - x) u(x) \; dx$$

Bibliography

[1] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] Alberto Bressan, *Lecture notes on functional analysis*, Graduate Studies in Mathematics, vol. 143, American Mathematical Society, Providence, RI, 2013. With applications to linear partial differential equations.

[3] Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[4] Haïm Brezis and Felix Browder, *Partial differential equations in the 20th century*, Adv. Math. **135** (1998), no. 1, 76–144.

[5] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[6] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Revised edition, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015.

[7] Gerald B. Folland, *Introduction to partial differential equations*, 2nd ed., Princeton University Press, Princeton, NJ, 1995.

[8] ———, *Real analysis*, 2nd ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999. Modern techniques and their applications; A Wiley-Interscience Publication.

[9] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[10] Qing Han and Fanghua Lin, *Elliptic partial differential equations*, 2nd ed., Courant Lecture Notes in Mathematics, vol. 1, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011.

[11] Qing Han, *A basic course in partial differential equations*, Graduate Studies in Mathematics, vol. 120, American Mathematical Society, Providence, RI, 2011.

[12] Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis; Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[13] Jürgen Jost, *Partial differential equations*, 2nd ed., Graduate Texts in Mathematics, vol. 214, Springer, New York, 2007.

[14] S. Kesavan, *Topics in functional analysis and applications*, John Wiley & Sons, Inc., New York, 1989.

[15] Giovanni Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009.

[16] Jeffrey Rauch, *Partial differential equations*, Graduate Texts in Mathematics, vol. 128, Springer-Verlag, New York, 1991.

[17] Michael Renardy and Robert C. Rogers, *An introduction to partial differential equations*, Texts in Applied Mathematics, vol. 13, Springer-Verlag, New York, 1993.

[18] Walter Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.

[19] ———, *Functional analysis*, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[20] ———, *Principles of mathematical analysis*, 3rd ed., 1976. International Series in Pure and Applied Mathematics.

[21] Robert S. Strichartz, *A guide to distribution theory and Fourier transforms*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1994.

[22] Michael E. Taylor, *Partial differential equations I. Basic theory*, 2nd ed., Applied Mathematical Sciences, vol. 115, Springer, New York, 2011.

[23] William P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.