Ideal theory of 2-dimensional normal local rings using resolution of singularities and a new characterization of rational singularities via core of ideals

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This is a joint work with Tomohiro Okuma (Yamagat Univ.) and Ken-ichi Yoshida (Nihon Univ.).

In this talk, let $(A, \mathfrak{m})$ be a local ring of a 2-dimensional algebraic variety over an algebraically closed field k (of any characteristic) and let $f : X \to \mathrm{Spec}(A)$ be a resolution of singulaities with $f^{-1}(\mathfrak{m}) = \cup_{i=1}^{r} E_i$. Take an integrally closed $\mathfrak{m}$ primary ideal I of A, then we can find f so that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ will be invertible, where $Z = \sum_{i=1}^{r} n_i E_i$. Thus we can translate ideal theory of A into theory of cycles on X. We denote $I = I_Z$ in this case.

Now, $p_g(A) := \ell_A(H^1(X, \mathcal{O}_X))$ is an invariant of A, called the "geometric genus" of A. If we put $q(I_Z) = \ell_A(H^1(X, \mathcal{O}_X(-Z)))$, then $q(I_Z)$ is an invariant of ideal I_Z. We can show that $0 \le q(I_Z) \le p_g(A)$ and in particular, we call $I = I_Z$ a p_g-ideal if $q(I_Z) = p_g(A)$. We will show that p_g-ideals have very nice properties and behaves like integrally closed ideals of rational singularities (note that, by our definition, A is a rational singularity if and only if every integrally closed ideal is a p_g-ideal).

The core of an ideal I is defined by the intersection of all the reductions of I (an ideal $J \subset I$ is a reduction of I if $I^{r+1} = I^r J$ for some $r \ge 1$) and investigated by meny authors (D. Rees, J. Lipman, C. Huneke, I. Swanson, N.V. Trung, E. Hyry, K. Smith, A. Corso, C. Polini, B. Ulrich, ...).

We will show that we have an explicit description of core(I) using the resolution of I

We ask the following question

Question 1. Let $(A, \mathfrak{m})$ be a normal 2-dimensional local ring and $I' \subset I$ are integrally closed ideals. Then is it always true that $\mathrm{core}(I') \subset \mathrm{core}(I)$

We answer this question by the following Theorem.

Theorem 1. Let $(A, \mathfrak{m})$ be a normal 2-dimensional local ring. Then the following conditions are equivalent.

(1) For any integrally closed $\mathfrak{m}$ primary ideals $I' \subset I$, we have $\mathrm{core}(I') \subset \mathrm{core}(I)$.
(2) A is a rational singularity.

More precisely, we will show the following Theorem. In fact we will give

Theorem 2. Let $(A, \mathfrak{m})$ be a normal 2-dimensional local ring.

(1) If $I' \subset I$ are p_g-ideals, then $\mathrm{core}(I') \subset \mathrm{core}(I)$.

(2) If I is an integrally closed ideal with $q(I) < p_g(A)$, assume that I^2 is integrally closed and $QI \ne I^2$ for some minimal reduction Q of I. Then there is a p_g-ideal $I' \subset I$ such that $\mathrm{core}(I') \not\subset \mathrm{core}(I)$.