Pick–Nevanlinna interpolation problem: complex-analytic methods in special domains

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Abstract: The Pick–Nevanlinna interpolation problem in its fullest generality is as follows: Given domains D_1, D_2 in complex Euclidean spaces, and a set $\{(z_i, w_i) : 1 \leq i \leq N \subsetneq D_1 \times D_2\}$, where z_i are distinct and N is a positive integer ≥ 2, find necessary and sufficient conditions for the existence of a holomorphic map F from D_1 into D_2 such that $F(z_i) = w_i$, $1 \leq i \leq N$. When such a map F exists, we say that F is an interpolant of the data. Of course, this problem is intractable at the above level of generality. However, two special cases of the problem have been of lasting interest:
Interpolation from the polydisc to the unit disc: This is the case $D_1 = \mathbb{D}^n$ and $D_2 = \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc in the complex plane and $n \in \mathbb{Z}_+$. The problem itself originates with Georg Pick's well-known theorem (independently discovered by Nevanlinna) for the case $n = 1$. Much later, Sarason gave another proof of Pick's result using an operator-theoretic approach, which is very influential. Using this approach for $n \geq 2$, Agler–McCarthy provided a solution to the problem with the restriction that the interpolant is in the Schur–Agler class. This is notable because when $n = 2$ the latter result completely solves the problem for the case $D_1 = \mathbb{D}^2, D_2 = \mathbb{D}$. However, Pick's approach can also be effective for $n \geq 2$. In this talk, I'll present a result on the existence of a 3-point interpolant based on Pick's approach and involving the study of rational inner functions.

Cole, Lewis and Wermer lifted Sarason's approach to uniform algebras leading to a characterization for the existence of an interpolant in terms of the positivity of a large, rather abstractly-defined family of $(N \times N)$-matrices. McCullough later refined their result by identifying a smaller family of matrices. The second result that I present in this talk is in the same vein, namely: it provides a characterization of those data that admit a $\mathbb{D}^n$-to-$\mathbb{D}$ interpolant in terms of the positivity of a family of matrices parametrized by a class of polynomials.
Interpolation from the unit disc to the spectral unit ball: This is the case $D_1 = \mathbb{D}$ and D_2 is the set of all $(n \times n)$-matrices with spectral radius less than 1. Agler–Young established a relation between the interpolation problem in the spectral unit ball and that in the symmetrized polydisc leading to a necessary condition for the existence of an interpolant. Bharali later provided a new inequivalent necessary condition for the existence of an interpolant for any n and N=2. There is a very natural connection between the interpolation problem in this case and a Schwarz lemma for *holomorphic correspondences*. In this talk, I will also present a Schwarz lemma for holomorphic correspondences leading to a necessary condition for the existence of an interpolant.