HILBERT-KUNZ MULTIPLICITY AND RELATED TOPICS

KEI-ICHI WATANABE

1. General Property

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d, which contains a perfect field $k \cong A/\mathfrak{m}$ of characteristic $p > 0$ and let I be an $\mathfrak{m}$-primary ideal of A.

We denote $I^{[q]} = (a^q \mid a \in I)$, where $q = p^e$ is a power of p.

We define $e_{HK}(I) = \lim_{q=p^e \to \infty} \dfrac{\ell_A(A/I^{[q]})}{q^d}$ to be the Hilbert-Kunz multiplicity of I and also we define

$$e_{HK}(A) = e_{HK}(\mathfrak{m}).$$

We will denote usual (Hilbert-Samuel) multiplicity of I by $e(I)$.

We list some general properties of $e_{HK}(I)$ (cf. [Hu], [Mo1]).

(1) $\frac{1}{d!}e(I) \le e_{\mathrm{HK}}(I) \le e(I)$.
(2) For $I \subset I'$, $e_{\mathrm{HK}}(I') \le e_{\mathrm{HK}}(I)$ and equality holds iff $I' \subset I^*$ (tight closure of I).
(3) In (1), we have $\frac{1}{d!}e(I) < e_{\mathrm{HK}}(I)$ but there are examples that both sides are arbitarary near.

To calculate concrete examples, the following Proposition is very useful and in general, to determine the value of $e_{HK}(I)$ is very difficult.

Proposition 1.1. *Let $(A, \mathfrak{m}) \subset (B, \mathfrak{n})$ be an extension of local domains where B is a finite A-module of rank r and $A/\mathfrak{m} = B/\mathfrak{n}$. Then for every $\mathfrak{m}$-primary ideal I, $e_{HK}(I, A) = \dfrac{1}{r} e_{HK}(IB, B)$.*

In particular, if B is regular, then $e_{HK}(I) = \dfrac{1}{r} l_B(B/IB)$.

2. Criterion of Regularity

Let A be an unmixed local ring. Then Nagata showed that A is regular off $e(A) = 1$. The same holds for HK-multiplicity.

Theorem 2.1. *([WY1], [HuYa]) $e_{HK}(A) \ge 1$ always and $e_{HK}(A) = 1$ iff A is regular.*

3. Case of dimension 2, 3.

In dimension 2, 3, we enjoy many nice properties of $e_{HK}(I)$.

Proposition 3.1. *(([WY2]) Assume* $\dim A = 2$ *and* A *is not regular.*

1. *If* $e_{HK} < 2$*, then* $e_{HK}(A) = 2 - 1/n$ *for some* $n \in \mathbb{Z}, n > 0$*.*
2. *If* $e_{HK}(A) = 2$*, then either* $e(A) = 2$ *and* A *is not F-rational or* $\hat{A} \cong (k[[X,Y]])^{[3]}$ *(3rd Veronese subring) if* $A/\mathfrak{m}$ *is algebraically closed.*
3. *For any* $\mathfrak{m}$*-primary ideal* I*, we have* $e_{HK}(I) \geq (e(I)+1)/2$*.*

Theorem 3.2. *([WY4]) Assume* $\dim A = 3$ *and* A *is not regular.*

1. $e_{HK}(A) \geq 4/3$ *and equality holds iff* $G_{\mathfrak{m}}(A) \cong k[X,Y,Z,W]/(XW - YZ)$ *(if* k *is algebraically closed).*
2. *If* $e_{HK}(A) < 2$*, then* A *is F-rational.*

4. Non-regular rings with minimal HK multiplicity

Fix d and consider the set $e_{HK}(A)$ of all non-regular local rings of dimension d. If $d = 2, 3$ the minimal values are $3/2$ if $d = 2$ and $4/3$ if $d = 3$.

Conjecture 4.1. Fix d and a prime number p and assume for simplicity, $p > 2$. Put

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2).$$

Then for all non-regular local ring A of dimension d, $e_{HK}(A) \geq e_{HK}(A_{p,d})$.

This conjecture was solved in [WY2],[WY5] if $d \leq 4$ and in [AbE*] if $d \leq 6$.

Monsky has a surprising result about the limit of $e_{HK}(A_{p,d})$.

Theorem 4.2. *([GM*]) Let* $\sum_{d\geq 0}(c_d/d!)x^d$ *be the Maclaurin expansion of* $\tan x + \sec x$*. Then*

$$\lim_{p\to\infty} e_{\mathrm{HK}}(A_{p,d}) = 1 + \frac{c_d}{d!}.$$

5. Minimal Relative HK multiplicity and Relations with F-signature

We are interested to estimate the difference $e_{HK}(I) - e_{HK}(I')$, where $I \subset I'$ with $\ell_A(I'/I) = 1$.

Definition 5.1. (Ket E be the injective envelope of $A/\mathfrak{m}$ and z be a generator of the socle $[0:\mathfrak{m}]_E$ of E. We define

$$m_{hHK}(A) := \liminf_{e\to\infty} \frac{l_A(A/\mathrm{Ann}_A(z^{p^e}))}{p^{ed}},$$

and call it the Minimal relative Hilbert-Kunz multiplicity of A.

Proposition 5.2. *If A is Gorenstein, we have the following facts;*

(1) $m_{HK}(A) > 0$ *if and only if A is F-regular*
(2) *For every full parameter ideal Q of A, if we put $I = [Q:\mathfrak{m}]$, then $m_{HK}(A) = e_{HK}(Q) - e_{HK}(I)$.*
(3) $m_{HK}(A)$ *coincides with F-signature defined by Huneke and Leuschke (*[HuLe]*).*
(4) *If A is an invariant subring of a finite group G whose order is relatively prime to p and acting on a regular local ring without non-trivial pseudo-reflaection, then $m_{HK}(A) = 1/|G|$.*

6. Generalized HK multiplicity

The concept of generalized HK multiplicity was first defined in [EY] and the studied in [DS],[DW].

Definition 6.1. Let $(A,\mathfrak{m})$ as above. For a finitely generated A module M, we put $F_A^n(M) = M\otimes_A F^n(A)$ and put $f_{gHK}^M(n) := \ell_A(H^0_{\mathfrak{m}}(F_A^n(M))$. Then we define

$$e_{gHK}(M) = \lim_{n\to\infty} \frac{f_{HK}^M(n)}{p^{nd}}$$

Some interesting properties are;

Proposition 6.2. (1) *Let A be a normal local ring of dimension 2 and let I be a reflexive ideal in A. Then I is principal if and only if $e_{gHK}(A/I) = 0$.*
(2) *If A is F-regular, then $e_{gHK}(M) = 0$ if and only if depth$(F^n(M)) > 0$ for every n.*
(3) *Assume A is F-regular with $\dim A \geq 2$ and I is a reflexive ideal locally free at $\mathrm{Spec}(A)\setminus\{\mathfrak{m}\}$. Then $e_{gHK}(A/I) = 0$ if and only if I is principal.*

I put references concerning HK multiplicity. But the references with * are beyond my scope of this lecture.

References

[AbE*] I. M. Aberbach and F. Enescu, New estimates of Hilbert-Kunz multiplicities for local rings of fixed dimension, Nagoya Math. J. **212** (2013), 59–85.

[AbL*] I. M. Aberbach and G. Leuschke, The F-signature and strong F-regularity, Math. Res. Lett. **10** (2003), 51–56.

[Br*] H. Brenner, The rationality of Hilbert-Kunz multiplicity in graded dimension two, Math. Ann. **334** (2006), 91–110.

[Br2*] H. Brenner, Irrational Hilbert-Kunz multiplicities, arXiv:1305.5873.

[BM*] H. Brenner and P. Monsky, Tight closure does not commute with localization, Ann. of Math. (2) **171** (2010), no. 1, 571–588.

[EnSh*] F. Enescu, K. Shimomoto, On the upper semi-continuity of the Hilbert-Kunz multiplicity, J. Algebra **285** (2005), 222–237.

[DS] H. Dao, I. Smirnov, *On generalized HIlbert-Kunz function and multiplicity*, arxiv:1305.1833.

[DW] H. Dao and K. Watanabe, Some computations of generalized Hilbert-Kunz function and multiplicity, Proc. AMS, **144** (2016), 3199-3206.

[EY] N. Epstein, Y. Yao, *Some extensions of Hilbert-Kunz multiplicity*, Collect. Math. (2016). doi:10.1007/s13348-016-0174-2; arXiv:1103.4730,.

[GM*] I. Gessel and P. Monsky, The limit as $p \to \infty$ of the Hilbert-Kunz multiplicity of $\sum x_i^{d_i}$, arXiv:1007.2004.

[Han] D. Hanes, Notes on Hilbert-Kunz function, J. Algebra **265** (2003), 619–630.

[HaMo*] C. Han and P. Monsky, Some surprising Hilbert-Kunz functions, Math. Z. **214** (1993), 119–135.

[Hu] C. Huneke, Tight closure, parameter ideals, and geometry, Six lectures on commutative algebra (Bellaterra, 1996), 187–239, Progr. Math., 166, Birkhäuser, Basel, 1998.

[HuLe] C. Huneke and G. Leuschke, Two theorems about maximal Cohen-Macaulay modules, Math. Ann. **324** (2002), 391–404.

[HMM] C. Huneke, M. A. McDermott and P. Monsky, Hilbert-Kunz functions for normal rings, Math. Res. Lett. **11** (2004), 539–546.

[HuYa] C. Huneke and Y. Yao, Unmixed local rings with minimal Hilbert-Kunz multiplicity are regular, Proc. Amer. Math. Soc. **130** (2002), 661–665.

[Ku] E. Kunz, Characterizations of regular local rings of characteristic p, Amer. J. Math. **91** (1969), 772–784.

[Ku2] E. Kunz, On Noetherian rings of characteristic p, Amer. J. Math. **98** (1976), 999–1013.

[Kur*] K. Kurano, The singular Riemann-Roch theorem and Hilbert-Kunz functions, J. Algebra **304** (2006), 487–499.

[Le] C. Lech, Inequalities related to certain complexes of local rings, Acta Math. **112** (1964), 69-89.

[Mo1] P. Monsky, The Hilbert-Kunz function, Math. Ann. **263** (1983), 43–49.

[Mo2*] P. Monsky, Rationality of Hilbert-Kunz multiplicities: A likely counterexample, Michigan Math. J. **57** (2008), 605–613.

[Se] G. Seibert, The Hilbert-Kunz function of rings of finite Cohen-Macaulay type, Arch. Math. **69** (1997), 286–296.

[Tr] V. Trivedi, Stability and Hilbert-Kunz multiplicities for curves, J. Algebra **284** (2005), 627–644.

[Tuc] K. Tucker, F-signature exists, Invent. math **190** (2012), 743–765.

[Wa] K.-i. Watanabe, Hilbert-Kunz multiplicity of toric rings, Proc. INst. Natural Sci. Nihon Univ., **35** (2000), 173–177.

[WY1] K.-i. Watanabe and K.-i. Yoshida, Hilbert-Kunz multiplicity and an inequality between multiplicity and colength, J. Algebra **230** (2000), 295–317.

[WY2] K.-i. Watanabe and K.-i. Yoshida, Hilbert-Kunz multiplicity of two-dimensional local rings, Nagoya Math. J. **162** (2001), 87–110.

[WY3] K.-i. Watanabe and K.-i. Yoshida, Hilbert-Kunz multiplicity, McKay correspondence and Good ideals in 2-dimensional Rational Singularities, Manuscripta Math., 104 (2001), 275-294

[WY4] K.-i. Watanabe and K.-i. Yoshida, Minimal relative Hilbert-Kunz multiplicity, Illinois J. Math. **48** (2004) 273–294.

[WY5] K.-i. Watanabe and K.-i. Yoshida, Hilbert-Kunz multiplicity of three dimensional local rings, Nagoya Math. J. **177** (2005), 47–75.

Department of Mathematics, College of Humanities and Sciences, Nihon University, Setagaya-Ku, Tokyo 156-0045, Japan

E-mail address: `watanabe@math.chs.nihon-u.ac.jp`