End Semester Exam

MA5101

18-Nov-15, Full Marks 30, Time 3 hours

Each question carries 3 marks. Assume all rings to be noetherian.

1. Prove that a local ring cannot be a product of two nonzero rings.

2. Let A be a ring and $S \subset A$ a multiplicative set. Prove that the prime ideals of $S^{-1}A$ are in one-to-one correspondence ($\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p}$) with the prime ideals of A which don't meet S.

3. Let A be a matrix in Jordan canonical form over an algebraically closed field, and let λ be an eigenvalue of A. Prove that the size of the largest Jordan block corresponding to the eigenvalue λ is equal to the exponent of $(X - \lambda)$ in the expression of the minimal polynomial of A over k.

4. Can there be a 3×3 matrix A over $\mathbb{Q}$ such that $A^8 = I$ but $A^4 \neq I$?

5. Prove the following. Let k be algebraically closed and A any $n \times n$ matrix over k. Then there are a semisimple matrix S and a nilpotent matrix N such that (i) $A = S + N$, (ii) both S and N are polynomials in A (in particular they commute with A). Such a decomposition is unique.

6. Let A be a noetherian ring which is reduced (i.e. there are no nonzero nilpotents). Then prove that (0) does not have any embedded prime ideals.

7. Prove or give counterexample: Every primary ideal is a power of a prime ideal.

8. Prove or give counterexample: Every power of a prime is a primary ideal.

9. Prove the "going-up theorem": Let $A \subset B$ be rings, B integral over A, and let $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m$ $(m < n)$ a chain of prime ideals of B such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ $(1 \leq i \leq m)$. Then the chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m$ can be extended to a chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq n$.

10. Let A be a domain. Prove that the following are equivalent:

    (i) A is integrally closed.

    (ii) $A_{\mathfrak{p}}$ is integrally closed, for each prime ideal $\mathfrak{p}$.

    (iii) $A_{\mathfrak{m}}$ is integrally closed, for each maximal ideal $\mathfrak{m}$.