1. Let $A$ be a commutative ring with unity in which every ideal is a principal ideal. Is $A$ necessarily a domain?

2. In the structure theorem of modules over a pid, we saw that the condition of the ring being a pid is necessary (remember $(x, y)$ in $\mathbb{C}[x, y]$). Is the condition of the module being finitely generated necessary as well?

3. As an application of the structure theorem, show that if $A$ is a domain all whose local rings $A_{\mathfrak{p}}$ are principal ideal domains, then any finitely generated torsion free module over $A$ is locally free.

4. Let $A$ be a commutative noetherian domain in which every prime ideal is principal. Prove that $A$ is a pid.

   Hint. Choose a maximal non principal ideal $\mathfrak{m}$. Prove that it is prime in the following way. Choose $a \notin \mathfrak{m}$, $b \notin \mathfrak{m}$, $ab \in \mathfrak{m}$, $(\mathfrak{m}, a) = (a)$. (This can be chosen: start with any $a_1, b_1$ outside $\mathfrak{m}$, such that $a_1 b_1 \in \mathfrak{m}$. The ideal $(\mathfrak{m}, a_1) = (a_2)$. If the condition is not satisfied, then $a_1 = a_2 a_3$ with $a_3$ non-unit. Then $b_1 a_2 a_3 \in \mathfrak{m}$. If $b_1 a_3 \notin \mathfrak{m}$, then $a_2, b_1$ is the pair needed. Otherwise replace $a_1$ by $a_3$ and repeat the process. By noetherianness, the process ends.) Then the ideal $\{x \in A : xa \in \mathfrak{m}\}$ properly contains $\mathfrak{m}$, hence is generated by a single element $c$, say. Then $\mathfrak{m} = (ac)$.

5. Prove that a square matrix over an algebraically closed field is diagonalizable if and only if its minimal polynomial is separable.

6. Let $k$ be a field, and consider the $n \times n$ matrix all whose entries are 1. What is its Jordan canonical form?

7. Can there be a $3 \times 3$ matrix $A$ over $\mathbb{Q}$ such that $A^8 = I$ but $A^4 \neq I$?

8. Using Jordan canonical form, prove the following. Let $k$ be algebraically closed and $A$ any $n \times n$ matrix over $k$. Then there are a semisimple matrix $S$ and a nilpotent matrix $N$ such that (i) $A = S+N$, (ii) both $S$ and $N$ are polynomials in $A$ (in particular they commute with $A$). Such a decomposition is unique.

9. Suppose $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 + I = 0$. Prove that $n$ is even and $A$ is similar over $\mathbb{R}$ to the following matrix.

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$