Rings and modules

1. Is there any finite group G such that $\mathbb{H}$ is isomorphic as an $\mathbb{R}$-algebra to $\mathbb{R}[G]$?

2. Let R be a ring. Then there is a canonical isomorphism of R-algebras
$$R[x]/(x^n - 1) \to R[\mathbb{Z}/n\mathbb{Z}].$$

3. The real quaternions $\mathbb{H}$ and $M_2(\mathbb{R})$ both are $\mathbb{R}$-algebras and have the same underlying real vector spaces. But they are not isomorphic as $\mathbb{R}$-algebras.

4. Let k be a commutative ring and $R = M_n(k)$. Then $R \to R^{op} : A \mapsto {}^tA$ is an isomorphism of rings.

5. Let k be a commutative ring, G a finite group, and $R = k[G]$. Then $g \mapsto g^{-1}$ induces an isomorphism of rings $R \xrightarrow{\simeq} R^{op}$.

6. Let $\mathbb{H}$ be the quaternion ring over $\mathbb{R}$. The quaternionic conjugation $a + bi + cj + dk \mapsto a - bi - cj - dk$ induces an isomorphism of rings $\mathbb{H} \xrightarrow{\simeq} \mathbb{H}^{op}$.

7. Let R be a ring such that there is an isomorphism of rings $\alpha : R^{op} \xrightarrow{\simeq} R$. If M is a left R-module, then the composite $R^{op} \xrightarrow{\alpha} R \xrightarrow{\phi} \mathrm{End}_{\mathbb{Z}}(M)$ gives rise to a right module structure on M. Similarly, a right R-module structure ρ on V gives rise to a left module structure by composing ρ with α^{-1}. In particular if R is a commutative ring, then a left R-module is automatically a right R-module.

8. A $\mathbb{Z}/n\mathbb{Z}$-module is given by an abelian group M s.t. for each $m \in M$, $nm = 0$.

9. Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of R-modules. If $M \to M''$ has an R-linear section s, then $M' \oplus M'' \to M : (x, y) \mapsto x + s(y)$ is an R-module isomorphism with an inverse given by $M \to M' \oplus M'' : z \mapsto (z - sp(z), p(z))$.

10. Let M be a finitely generated R-module and let $\phi : M \to R^n$ be a surjective R-linear map. Show that $\ker \phi$ is finitely generated. [Hint: Use the previous exercise. Show that $\ker \phi$ is a direct summand of M, hence there is a surjection $M \to \ker \phi$.]

*11 Let L/k be a Galois extension of fields with $G = Gal(L/k)$. Then L is actually a free $k[G]$-module of rank 1. (Hint: Normal basis theorem)

*12 If G is a finite group whose order is invertible in a field k, then $k[G]$ is semisimple in the following sense. Let $M' \subset M$ be a submodule. Then there is another submodule $M'' \subset M$ such that $M = M' \oplus M''$.

13. Give an example to show the following. Suppose M' and M'' are two R-modules such that $M' \oplus M''$ is free. It does not necessarily imply that M' and M'' are both free. [Hint: Look at $\mathbb{C}$ as a $\mathbb{C} \times \mathbb{C}$-module]

∗14 Prove that $\mathbb{Q}$ as a $\mathbb{Z}$-module is not projective. [Hint: Otherwise $\mathbb{Q}$ would be a direct summand of a free module over $\mathbb{Z}$. Taking composition with one of the projections to $\mathbb{Z}$, get at least one nonzero map $\mathbb{Q} \to ZZ$. Such a map cannot exist.]

[Note: The last two problems show that projective modules may not be free, while flat modules need not be projective.]

Tensor product

15. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, where $d = gcd(m, n)$.

16. Let A be a local ring, M, N finitely generated modules over A. Suppose $M \otimes_A N = 0$. Then either $M = 0$ or $N = 0$.

17. Let A be a nonzero commutative ring. Let $\phi : A^{\oplus m} \to A^{\oplus n}$ be an A-module homomorphism.

    (a) If ϕ is surjective then $m \geq n$.

    (b) If ϕ is injective then $m \leq n$. [Hint: Assume $m > n$. Compose ϕ with the inclusion $i : A^n \hookrightarrow A^m : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$. Then $i\phi : A^m \to A^m$ is injective. However, any element $(y_1, \ldots, y_m) \in A^m$ with $y_j = 0$ for $j \leq n$ lies in the kernel of $i\phi$.]

18. Let A be a commutative ring, and let $\phi : A^n \to A^n$ be an A-linear map, which is surjective. Prove that ϕ is injective as well. [Hint: Enough to consider A local. By tensoring with the residue field k, we see that $\det(\phi) \notin \mathfrak{m}$, hence $\det(\phi)$ is invertible. Now, $Adj(\phi)\phi = \det(\phi)I$, which is injective because $\det(\phi)$ is invertible, hence ϕ is injective.]

19. Let L and E be two field extensions of the same field k. Then they are both subfields of a common field extension $F \supset L, E$. [Hint: Consider a maximal ideal $\mathfrak{m} \subset L \otimes_k E$, and define $F := L \otimes_k E/\mathfrak{m}$.]

∗20 Let L/k be a finite extension of fields. Then L/k is separable if and only if there is a field extension E/k such that $L \otimes_K E$ is isomorphic as an E-algebra to a product of copies of E (i.e. $E \times \cdots \times E$).

Localisation

21. Let S be a multiplicative set in a ring A. Prove that there is at least one prime ideal $\mathfrak{p}$ in A such that $\mathfrak{p} \cap S = \emptyset$.

22. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated module over A. If $m_1, \ldots, m_n \in M$ such that $\overline{m}_i$ span M/mM, then prove that m_i span M.

23. Prove that finitely generated projective modules over a local ring are free.

24. Prove that a local ring cannot be a product of two nonzero rings.

25. Let $M' \to M \to M''$ be two A-module homomorphisms. Then show that the following are equivalent.

    (1) $M' \to M \to M''$ is exact, (2) $M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$ is exact for all prime ideals $\mathfrak{p}$, (3) $M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}}$ is exact for all maximal ideals $\mathfrak{m}$.

26. Let $\mathfrak{q} \subset \mathfrak{p}$ be two prime ideals of A. Show that the localization of $A_{\mathfrak{p}}$ at the prime ideal corresponding to $\mathfrak{q}$ is isomorphic to $A_{\mathfrak{q}}$.

27. Let M, N be two A-modules, with M finitely generated. Then the natural map
$$S^{-1}Hom_A(M, N) \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$$
    is an isomorphism.

    (Note: The statement is not true without M being finitely generated. For example, take $A = \mathbb{C}[X]$, $M = \oplus_{n\geq 1}A$, $N = \oplus_{n\geq 1}\mathbb{C}[X]/(X^n)$, and $S = \{X^n : n \geq 0\}$. Take $f : M \to N$ defined at the n-th coordinate by the map $\mathbb{C}[X] \to \mathbb{C}[X]/X^n$. Then $f/1$ is nonzero in $S^{-1}Hom_A(M, N)$. However, the whole right hand side is zero, because $S^{-1}N = 0$.)

28. Suppose A is a ring such that for each prime ideal $\mathfrak{p}$, $A_{\mathfrak{p}}$ is a domain. Is A necessarily a domain?