A will stand for a commutative ring with unity, and will be nonzero unless stated otherwise. A "ring" will always mean a commutative ring with unity.

Multiplicative set. A subset $S \subset A$ is called a *multiplicative set* if $1 \in S$, $0 \notin S$, whenever $a, b \in S$, $ab \in S$.

Examples of multiplicative sets. (1) Let $a \in A$ such that $a^n \neq 0$ for $n \geq 0$. Then the set $S = \{a^n : n \geq 0\}$ is a multiplicative set.

(2) $S = A \backslash \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal.

(3) $S = A \backslash \cup_i \mathfrak{p}_i$, where $\mathfrak{p}_i$ are prime ideals.

(4) $S = 1 + I$, where $I \neq A$ is any ideal.

Localization of modules. Let M be an A-module. Then consider the relation $\sim$ defined on the set $M \times S$ defined by $(m, s) \sim (m', s')$ if and only if there is $u \in S$ such that $u(s'm - sm') = 0$. This is an equivalence relation. Reflexivity and symmetry are obvious. If $(m, s) \sim (m', s') \sim (m'', s'')$, then let $u, v \in S$ be such that $u(s'm - sm') = 0$, $v(s''m' - s'm'') = 0$. Then $0 = uvs''(s'm - sm') + vus(s''m' - s'm'') = uvs'(s''m - sm'')$, and $uvs' \in S$ because S is multiplicative. So $(m, s) \sim (m'', s'')$.

Write $S^{-1}M = M \times S/\sim$, and $s^{-1}m$ of m/s for the equivalence class determined by (m, s). The following operation is well defined in $S^{-1}M$:

$$m/s + m'/s' = (s'm + sm')/ss'.$$

This makes $S^{-1}M$ an abelian group.

With $M = A$, there is another well defined operation on $S^{-1}A$, namely,

$$a/s \cdot a'/s' = aa'/ss'.$$

It is easy to see that with these operations, $S^{-1}A$ becomes a commutative ring, where $1/1$ is the unity.

There is a well defined action of $S^{-1}A$ on $S^{-1}M$ by the following formula:

$$(a/s) \cdot (m/t) = am/st.$$

This way, $S^{-1}M$ becomes a module under $S^{-1}A$ in a natural way.

If $f : M \to N$ is an A-module homomorphism, then we have a natural morphism $(S^{-1}f) : S^{-1}M \to S^{-1}N$ of $S^{-1}A$-modules. Therefore, $M \mapsto S^{-1}M$ determines a functor between the abelian categories $S^{-1}(-) : A\text{-}mod \to S^{-1}A\text{-}mod$.

We have the natural homomorphism of rings $A \to S^{-1}A$. This way, $S^{-1}A$ is an A-algebra in a natural way. This is the forgetful functor $F : S^{-1}A\text{-}mod \to A\text{-}mod$. Then $S^{-1}(-)$ is a left adjoint to F.

The field of fractions. Let D be a commutative domain (i.e. one without any zero divisors). Then $S = D - 0$ is a multiplicative set and $K = S^{-1}D$ is a field, called the field of fractions of D. In case $D = \mathbb{Z}$, we have $K = \mathbb{Q}$.

The universal property. Let $S \subset A$ be any multiplicative subset, let B be any ring, and let $f : A \to B$ be any ring homomorphism. Then f factors through a unique map $f' : S^{-1}A \to B$ if an only if for each $s \in S$, $f(s)$ is a unit in B.

Description of submodules. Let $N' \subset S^{-1}M$ be a $S^{-1}A$-submodule of $S^{-1}M$. Then there is an A-submodule $N \subset M$ such that $S^{-1}N$. Indeed, set $N := \{m \in$

$M : m/1 \in N'\}$. Then consider the $S^{-1}A$-module homomorphism $f : S^{-1}N \to N' : m/1 \mapsto m/1$. This is surjective: any element in N' is of the form $m' = m/s$, so $sm' = m/1 \in N'$, so $m \in N$. This means $m' = f(m/s)$. It is obviously injective, hence f is an isomorphism.

Note that the adjunction map is an A-module homomorphism $g : M \to S^{-1}M$, and if $N' \subset S^{-1}M$ is a submodule then $N' = S^{-1}N$, where $N = f^{-1}(N')$.

Description of ideals. Ideals in a ring B are nothing but B-submodules of B. Therefore any ideal J of $S^{-1}A$ is of the form $S^{-1}I$, where $I = g^{-1}(J)$, $g : A \to S^{-1}A$ being the natural map.

Localization preserves noetherian property.

Suppose A is noetherian, and $S \subset A$ is any multiplicative subset. Then $S^{-1}A$ is also noetherian.

Proof. Let $J_1 \subset J_2 \subset \cdots$ be an increasing family of ideals in $S^{-1}A$. Then setting $I_\alpha := g^{-1}(J_\alpha)$, where $g : A \to S^{-1}A$, we see that $I_1 \subset I_2 \subset \cdots$ is an increasing family of ideals in A, hence stabilises. Let $r > 0$ be such that $I_{r+n} = I_r$ for all $n \geq 0$. Then since $J_\alpha = S^{-1}I_\alpha$, we see that $J_{r+n} = J_r$ for all $n \geq 0$. □

Localization preserves exactness.

Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence of A-modules. Then the sequence $S^{-1}M'' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is also exact.

Proof. Since $S^{-1}(-)$ is a functor, and since $g \circ f = 0$, we see that $S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$. Let $m/s \in ker(S^{-1}g)$, in other words, $S^{-1}f(m/s) = f(m)/s = 0$ in $S^{-1}M''$. So there is $u \in S$ such that $uf(m) = 0$. But this means $f(um) = 0$, so $um \in \ker g$. Therefore $um = f(m')$ for some $m' \in M'$. This means $m/s = f(m')/us = S^{-1}f(m'/us)$. □

Localization commutes with direct sum.

Let M_i be a collection of A-modules. Then the following canonical $S^{-1}A$-module homomorphism is an isomorphism.

$$f : S^{-1}(\bigoplus_i M_i) \to \bigoplus_i S^{-1}M_i : (\sum_i m_i)/s \mapsto \sum_i m_i/s,\ m_i \in M_i$$

Proof. Indeed, let $(\sum_i m_i)/s \in \ker f$, so $\sum_i m_i/s = 0$. This means $m_i/s = 0$ in $S^{-1}M_i$. There are some $u_i \in S$ such that $u_i m_i = 0$. Since only finitely many m_i are nonzero, the element $u = \prod_i u_i$ is well defined. Then $u(\sum_i m_i) = \sum_i um_i = 0$, hence $(\sum_i m_i)/s = 0$. For surjectivity, let $\sum_i m_i/s_i$ be an element in the RHS. Then choosing $s = \prod_i s_i$, and $t_i = \prod_{j \neq i} s_j$, we see that $\sum_i m_i/s_i$ is the image under f of $(\sum_i t_i m_i)/s$. □

Prime ideals of $S^{-1}A$.

Lemma 0.1. *The only prime ideals of $S^{-1}A$ are $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Thus, prime ideals of $S^{-1}A$ are in bijective correspondence with the prime ideals of A that do not intersect S.*

Proof. For any prime ideal $\mathfrak{p}$ of A that does not intersect S, the ideal $S^{-1}\mathfrak{p}$ is a prime in $S^{-1}A$. Indeed, $a/s \cdot b/t \in S^{-1}\mathfrak{p}$ implies that $ab/st = c/u$ for some $c \in \mathfrak{p}$ and $u \in S$. This implies $v(abu - cst) = 0$ for some $v \in S$, so $abuv = cstv \in \mathfrak{p}$, which gives $ab \in \mathfrak{p}$, as $\mathfrak{p} \cap S = \emptyset$. Therefore either $a \in \mathfrak{p}$, so that $a/s \in S^{-1}\mathfrak{p}$, or $b \in \mathfrak{p}$, so that $b/t \in S^{-1}\mathfrak{p}$.

Let $\mathfrak{q}$ be a prime ideal of $S^{-1}A$. We know that if $g : A \to S^{-1}A$ is the canonical map and $\mathfrak{a} = g^{-1}(\mathfrak{q})$, then $\mathfrak{q} = S^{-1}\mathfrak{a}$. Now, the inverse image of a prime ideal is always a prime ideal, so $\mathfrak{a}$ is a prime ideal. □

Localization at a prime ideal.

Set $S = A - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal and write $A_\mathfrak{p} = S^{-1}A$. We say that $A_\mathfrak{p}$ is obtained by localizing A at the prime ideal $\mathfrak{p}$. Similarly, if M is an A-module, then $S^{-1}M$ is written as $M_\mathfrak{p}$. It is an $A_\mathfrak{p}$-module.

A ring R is called a *local ring* if it has a *unique* maximal ideal.

If R is a local ring with $\mathfrak{m}$ as its unique maximal ideal, then $A - \mathfrak{m}$ is the set of all invertible elements. In other words, $\mathfrak{m}$ consists of those elements which are not invertible.

If $\mathfrak{p}$ is a prime ideal of A, then $A_\mathfrak{p}$ is a local ring, whose maximal ideal is $S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$.

Local-global principles.

Lemma 0.2. *Suppose M is an A-module. Then the following are equivalent.*

(1) $M = 0$, (2) $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$, (3) $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$.

(1)⇒(2)⇒(3) is clear. We want (3)⇒(1). Suppose (3) holds, and suppose $x \in M$ is any element. Then for each maximal ideal $\mathfrak{m}$, there is $u_\mathfrak{m} \notin \mathfrak{m}$ such that $u_\mathfrak{m} x = 0$. Now, take the ideal $\mathfrak{a}$ generated by all $u_\mathfrak{m}$. If $\mathfrak{a}$ is not unital, then there is some maximal ideal $\mathfrak{q}$ such that $\mathfrak{a} \subset \mathfrak{q}$. But, $u_\mathfrak{q} \notin \mathfrak{a}$, contradiction. So there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ and elements $\beta_1, \ldots, \beta_r \in A$ such that if $\alpha_i = u_{\mathfrak{m}_i}$, then $\sum_i \alpha_i \beta_i = 1$. Therefore $x = \sum_i \alpha_i \beta_i x = 0$. □

Lemma 0.3. *Let $f : M \to N$ be an A-module homomorphism. Then the following are equivalent.*

(1) f is injective (or surjective, or bijective, or zero), (2) $f_\mathfrak{p}$ is injective (resp., surj., bij., zero) for all prime ideals $\mathfrak{p}$, (3) $f_\mathfrak{m}$ is injective (resp., surj., bij., zero) for all maximal ideals $\mathfrak{m}$.

The map f gives rise to the following sequence of maps.

$$\ker(f) \hookrightarrow M \twoheadrightarrow \operatorname{im}(f) \hookrightarrow N \twoheadrightarrow \operatorname{coker}(f).$$

The map is injective iff $\ker(f) = 0$, surjective iff $\operatorname{coker}(f) = 0$, and zero iff $\operatorname{im}(f) = 0$. Then apply the above lemma. □

Locally free modules.

Lemma 0.4. *Let M be a finitely generated A-module, let $\mathfrak{p}$ be a prime ideal of A. Suppose $M_{\mathfrak{p}} = 0$. Then there is some $s \notin \mathfrak{p}$ such that $M_s = 0$.*

Let $m_1, \ldots, m_n$ be a generating set of M. Then since $M_{\mathfrak{p}} = 0$, we have $s_i \notin \mathfrak{p}$ such that $s_i m_i = 0$. Setting $s = s_1 \ldots s_n$, we have $sm_i = 0$ for each i, hence $M_s = 0$. □

Lemma 0.5. (**Nakayama lemma**) *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $M/\mathfrak{m}M = 0$. Then $M = 0$.*

Let $x_1, \ldots, x_r$ be a set of generaters for M. The hypothesis implies that $M = \mathfrak{m}M$. Therefore there are elements $\alpha_{ij} \in A$ such that

$$x_i = \sum_j \alpha_{ij} x_j.$$

If $A_{ij} = \delta_{ij} - \alpha_{ij}$, then $(A_{ij})\ {}^t(x_1, \ldots, x_r) = 0$. One knows that $Adj(A_{ij}) \cdot (A_{ij}) = det(A_{ij}) I_r$. Hence, multiplying by $Adj(A_{ij})$, we see that $det(A_{ij})\ {}^t(x_1, \ldots, x_r) = 0$, which means $det(A_{ij})M = 0$. Now, Since $A \to A/\mathfrak{m}$ is a ring homomorphism, $\overline{det(A_{ij})} = det(\overline{A}_{ij}) = 1$, hence $det(A_{ij}) = 1 + \alpha$, where $\alpha \in \mathfrak{m}$, which must be invertible. Therefore an invertible element annihilates M. □

Lemma 0.6. *Let M be a finitely generated A-module. Suppose for a prime ideal $\mathfrak{p}$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module. Then there is $s \notin \mathfrak{p}$ such that M_s is A_s-free.*

Let $\phi : A_{\mathfrak{p}}^{\oplus n} \to M_{\mathfrak{p}}$ be an isomorphism of $A_{\mathfrak{p}}$-modules. Let $m_i/s_i = \phi(e_i)$, where $m_i \in M$, $s_i \notin \mathfrak{p}$. Then setting $t = s_1 \cdots s_n$, we see that m_i/t also form a basis for $M_{\mathfrak{p}}$. Now, $\phi' : A_t^{\oplus n} \to M_t : e_i \mapsto m_i/t$ is well defined, and also passes to an isomorphism when localized at $\mathfrak{p}$. Note that if $V = \ker \phi'$ and $W = \mathrm{coker}\phi'$, then $V_{\mathfrak{p}} = 0$, $W_{\mathfrak{p}} = 0$. By the above, there is some $u \notin \mathfrak{p}$ such that $V_u = 0$, $W_u = 0$. Then $s = tu$ works. □

Lemma 0.7. *Suppose M is a finitely generated module. Then the following conditions are equivalent.*

(1) For each prime ideal $\mathfrak{p}$, there is an element $s \notin \mathfrak{p}$ such that M_s is A_s-free. (2) For each prime ideal $\mathfrak{p}$, $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free. (3) There are elements $x_1, \ldots, x_r$ such that the ideal $(x_1, \ldots, x_r)$ is unital, and each M_{x_i} is A_{x_i}-free.

(2)⇒(1) is the lemma above.

(3)⇒(1) Suppose $\mathfrak{p}$ is a prime ideal. Since $(x_1, \ldots, x_r)$ is unital, for some i, $x_i \notin \mathfrak{p}$. Choose $s = x_i$.

(1)⇒(3) For each prime ideal $\mathfrak{p}$, choose an $s(\mathfrak{p})$ such that $M_{s(\mathfrak{p})}$ is $A_{s(\mathfrak{p})}$-free. Take the ideal $\mathfrak{a}$ generated by all the $s(\mathfrak{p})$. If $\mathfrak{a}$ does not contain 1, then there must be a maximal ideal containing $\mathfrak{m}$, and $s(\mathfrak{m}) \notin \mathfrak{m}$, while $s(\mathfrak{m}) \in \mathfrak{a} \subset \mathfrak{m}$, a contradiction. Hence $\mathfrak{a} = A$. This gives a finite number of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ such that if $x_i = s(\mathfrak{p}_i)$, then are $y_i \in A$ with $\sum x_i y_i = 1$. □

Any finitely generated module M that satisfies the above equivalent conditions is called *locally free*.