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Topics in Physical Mathematics: Geometric Topology and Field Theory

by

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Dedicated to the memory of my mother,
Indumati (1920 - 2005)
who passed on to me her love of learning.

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Memories

Your voice is silent now.
But the sound of your soft
Music will always resonate
In my heart.

The wisp of morning incense
Floats in the air no more.
It now resides only in my
Childhood memories.

Abstract

In recent years the interaction between geometic topology and classical and quantum field theories has attracted a great deal of attention from both the mathematicians and physicists. We discuss some topics from low dimensional topology where this has led to new viewpoints as well as new results. They include categorification of knot polynomials and a special case of the gauge theory to string theory correspondence in the Euclidean version of the theories, where exact results are available. We show how the Witten-Reshetikhin-Turaev invariant in $SU(n)$ Chern-Simons theory on S^3 is related via conifold transition to the all-genus generating function of the topological string amplitudes on a Calabi-Yau manifold. This result can be thought of as an interpretation of TQFT as TQG (Topological Quantum Gravity). A brief discussion of Perelman's work on the geometrization conjecture and its relation to gravity is also included.

1 Introduction

This paper is based in part on my special lectures in the 2006 summer semester of the International Mathematical Physics Research School as well as several other recent seminars given at the Max Planck Institute for Mathematics in the Sciences, and at other institutes, notably at the Universitá di Firenze, University of Florida at Gainsville, Inter University Center for Astronomy and Astrophysics, University of Pune, India and conferences given at the XXIV workshop on Geometric Methods in Physics ,Poland [58] and the Blaubeuren 2005 workshop "Mathematical and Physical Aspects of Quantum Gravity" [59]. In my lectures on the mathematical and physical aspects of gauge theories in New York and Florence in the early 1980s, I began using the phrase gauge theoretic topology and geometry to describe a rapidaly developing area of mathematics, where unexpected advances were made with essential use of gauge theory. By the late 1990s it was evident that in addition to gauge theory, many other parts of theoretical physics were contributing new ideas and methods to the study of topology, geometry, algebra and other fields of mathematics. I then began using the phrase "Physical Mathematics" to collectively denote the areas of mathematics benefitting from an infusion of ideas from physics. It appears in print for the first time in [56] and more recently, in [57] and is the theme of my forthcoming book "Topics in Physical Mathematics".

During the past two decades a surprising number of new structures have appeared in the geometric topology of low-dimensional manifolds. Chiral, Vertex, Affine and other infinite dimensional algebras are related to 2d CFT and string theory as well as to sporadic finite groups such as the monster. In three dimensions there are the polynomial link invariants of Jones, Kaufman. HOMFLY and others, Witten-Reshetikhin-Turaev invariants of 3-manifolds, Casson invarinats of homology spheres and Fukaya-Floer instanton homologies. In 4 dimensions we have the instanton invariants of Donaldson and the monopole invariants of Seiberg-Witten and the list continues to grow. These invariants may be roughly split into two groups. Those in the first group arise from combinatorial (algebraic or topological) considerations and can be computed algorithmically. Those in the second group arise from the study of moduli spaces of solutions of partial differential equations which have their origin in physical field theories. Here the computations generally depend on special conditions or extra structures. The main aim of these lectures is to study some of the relations that have been found between the invariants from the two groups and more generally, to understand the influence of ideas from field theories in geometric topology and vice versa. For example, many physicists consider supersymmetric string theory to be the most promising candidate to lead to the so called grand unification of all four fundamental forces. This goal seems distant at this time, since even the physical foundations for such unification are not yet clear. However, in mathematics it has led to new areas such as mirror symmetry, Calabi-yau spaces, Gromov-Witten theory, and Gopakumar-Vafa invariants. The earliest and the best understood example of the relationship between invariants from the two groups is illustrated by the Casson invariant which was defined by using combinatorial topological methods. Taubes found a gauge theoretic interpretation of the Casson invariant as the Euler characteristic by using the generalized Poincaré-Hopf index which can also be obtained by using Floer's instanton homology. Yet there is no algorithm for computing the homology groups themselves.

As a student I was fascinated by Galileo's life and work. Then for nearly three decades while working at Florence university, I was previledged to stay at "Villa Arrighetti" an Italian national monument which once belonged to Galileo's friend and desciple Senator Arrighetti, and which is just a stone's throw from the house of Galileo. He was condemned by the Vatican to spend the last years of his life at his home. Now after nearly 400 years his conviction has been overturned by the Vatican and his home will become a museum dedicated to one of the greatest scientists and philosophers of all time. In this and other works, I have tried to follow Galileo's most important advice: "To read the book of Universe which is wide open in front of our eyes, you must know the language in which it is written. This is the language of Geometry." I will discuss the geometric setting which has had great success in the study of fundamental particles and forces. I will also follow Gallileo's preference and discuss "a small truth" which shows a beautiful and quite unexpected relationship between topological quantum field theory and string theory amplitudes , both calculated in the Euclidean version of the theories. The result is thus primarily of interest in geometric toplogy. Quantum group computations initiated by Reshetikhin and Turaev and Kohno's special functions corresponding to representations of mapping class groups in the space of conformal blocks also lead to the same result.

Topological quantum field theory was ushered in by Witten in his 1989 paper [85] "QFT and the Jones' polynomial". WRT invariants arose as a byproduct of the quantization of Chern-Simons theory used to characterize the Jones' polynomial. At this time, it is the only known geometric characterization of the Jones' polynomial, although the Feynman integrals used by Witten do not yet have a mathematically acceptable definition. Spacetime manifolds in such theories are compact Riemannina manifolds. They are referred to as Euclidean theories in the physics literature. Their role in physically interesting theories is not clear at this time and they should be regarded as toy models.

The unification of electric and magnetic fields by Maxwell is one of the most important chapters in mathematical physics. It is the only field theory which has had great predictive success in classical physics and an extension to the quantum domain. Predictions of Quantum Electrodynamics are in agreement with experimental observations to a very high degree of accuracy. Yang-Mills theory predicted massless particles and was unused for over two decades until the mechanism of symmetry breaking led to the electro-weak theory and the standard model. String theory and certain other supersymmetric theories seem to be the most promising candidates to lead to the so called grand unification of all four fundamental forces. Unifying different string theories into a single theory (such as M-theory) would seem to be the natural first step. Here even the physical foundations are not yet clear. From a mathematical point of view we would be lucky if in a few years we know what are the right questions to ask.

Last year (2005) we celebrated a number of special years. The Gauss' year and the 100th anniversary of Einstein's "Annus Mirabilis" (the miraculous year) are the most important among these. Indeed, Gauss' "Disquisitiones generale circa superficies curvas" was the basis and inspiration for Riemann's work which ushered in a new era in geometry. It is an extension of this geometry that is the cornerstone of relativity theory. More recently, we have witnessed the marriage between Gauge Theory and the Geometry of Fiber Bundles from the sometime warring tribes of Physics and Mathematics. Marriage brokers were none other than Chern and Simons. The 1975 paper by Wu and Yang [88] can be regarded as the announcement of this union. It has led to many wonderful offspring. The theories of Donaldson, Chern-Simons, Floer-Fukaya, Seiberg-Witten, and TQFT are just some of the more famous members of their extended family. Quantum Groups, CFT, Supersymmetry (SUSY), String Theory, Gromov-Witten theory and Gravity also have close ties with this family. Later in this paper we will discuss one particular relationship between gauge theory and string theory, that has recently come to light. The qualitative aspects of Chern-Simons theory as string theory were investigated by Witten [87] almost ten years ago. Before recounting the main idea of this work we review the Feynman path integral method of quantization which is particularly suited for studying topologial quantum field theories. For general background on gauge theory and geometric topology see, for example, [55, 56].

We now give a brief description of the contents of the paper. In section 2 we discuss Gauss' Formula for Linking Number of knots, the earliest example of TFT (Topological Field Theory) and its recent extension to self linking invariants. Witten's fundamental work on supersymmetry and Morse theory is covered in section 3. Chern-Simons theory is introduced in section 4. Its relation to Casson invariant via the moduli space of flat connections is explained in section 5. Ideas from sections 3 and 4 are used in section 6 to define the Fukaya-Floer homology. This homology provides the categorification of the Casson invariant. Knot polynomials and their categorification are discussed in sections 7 and 8 respectively. Section 9 is devoted to a general discussion of TQFT and its applications to invariants of links and 3-manifolds. Atiyah-Segal axioms for TQFT are introduced in subsection 9.1. In subsection 9.2 we define quantum observables and introduce the Feynman path integral approach to QFT. The Euclidean version of this theory is applied in subsection 9.3 to the Chern-Simons Lagrangian to obtain the skein relations for the Jones-Witten polynomial of a link in S^3. A by product of this is the family of WRT invariants of 3-manifolds. They are discussed in subsection

9.4. Section 10 is devoted to studying the relation between WRT invariants of S^3 with gauge group $SU(n)$ and the open and closed string amplitudes in generalized Calabi-Yau manifolds. Change in geometry and topology via conifold transition which plays an important role in this study is introduced in subsection 10.1 in the form needed for our specific problem. Expansion of free energy and its relation to string amplitudes is given in subsection 10.2 This result is a special case of the general program introduced by Witten in [87]. A realization of this program even within Euclidean field theory promises to be a rich and rewarding area of research. We have given some indication of this at the end of this section. Links between Yang-Mills, gravity and string theory are considered in the concluding section 11. Relation of Yang-Mills equations with Einstein's equations for gravitational field in the Euclidean setting is considered in subsection 11.1. Various formulations of Einstein's equations for gravitational field are discussed in subsection 11.2. They also make a surprising appearance in Perelaman's proof of Thurston's Geometrization conjecture. A brief indication of this is given in subsection 11.3. A discussion of the topology and geometry of 4-manifolds is included in subsection 11.4. No physical application of Euclidean gravitational instantons is known at this time. So there appearance in supersymmetric string theory does not give the usual field equations of gravitation. However, topological amplitudes calculated in this theory can be thought of as Euclidean TQG. We give some further arguments in the concluding section 12 in support of supersymmetric string theory as a candidate for the unification of all four fundamental forces.

We have included some basic material and given more details than necessary to make the paper essentially self-contained. A fairly large number of references ranging from January 1833 to June 2006, when IMPRS special lectures were given, are included to facilitate further study and research in this exciting and rapidaly expanding area.

2 Gauss' Formula for Linking Number of knots

Knots have been known since ancient times but knot theory is of quite recent origin. One of the earliest investigations in combinatorial knot theory is contained in several unpublished notes written by Gauss between 1825 and

1844 and published posthumously as part of his Nachlaß(estate). They deal mostly with his attempts to classify "Tractfiguren" or plane closed curves with a finite number of transverse self-intersections. However, one fragment deals with a pair of linked knots. We reproduce a part of this fragment below.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie $r = (x, y, z)$*; der zweiten* $r' = (x', y', z')$ *und*

$$\int\int \frac{(r' - r) \cdot (dr \times dr')}{|r' - r|^3} = V$$

dann ist dies Integral durch beide Linien ausgedehnt $= 4\pi m$ *und* m *die Anzahl der Umschlingungen. Der Werth ist gegenseitig, d.i. er bleibt derselbe, wenn beide Linien gegen einander umgetauscht werden* *1833. Jan. 22.*

In this fragment of a note from his Nachlaß, Gauss had given an analytic formula for the linking number of a pair of knots. This number is a combinatorial topological invariant. As is quite common in Gauss's work, there is no indication of how he obtained this formula. The title of the section where the note appears, "Zur Electrodynamik" ("On Electrodynamics") and his continuing work with Weber on the properties of electric and magentic fields leads us to guess that it originated in the study of magnetic field generated by an electric current flowing in a curved wire.

Maxwell knew Gauss's formula for the linking number and its topological significance and its origin in electromagentic theory. In fact, before he knew of Gauss's formula, he had rediscovered it. He mentions it in a letter to Tait dated December 4, 1867. He wrote several manuscripts which study knots, links and also addressed the problem of their classification. In these and other topological problems his approach was not mathematically rigorous but was rather based on his deep understanding of physics. Indeed this situation persists today in several mathematical results obtained by physical reasoning. Like Maxwell, Tait used his physical intuition to correctcly classify all knots up to seven crossings and made a number of conjectures, the last of which remained open for over hundred years.

In obtaining a topological invariant by using a physical field theory, Gauss had anticipated Topological Field Theory by almost 150 years. Even the term topology was not used then. It was introduced in 1847 by J. B. Listing, a student and protegé of Gauss, in his essay "Vorstudien zur Topologie". Gauss's linking number formula can also be interpreted as the equality of topological and analytic degree of the function λ defined by

$$\lambda(\vec{r}, \vec{r'}) := \frac{(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|}, \ \forall (\vec{r}, \vec{r'}) \in C \times C'$$

It is well defined by the disjointness of C and C'. If ω denotes the standard volume form on S^2, then the pull back $\lambda^*(\omega)$ of ω to $C \times C'$ is precisely the integrand in the Gauss formula and $\int \omega = 4\pi$. One can check that the topological degree of λ equals the linking number m.

Recently, Bott and Taubes have used these ideas to study a self-linking invariant of knots [11]. It turns out that this invariant belongs to a family of knot invariants, called finite type invariants, defined by Vassiliev. Gauss forms with different normalization are used by Kontsevich [42] in the formula for this invariant and it is stated that the invariant is an integer equal to the second coefficient of the Alexander-Conway polynomial of the knot. In [9, 10] Bott and Cattaneo obtain invariants of rational homology 3-spheres in terms of configuration space integrals. Kontsevich views these formulas as forming a small part of a very broad program to relate the invariants of low-dimensional manifolds, homotopical algebras, and non-commutative geometry with topological field theories and the calculus of Feynman diagrams. It seems that the full realization of this program would require the best efforts of mathematicians and physicists for years to come.

3 Supersymmetry and Morse Theory

Classical Morse theory on a finite dimensional, compact, differentiable manifold M relates the behaviour of critical points of a suitable function on M with topological information about M. The relation is generally stated as an equality of certain polynomials as follows. Recall first that a smooth function $f : M \to \mathbb{R}$ is called a **Morse function** if its critical points are isolated and non-degenerate. If $x \in M$ is a critical point (i.e. $df(x) = 0$), then by Taylor expansion of f around x, we obtain the Hessian of f at x defined by

$$\{\frac{\partial^2 f}{\partial x^i \partial x^j}(x)\}.$$

Then the non-degeneracy of the critical point x is equivalent to the non-degeneracy of the quadratic form determined by the Hessian. The dimension of the negative eigenspace of this form is called the **Morse index**, or simply index, of f at x and is denoted by $\mu_f(x)$ or simply $\mu(x)$ when f is understood. It can be verified that these definitions are independent of the choice of the local coordinates. Let m_k be the number of critical points with index k. Then the **Morse series** of f is the formal power series

$$\sum_k m_k t^k \text{ , where } m_k = 0, \forall k > \dim M.$$

Recall that the Poincaré series of M is given by $\sum_k b_k t^k$, where $b_k \equiv b_k(M)$ is the k-th Betti number of M. The relation between the two series is given by

$$\sum_k m_k t^k = \sum_k b_k t^k + (1+t)\sum_k q_k t^k, \tag{1}$$

where q_k are non-negative integers. Comparing the coefficients of the powers of t in this relation leads to the well-known **Morse inequalities**

$$\begin{aligned} \sum_{k=0}^{i} m_{i-k}(-1)^k &\geq \sum_{k=0}^{i} b_{i-k}(-1)^k \ , \quad 0 \leq i \leq n-1, \\ \sum_{k=0}^{n} m_{n-k}(-1)^k &= \sum_{k=0}^{n} b_{n-k}(-1)^k \ . \end{aligned}$$

The Morse inequalities can also be obtained from the following observation. Let C^* be the graded vector space over the set of critical points of f. Then the Morse inequalities are equivalent to the existence of a certain coboundary operator $\partial : C^* \to C^*$ so that $\partial^2 = 0$ and the cohomology of the complex (C^*, ∂) coincides with the deRham cohomology of M.

In his fundamental paper [84], Witten arrives at precisely such a complex by considering a suitable supersymmetric quantum mechanical Hamiltonian. Witten showed how the standard Morse theory (see, for example, Milnor [60]) can be modified by considering the gradient flow of the Morse function f between pairs of critical points of f. One may think of this as a sort of relative Morse theory. He was motivated by the phenomenon of the quantum mechanical tunneling. We now discuss this approach. From a mathematical point of view, supersymmetry may be regarded as a theory of operators on a Z_2-graded Hilbert space. In recent years this theory has attracted a great deal of interest from theoretical point of view even though as yet there is no physical evidence for its existence.

Graded Algebraic Structures

In this section we recall briefly a few important properties of graded vector spaces and graded operators in a slightly more general situation than is immediately needed. We will use this information again in studying Khovanov homology. Graded algebraic structures appear naturally in many mathematical and physical theories. We shall restrict our considerations only to $\mathbb{Z}$- and $\mathbb{Z}_2$-gradings. The most basic such structure is that of a graded vector space which we now describe. Let V be a vector space. We say that V is **$\mathbb{Z}$-graded** (resp. **$\mathbb{Z}_2$-graded**) if V is the direct sum of vector subspaces V_i, indexed by the integers (resp. integers mod. 2), i.e.

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \quad (\text{ resp. } V = V_0 \oplus V_1).$$

The elements of V_i are said to be **homogeneous** of **degree** i. In the case of $\mathbb{Z}_2$-grading it is customary to call the elements of V_0 (resp. V_1) **even** (resp. **odd**). If V and W are two $\mathbb{Z}$-graded vector spaces, a linear transformation $f : V \to W$ is said to be **graded** of **degree** k if $f(V_i) \subset W_{i+k}, \ \forall i \in \mathbb{Z}$. If V and W are $\mathbb{Z}_2$-graded , then a linear map $f : V \to W$ is said to be **even** if $f(V_i) \subset W_i, \ i \in \mathbb{Z}_2$ and is said to be **odd** if $f(V_i) \subset W_{i+1}, \ i \in \mathbb{Z}_2$. An **algebra** A is said to be **$\mathbb{Z}$-graded** if A is $\mathbb{Z}$-graded as a vector space, i.e.

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

and $A_i A_j \subset A_{i+j}, \ \forall i, j \in \mathbb{Z}$. An ideal $I \subset A$ is called a **homogeneous ideal** if

$$I = \bigoplus_{i \in \mathbb{Z}} (I \cap A_i).$$

A similar definition can be given for a $\mathbb{Z}_2$-graded algebra. In the physical literature a **$\mathbb{Z}_2$-graded algebra** is referred to as a **superalgebra**. Other algebraic structures (such as Lie, commutative etc.) have their superalgebra counterparts. An example of a $\mathbb{Z}$-graded algebra is given by the exterior algebra of differential forms $\Lambda(M)$ of a manifold M if we define $\Lambda^i(M) = 0$ for $i < 0$. The exterior differential d is a graded linear transformation of degree 1 of $\Lambda(M)$. The graded or quantum dimension of V is defined by

$$\dim_q V = \sum_{i \in \mathbb{Z}} q^i (\dim(V_i)) \quad ,$$

where q is a formal variable. If we write $q = exp 2\pi i z, z \in \mathbb{C}$ then $\dim_q V$ can be regarded as the Fourier expansion of a complex function. A spectacular application of this occurs in the study of finite groups. We discuss this briefly in the next paragraph. It is not needed in the rest of the paper. However, it has surprising connections with conformal field theory and vertex algebras.

Monstrous Moonshine

It was his study of Kepler's sphere packing conjecture, that led John Conway to the discovery of his sporadic simple group. Soon thereafter the last holdouts in the complete list of the 26 finite sporadic simple groups were found. All the infinite families of finite simple groups (such as the groups $\mathbb{Z}_p$, for p a prime number and alternating groups $A_n, n > 4$ that we study in the first course in algebra) were already known. So the classification of finite simple groups was complete. It ranks as the greatest achievement of twentieth century mathematics. Hundreds of mathematicians contributed to it. The various parts of the classification together fill more than ten thousand pages. Conway's group and other sporadic simple groups are closely related to the symmetries of lattices. The study of representations of the largest of these groups (called the Friendly Giant or Fisher-Griess Monster) has led to the creation of a new field of mathematics called Vertex algebras. They turn out to be closely related to the chiral algebras in conformal field theory. These and other ideas inspired by string theory have led to a proof of Conway and Norton's Moonshine conjectures (see, for example, Borcherds [8], and the book [26] by Frenkel, Lepowski, Meurman). The monster Lie algebra is the simplest example of a Lie algebra of physical states of a chiral string on a 26-dimensional orbifold. This algebra can be defined by using the infinite dimensional graded representation V of the monster simple group. Its quantum dimension is related to Jacobi's $SL(2, \mathbb{Z})$ hauptmodule (elliptic modular function of genus 0) $j(q)$, where $q = e^{2\pi i z}, z \in \mathbb{H}$ by

$$\dim_q V = j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots$$

The above formula is one small part in the proof of the moonshine conjectures.

SUSY Quantum Theory

The Hilbert space E of a supersymmetric theory is Z_2-graded, i.e. $E = E_0 \oplus E_1$, where the even (resp. odd) space E_0 (resp. E_1) is called the space of bosonic (resp. fermionic) states. These spaces are distinguished by an operator $S : E \to E$ defined by

$$Su = u, \quad \forall u \in E_0,$$

$$Sv = -v, \quad \forall v \in E_1.$$

The operator S is interpreted as counting the number of fermions modulo 2. A supersymmetric theory begins with a collection $\{Q_i \mid i = 1, \ldots, n\}$ of supercharge (or supersymmetry) operators on E which are of odd degree, i.e. anti-commute with S

$$SQ_i + Q_iS = 0, \quad \forall i \tag{2}$$

and satisfy the following anti-commutation relations

$$Q_iQ_j + Q_jQ_i = 0, \quad \forall i \neq j. \tag{3}$$

The dynamics is introduced by the Hamiltonian operator H which commutes with the supercharge operators and is usually required to satisfy additional conditions. For example, in the simplest non-relativistic theory one requires that

$$H = Q_i^2, \quad \forall i. \tag{4}$$

In fact this simplest supersymmetric theory has surprising connections with Morse theory which we now discuss.

Let M be a compact differentiable manifold and define E by

$$E := \Lambda(M) \otimes \mathbb{C}.$$

The natural grading on $\Lambda(M)$ induces a grading on E. We define

$$E_0 := \bigoplus_j \Lambda^{2j}(M) \otimes \mathbb{C} \quad (\text{resp. } E_1 := \bigoplus_j \Lambda^{2j+1}(M) \otimes \mathbb{C})$$

the space of complex-valued even (resp. odd) forms on M. The exterior differential d and its formal adjoint δ have natural extension to odd operators on E and thus satisfy (2). We define supercharge operators $Q_j,\ j = 1, 2$, by

$$Q_1 = d + \delta, \tag{5}$$

$$Q_2 = i(d - \delta). \tag{6}$$

The Hamiltonian is taken to be the Hodge-deRham operator extended to E, i.e.

$$H = d\delta + \delta d. \tag{7}$$

The relations $d^2 = \delta^2 = 0$ imply the supersymmetry relations (3) and (4). We note that in this case bosonic (resp. fermionic) states correspond to even (resp. odd) forms. The relation to Morse theory arises in the following way. If f is a Morse function on M, define a one-parameter family of operators

$$d_t = e^{-ft} d e^{ft}, \quad \delta_t = e^{ft} \delta e^{-ft}, \quad t \in \mathbb{R} \tag{8}$$

and the corresponding supersymmetry operators

$$Q_{1,t} = d_t + \delta_t, \quad Q_{2,t} = i(d_t - \delta_t), \quad H_t = d_t\delta_t + \delta_t d_t.$$

It is easy to verify that $d_t^2 = \delta_t^2 = 0$ and that $Q_{1,t}$, $Q_{2,t}$, H_t satisfy the supersymmetry relations (3) and (4). The parameter t interpolates between the deRham cohomology and the Morse indices as t goes from 0 to $+\infty$. At $t = 0$, the number of linearly independent eigenvectors with zero eigenvalue is just the k-th Betti number b_k when $H_0 = H$ is restricted to act on k-forms. In fact these ground states of the Hamiltonian are just the harmonic forms. On the other hand, for large t the spectrum of H_t simplifies greatly with the eigenfunctions concentrating near the critical points of the Morse function. It is in this way that the Morse indices enter into this picture. We can write H_t as a perturbation of H near the critical points. In fact, we have

$$H_t = H + t \sum_{j,k} f_{,jk}[\alpha^j, i_{X^k}] + t^2 \|df\|^2,$$

where $\alpha^j = dx^j$ acts by exterior multiplication, $X^k = \partial/\partial x^k$ and i_{X^k} is the usual action of inner multiplication by X^k on forms and the norm $\|df\|$ is the norm on $\Lambda^1(M)$ induced by the Riemannian metric on M. In a suitable neighborhood of a fixed critical point taken as origin, we can approximate H_t up to quadratic terms in x^j by

$$\overline{H}_t = \sum_j \left(-\frac{\partial^2}{\partial x_j^2} + t^2 \lambda_j^2 x_j^2 + t\lambda_j [\alpha^j, i_{X^j}] \right),$$

where λ_j are the eigenvalues of the Hessian of f. The first two terms correspond to the Hamiltonian of a harmonic oscillator with eigenvalues

$$t \sum_j \mid \lambda_j \mid (1 + 2N_j),$$

whereas the last term defines an operator with eigenvalues $\pm\lambda_j$. It commutes with the first and thus the spectrum of $\overline{H}_t$ is given by

$$t \sum_j [\mid \lambda_j \mid (1 + 2N_j) + \lambda_j n_j],$$

where N_j's are non-negative integers and $n_j = \pm 1$. Restricting H to act on k-forms we can find the ground states by requiring all the N_j to be 0 and by choosing n_j to be 1 whenever λ_j is negative. Thus the ground states (zero eigenvalues) of H correspond to the critical points of Morse index k. All other eigenvalues are proportional to t with positive coefficients. Starting from this observation and using standard perturbation theory, one finds that the number of k-form ground states equals the number of critical points of Morse index k. Comparing this with the ground state for $t = 0$, we obtain the weak Morse inequalities $m_k \geq b_k$. As we observed in the introduction the strong Morse inequalities are equivalent to the existence of a certain cochain complex which has cohomology isomorphic to $H^*(M)$, the cohomology of the base manifold M. Witten defines C_p, the set of p-chains of this complex, to be the free group generated by the critical points of Morse index p. He then argues that the operator d_t defined in (8) defines in the limit as $t \to \infty$ a coboundary operator

$$d_\infty : C_p \to C_{p+1}$$

and that the cohomology of this complex is isomorphic to the deRham cohomology of Y.

Thus we see that in establishing both the weak and strong form of Morse inequalities a fundamental role is played by the ground states of the supersymmetric quantum mechanical system (5), (6), (7). In a classical system the transition from one ground state to another is forbidden, but in a quantum mechanical system it is possible to have tunneling paths between two ground states. In gauge theory the role of such tunneling paths is played by instantons. Indeed, Witten uses the prescient words "instanton analysis" to describe the tunneling effects obtained by considering the gradient flow of the Morse function f between two ground states (critical points). If β (resp. α)

is a critical point of f of Morse index $p+1$ (resp. p) and Γ is a gradient flow of f from β to α, then by comparing the orientation of negative eigenspaces of the Hessian of f at β and α, Witten defines the signature n_Γ of this flow. By considering the set S of all such flows from β to α, he defines

$$n(\alpha, \beta) := \sum_{\Gamma \in S} n_\Gamma.$$

Now defining $\delta_{\mathbb{Z}}$

$$\delta_{\mathbb{Z}} : C_p \to C_{p+1} \text{ by } \alpha \mapsto \sum_{\beta \in C_{p+1}} n(\alpha, \beta)\beta, \tag{9}$$

he shows that $(C_*, \delta_{\mathbb{Z}})$ is a cochain complex with integer coefficients. Witten conjectures that the integer-valued coboundary operator $\delta_{\mathbb{Z}}$ actually gives the integral cohomology of the manifold M. The complex $(C_*, \delta_{\mathbb{Z}})$, with the coboundary operator defined by (9), is referred to as the **Witten complex**. As we will see later, Floer homology is the result of such "instanton analysis" applied to the gradient flow of a suitable Morse function on the moduli space of gauge potentials on an integral homology 3-sphere. Floer has also used these ideas to study a "symplectic homology" associated to a manifold. A corollary of this theory proves the Witten conjecture for finite dimensional manifolds (see [70] for further details), namely

$$H^*(C_*, \delta_{\mathbb{Z}}) = H^*(M, \mathbb{Z}).$$

A direct proof of the conjecture may be found in the appendix to K. C. Chang [14]. A detailed study of the homological concepts of finite dimensional Morse theory in anology with Floer homology may be found in M. Schwarz [72]. While many basic concepts of "Morse homology" can be found in the classical investigations of Milnor, Smale and Thom, its presentation as an axiomatic homology theory in the sense of Eilenberg and Steeenrod [20] is given for the first time in [72]. One consequence of this axiomatic approach is the uniqueness result for "Morse homology" and its natural equivalence with other axiomatic homology theories defined on a suitable category of topological spaces. Witten conjecture is then a corollary of this result. A discussionof the relation of equivariant cohomolgy and supersymmetry may be found in Guillemin and Sternberg's book [27].

4 Chern-Simons Theory

Let M be a compact manifold of dimension $m = 2r + 1,\ r > 0$, and let $P(M, G)$ be a principal bundle over M with a compact, semisimple Lie group G as its structure group. Let $\alpha_m(\omega)$ denote the Chern-Simons m-form on M corresponding to the gauge potential (connection) ω on P; then the Chern-Simons action $\mathcal{A}_{CS}$ is defined by

$$\mathcal{A}_{CS} = c(G) \int_M \alpha_m(\omega), \tag{10}$$

where $c(G)$ is a coupling constant whose normalization depends on the group G. In the rest of this paragraph we restrict ourselves to the case $r = 1$ and $G = SU(n)$. The most interesting applications of the Chern-Simons theory to low dimensional topologies are related to this case. It has been extensively studied by both physicists and mathematicians in recent years. In this case the action (10) takes the form

$$\begin{aligned} \mathcal{A}_{CS} &= \frac{k}{4\pi} \int_M tr(A \wedge F - \frac{1}{3} A \wedge A \wedge A) && (11) \\ &= \frac{k}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), && (12) \end{aligned}$$

where $k \in \mathbb{R}$ is a coupling constant, A denotes the pull-back to M of the gauge potential ω by a local section of P and $F = F_\omega = d^\omega A$ is the gauge field on M corresponding to the gauge potential A. A local expression for (11) is given by

$$\mathcal{A}_{CS} = \frac{k}{4\pi} \int_M \epsilon^{\alpha\beta\gamma} tr(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma), \tag{13}$$

where $A_\alpha = A^a_\alpha T_a$ are the components of the gauge potential with respect to the local coordinates $\{x_\alpha\}$, $\{T_a\}$ is a basis of the Lie algebra $su(n)$ in the fundamental representation and $\epsilon^{\alpha\beta\gamma}$ is the totally skew-symmetric Levi-Civita symbol with $\epsilon^{123} = 1$. We take the basis $\{T_a\}$ with the normalization

$$tr(T_a T_b) = \frac{1}{2} \delta_{ab}, \tag{14}$$

where δ_{ab} is the Kronecker δ function. Let $g \in \mathcal{G}$ be a gauge transformation regarded (locally) as a function from M to $SU(n)$ and define the 1-form θ by

$$\theta = g^{-1} dg = g^{-1} \partial_\mu g dx^\mu.$$

Then the gauge transformation A^g of A by g has the local expression

$$A^g_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g. \tag{15}$$

In the physics literature, the connected component of the identity, $\mathcal{G}_{id} \subset \mathcal{G}$ is called the group of **small gauge transformations**. A gauge transformation not belonging to $\mathcal{G}_{id}$ is called a **large gauge transformation**. By a direct calculation, one can show that the Chern-Simons action is invariant under small gauge transformations, i.e.

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A),\ \forall g \in \mathcal{G}_{id}.$$

Under a large gauge transformation g the action (13) transforms as follows:

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A) + 2\pi k \mathcal{A}_{WZ}, \tag{16}$$

where

$$\mathcal{A}_{WZ} := \frac{1}{24\pi^2}\int_M \epsilon^{\alpha\beta\gamma} tr(\theta_\alpha\theta_\beta\theta_\gamma) \tag{17}$$

is the **Wess-Zumino action functional**. It can be shown that the Wess-Zumino functional is integer valued and hence, if the Chern-Simons coupling constant k is taken to be an integer, then we have

$$e^{i\mathcal{A}_{CS}(A^g)} = e^{i\mathcal{A}_{CS}(A)}.$$

The integer k is called the **level** of the corresponding Chern-Simons theory. It follows that the path integral quantization of the Chern-Simons model is gauge-invariant. This conclusion holds more generally for any compact simple group if the coupling constant $c(G)$ is chosen appropriately. The action is manifestly covariant since the integral involved in its definition is independent of the metric on M. It is in this sense that the Chern-Simons theory is a topological field theory. We will consider this aspect of the Chern-Simons theory later.

In general, the Chern-Simons action is defined on the space $\mathcal{A}_{P(M,G)}$ of all gauge potentials on the principal bundle $P(M,G)$. But when M is 3-dimensional P is trivial (in a non-canonical way). We fix a trivialization to write $P(M,G) = M \times G$ and write $\mathcal{A}_M$ for $\mathcal{A}_{P(M,G)}$. Then the group of gauge transformations $\mathcal{G}_P$ can be identified with the group of smooth functions from M to G and we denote it simply by $\mathcal{G}_M$. For $k \in \mathbb{N}$, the transformation law (16) implies that the Chern-Simons action descends to the quotient $\mathcal{B}_M = \mathcal{A}_M/\mathcal{G}_M$ as a function with values in $\mathbb{R}/\mathbb{Z}$. We denote this function by f_{CS}, i.e.

$$f_{CS} : \mathcal{B}_M \to \mathbb{R}/\mathbb{Z} \text{ is defined by } [\omega] \mapsto \mathcal{A}_{CS}(\omega),\ \forall\ [\omega] = \omega\mathcal{G}_M \in \mathcal{B}_M. \quad (18)$$

The field equations of the Chern-Simons theory are obtained by setting the first variation of the action to zero as

$$\delta\mathcal{A}_{CS} = 0.$$

We shall discuss two approaches to this calculation. Consider first a one parameter family $c(t)$ of connections on P with $c(0) = \omega$ and $\dot{c}(0) = \alpha$. Differentiating the action $\mathcal{A}_{CS}(c(t))$ with respect to t and noting that differentiation commutes with integration and the tr operator, we get

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{A}_{CS}(c(t)) &= \frac{1}{4\pi}\int_M tr\,(2\dot{c}(t)\wedge dc(t) + 2(\dot{c}(t)\wedge c(t)\wedge c(t))) \\
&= \frac{1}{2\pi}\int_M tr\,(\dot{c}(t)\wedge(dc(t) + c(t)\wedge c(t))) \\
&= \frac{1}{2\pi}\int_M <\ \dot{c}(t)\ ,\ *F_{c(t)}\ >
\end{aligned}$$

where the inner product on the right is as defined in Definition 2.1. It follows that

$$\delta\mathcal{A}_{CS} = \frac{d}{dt}\,\mathcal{A}_{CS}(c(t))_{|t=0} = \frac{1}{2\pi}\int_M <\ \alpha\ ,\ *F_\omega\ > . \quad (19)$$

Since α can be chosen arbitrarily, the field equations are given by

$$*F_\omega = 0 \text{ or equivalently } F_\omega = 0. \quad (20)$$

Alternatively, one can start with the local coordinate expression of equation (13) as follows

$$\begin{aligned}
\mathcal{A}_{CS} &= \frac{k}{4\pi}\int_M \epsilon^{\alpha\beta\gamma} tr(A_\alpha\partial_\beta A_\gamma + \frac{2}{3}A_\alpha A_\beta A_\gamma) \\
&= \frac{k}{4\pi}\int_M \epsilon^{\alpha\beta\gamma} tr(A^a_\alpha\partial_\beta A^c_\gamma T_a T_b + \frac{2}{3}A^a_\alpha A^b_\beta A^c_\gamma T_a T_b T_c)
\end{aligned}$$

and find the field equations by using the variational equation

$$\frac{\delta\mathcal{A}_{CS}}{\delta A^a_\rho} = 0. \quad (21)$$

This method brings out the role of commutation relations and the structure constants of the Lie algebra $su(n)$ as well as the boundary conditions used in the integration by parts in the course of calculating the variation of the action. The result of this calculation gives

$$\frac{\delta \mathcal{A}_{CS}}{\delta A^a_\rho} = \frac{k}{2\pi} \int_M \epsilon^{\rho\beta\gamma} \left(\partial_\beta A^a_\gamma + A^b_\beta A^c_\gamma f_{abc} \right) \tag{22}$$

where f_{abc} are the structure constants of $su(n)$ with respect to the basis T_a. The integrand on the right hand side of the equation (22) is just the local coordinate expression of $*F_A$, the dual of the curvature, and hence leads to the same field equations.

The calculations leading to the field equations (20) also show that the gradient vector field of the function f_{CS} is given by

$$\text{grad } f_{CS} = \frac{1}{2\pi} * F \tag{23}$$

The gradient flow of f_{CS} plays a fundamental role in the definition of Floer homology. The solutions of the field equations (20) are called the **Chern-Simons connections**. They are precisely the flat connections. In the next paragraph we discuss flat connections on a manifold N and their relation to the homomorphisms of the fundamental group $\pi_1(N)$ into the gauge group.

Flat connections

Let H be a compact Lie group and $Q(N, H)$ be a principal bundle with structure group H over a compact Riemannian manifold N. A connection ω on Q is said to be **flat** if its curvature is zero, i.e. $F_\omega = 0$. The pair (Q, ω) is called a **flat bundle**. Let $\Omega(N, x)$ be the loop space at $x \in N$. Recall that the horizontal lift h_u of $c \in \Omega(N, x)$ to $u \in \pi^{-1}(x)$ determines a unique element of H. Thus we have the map

$$h_u : \Omega(N, x) \to H.$$

It is easy to see that ω flat implies that this map h_u depends only on the homotopy class of the loop c and hence induces a map (also denoted by h_u)

$$h_u : \pi_1(N, x) \to H.$$

It is this map that is related to the Bohm-Aharonov effect. It can be shown that the map h_u is a homomorphism of groups. The group H acts on the set $Hom(\pi_1(N), H)$ by conjugation sending h_u to $g^{-1}h_u g = h_{ug}$. Thus a flat bundle (Q, ω) determines an element of the quotient $Hom(\pi_1(N), H)/H$. If $a \in \mathcal{G}(Q)$, the group of gauge transformations of Q, then $a \cdot \omega$ is also a flat connection on Q and determines the same element of $Hom(\pi_1(N), H)/H$. Conversely, let $f \in Hom(\pi_1(N), H)$ and let (U, q) be the universal covering of N. Then U is a principal bundle over N with structure group $\pi_1(N)$. Define $Q := U \times_f H$ to be the bundle associated to U by the action f with standard fiber H. It can be shown that Q admits a natural flat connection and that f and $g^{-1}fg,\ g \in H$, determine isomorphic flat bundles. Thus the moduli space $\mathcal{M}_f(N, H)$ of flat H-bundles over N can be identified with the set $Hom(\pi_1(N), H)/H$. The moduli space $\mathcal{M}_f(N, H)$ and the set $Hom(\pi_1(N), H)$ have a rich mathematical structure which has been extensively studied in the particular case when N is a compact Riemann surface [2].

The **flat connection deformation complex** is the generalized deRham sequence with the usual differential d replaced by the covariant differential d^ω. The fact that in this case it is a complex follows from the observation that ω flat implies $d^\omega \circ d^\omega = 0$. By rolling up this complex, we can consider the rolled up deformation operator $d^\omega + \delta^\omega : \Lambda^{ev} \to \Lambda^{odd}$. By the index theorem, we have

$$Ind(d^\omega + \delta^\omega) = \chi(N) dim H$$

and hence

$$\sum_{i=0}^{n} (-1)^i b_i = \chi(N) dim H, \tag{24}$$

where b_i is the dimension of the i-th cohomology of the deformation complex. Both sides are identically zero for odd n. For even n, the formula can be used to obtain some information on the virtual dimension of $\mathcal{M}_f$ $(= b_1)$. For example, if $N = \Sigma_g$ is a Riemann surface of genus $g > 1$, then $\chi(\Sigma_g) = -2g + 2$, while, by Hodge duality, $b_0 = b_2 = 0$ at an irreducible connection. Thus, equation (24) gives

$$-b_1 = -(2g - 2) dim H.$$

From this it follows that

$$dim \mathcal{M}_f(\Sigma_g, H) = dim \mathcal{M}_f = (2g - 2) dim H. \tag{25}$$

In even dimensions greater than 2, the higher cohomology groups provide additional obstructions to smoothability of $\mathcal{M}_f$. For example, for $n = 4$, Hodge duality implies that $b_0 = b_4$ and $b_1 = b_3$ and (24) gives

$$b_1 = b_0 + (b_2 - \chi(N) dimH)/2.$$

Equation (25) shows that $dim\mathcal{M}_f$ is even. Identifying the first cohomology $H^1(\Lambda(M, adh), d^\omega)$ of the deformation complex with the tangent space $T_\omega\mathcal{M}_f$ to $\mathcal{M}_f$, the intersection form defines a map $\iota_\omega : T_\omega\mathcal{M}_f \times T_\omega\mathcal{M}_f \to \mathbb{R}$ by

$$\iota(X, Y) = \int_{\Sigma_g} X \wedge Y, \quad X, Y \in T_\omega\mathcal{M}_f. \tag{26}$$

The map ι_ω is skew-symmetric and bilinear. The map

$$\iota : \omega \mapsto \iota_\omega, \ \forall \omega \in \mathcal{M}_f, \tag{27}$$

defines a 2-form ι on $\mathcal{M}_f$. If $\mathbf{h}$ admits an H-invariant inner product, then this 2-form ι is closed and non-degenerate and hence defines a symplectic structure on $\mathcal{M}_f$. It can be shown that, for a Riemann surface with $H = PSL(2, \mathbb{R})$, the form ι, restricted to the Teichmüller space, agrees with the well-known Weil-Petersson form.

We now discuss an interesting physical interpretation of the symplectic manifold $(\mathcal{M}_f(\Sigma_g, H), \ \iota)$. Consider a Chern-Simons theory on the principal bundle $P(M, H)$ over the $2 + 1$-dimensional space-time manifold $M = \Sigma_g \times \mathbb{R}$ with gauge group H and with time independent gauge potentials and gauge transformations. Let $\mathcal{A}$ (resp. $\mathcal{H}$) denote the space (resp. group) of these gauge connections (resp. transformations). It can be shown that the curvature F_ω defines an $\mathcal{H}$-equivariant moment map

$$\mu : \mathcal{A} \to \mathcal{LH} \cong \Lambda^1(M, adP), \text{ by } \omega \mapsto *F_\omega,$$

where $\mathcal{LH}$ is the Lie algebra of $\mathcal{H}$. The zero set $\mu^{-1}(0)$ of this map is precisely the set of flat connections and hence

$$\mathcal{M}_f \cong \mu^{-1}(0)/\mathcal{H} := \mathcal{A}//\mathcal{H} \tag{28}$$

is the reduced phase space of the theory, in the sense of the Marsden-Weinstein reduction. We call $\mathcal{A}//\mathcal{H}$ the **symplectic quotient** of $\mathcal{A}$ by $\mathcal{H}$. Marsden-Weinstein reduction and symplectic quotient are fundamental constructions in geometrical mechanics and geometric quantization. They also arise in many other mathematical applications.

A situation similar to that described above, also arises in the geometric formulation of canonical quantization of field theories. One proceeds by analogy with the geometric quantization of finite dimensional systems. For example, $Q = \mathcal{A}/\mathcal{H}$ can be taken as the configuration space and T^*Q as the corresponding phase space. The associated Hilbert space is obtained as the space of L^2 sections of a complex line bundle over Q. For physical reasons this bundle is taken to be flat. Inequivalent flat $U(1)$-bundles are said to correspond to distinct sectors of the theory. Thus we see that at least formally these sectors are parametrized by the moduli space

$$\mathcal{M}_f(Q, U(1)) \cong Hom(\pi_1(Q), U(1))/U(1) \cong Hom(\pi_1(Q), U(1))$$

since $U(1)$ acts trivially on $Hom(\pi_1(Q), U(1))$.

5 Casson invariant and Flat Connections

Let Y be a homology 3-sphere. Let D_1, D_2 be two unitary, unimodular representations of $\pi_1(Y)$ in $\mathbb{C}^2$. We say that they are equivalent if they are conjugate under the natural $SU(2)$-action on $\mathbb{C}^2$, i.e.

$$D_2(g) = S^{-1} D_1(g) S, \ \forall g \in \pi_1(Y), \ S \in SU(2).$$

Let us denote by $\mathcal{R}(Y)$ the set of equivalence classes of such representations. It is customary to write

$$\mathcal{R}(Y) := \mathrm{Hom}\{\pi_1(Y) \to SU(2)\}/\mathrm{conj}. \tag{29}$$

The set $\mathcal{R}(Y)$ can be given the structure of a compact, real algebraic variety. It is called the $SU(2)$-representation variety of Y. Let $\mathcal{R}^*(Y)$ be the class of irreducible representations. Fixing an orientation of Y, Casson showed how to assign a sign $s(\alpha)$ to each element $\alpha \in \mathcal{R}^*(Y)$. He showed that the set $\mathcal{R}^*(Y)$ is 0-dimensional and compact and hence finite. Casson defined a numerical invariant of Y by counting the signed number of elements of $\mathcal{R}^*(Y)$ by

$$c(Y) := \sum_{\alpha \in \mathcal{R}^*(Y)} s(\alpha).1 \tag{30}$$

The integer $c(Y)$ is called the **Casson invariant** of Y.

Theorem 1 *The Casson invariant $c(Y)$ is well defined upto sign for any homology sphere Y and satisfies the following properties:*
i) $c(-Y) = -c(Y)$,
ii) $c(X\#Y) = c(X) + c(Y)$, *X a homology sphere,*
iii) $c(Y)/2 = \rho(Y) \bmod 2$, *ρ Rokhlin invariant.*

We now give a gauge theory description of $\mathcal{R}(Y)$ leading to Taubes' theorem. In [77] Taubes gives a new interpretation of the Casson invariant $c(Y)$ of an oriented homology 3-sphere Y, which is defined above in terms of the signed count of equivalence classes of irreducible representations of $\pi_1(Y)$ into $SU(2)$. As indicated above, this space can be identified with the moduli space $\mathcal{M}_f(Y, SU(2))$ of flat connections in the trivial $SU(2)$- bundle over Y. Recall that this is also the space of solutions of the Chern-Simons field equations (20) The map $F : \omega \mapsto F_\omega$ defines a natural 1-form on $\mathcal{A}/\mathcal{G}$ and the zeros of this form are just the flat connections. We note that since $\mathcal{A}/\mathcal{G}$ is infinite dimensional, it is necessary to use suitable Fredholm perturbations to get simple zeros and to count them with appropriate signs. Let Z denote the set of zeros of the perturbed vector field and let $s(a)$ be the sign of $a \in Z$. Taubes shows that Z is contained in a compact set and that

$$c(Y) = \sum_{a \in Z} s(a).1$$

The right hand side of this equation can be interpreted as the index of a vector field in the infinite diemensional setting. The classical Poincaré-Hopf theorem can also be generalized to interprete the index as Euler characteristic. A natural question to ask is if this Euler characteristic comes from some homology theory? An affirmative answer is provided by Floer's instanton homology. We discuss it in the next section.

Another approach to Casson's invariant involves symplectic geometry and topology. We conclude this section with a brief indication of this approach. Let $Y_+ \cup_{\Sigma_g} Y_-$ be a Heegaard splitting of Y along the Riemann surface Σ_g of genus g. The space $\mathcal{R}(\Sigma_g)$ of conjugacy classes of representations of $\pi_1(\Sigma_g)$ into $SU(2)$ can be identified with the moduli space $\mathcal{M}_f(\Sigma_g, SU(2))$ of flat connections. This identification endows it with a natural symplectic structure which makes it into a $(6g-6)$-dimensional symplectic manifold. The representations which extend to Y_+ (resp. Y_-) form a $(3g-3)$-dimensional Lagrangian submanifold of $\mathcal{R}(\Sigma_g)$ which we denote by $\mathcal{R}(Y_+)$ (resp. $\mathcal{R}(Y_-)$). Casson's invariant is then obtained from the intersection number of the Lagrangian submanifolds $\mathcal{R}(Y_+)$ and $\mathcal{R}(Y_-)$ in the symplectic manifold $\mathcal{R}(\Sigma_g)$.

How the Floer homology of Y fits into this scheme seems to be unknown at this time.

6 Fukaya-Floer Homology

The idea of instanton tunnelling and the corresponding Witten complex was extended by Floer to do Morse theory on the infinite dimensional moduli space of gauge potentials on a homology 3-sphere Y and to define new topological invariants of Y. Fukaya has generalized this work to apply to arbitrary oriented 3-manifolds. We shall refer to the invariants of Floer and Fukaya collectively as Fukaya-Floer Homology. Fukaya-Floer Homology associates to an oriented, connected, closed, smooth 3-dimensional manifold Y, a family of $\mathbb{Z}_8$-graded instanton homology groups $FF_n(Y),\ n \in \mathbb{Z}_8$. We begin by introducing Floer's original definition, which requires Y to be a homology 3-sphere. Let $\mathcal{R}(Y)$ be the $SU(2)$-representation variety of Y as defined in (29 and let $\mathcal{R}^*(Y)$ be the class of irreducible representations. We say that $\alpha \in \mathcal{R}^*(Y)$ is a **regular representation** if

$$H^1(Y, ad(\alpha)) = 0. \tag{31}$$

We identify $\mathcal{R}(Y)$ with the space of flat or Chern-Simons connections on Y. The Chern-Simon functional has non-degenerate Hessian at α if α is regular. Fix a trivialization P of the given $SU(2)$-bundle over Y. Using the trivial connection θ on $P = Y \times SU(2)$ as a background connection on Y, we can identify the space of connections $\mathcal{A}_Y$ with the space of sections of $\Lambda^1(Y) \otimes su(2)$. In what follows we shall consider a suitable Sobolev completion of this space and continue to denote it by $\mathcal{A}_Y$.

Let $c : I \to \mathcal{A}_Y$ be a path from α to θ. The family of connections $c(t)$ on Y can be identified as a connection A on $Y \times I$. Using this connection we can rewrite the Chern-Simons action (11) as follows

$$\mathcal{A}_{CS} = \frac{1}{8\pi^2} \int_{Y \times I} tr(F_A \wedge F_A). \tag{32}$$

We note that the integrand corresponds to the second Chern class of the pull-back of the trivial $SU(2)$-bundle over Y to $Y \times I$. Recall that the critical points of the Chern-Simons action are the flat connections. The gauge group $\mathcal{G}_Y$ acts on $\mathcal{A}_{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{A}_{CS}(\alpha^g) = \mathcal{A}_{CS}(\alpha) + \deg(g),\ g \in \mathcal{G}_Y.$$

It follows that $\mathcal{A}_{CS}$ descends to $\mathcal{B}_Y := \mathcal{A}_Y/\mathcal{G}_Y$ as a map $f_{CS} : \mathcal{B}_Y \to \mathbb{R}/\mathbb{Z}$ and we can take $\mathcal{R}(Y) \subset \mathcal{B}_Y$ as the critical set of f_{CS}. The gradient flow of this function is given by the equation

$$\frac{\partial c(t)}{\partial t} = *_Y F_{c(t)}. \tag{33}$$

Since Y is a homology 3-sphere, the critical points of the flow of *grad* f_{CS} and the set of reducible connections intersect at a single point, the trivial connection θ. If all the critical points of the flow are regular then it is a Morse-Smale flow. If not, one can perturb the function f_{CS} to get a Morse function.

In general the representation space $\mathcal{R}^*(Y) \subset \mathcal{B}_Y$ contains degenerate critical points of the Chern-Simons function f_{CS}. In this case Floer defines a set of perturbations of f_{CS} as follows. Let $m \in \mathbb{N}$ and let $\vee_{i=1}^m S_i^1$ be a bouquet of m copies of the circle S^1. Let Γ_m be the set of maps

$$\gamma : \bigvee_{i=1}^m S_i^1 \times D^2 \to Y$$

such that the restrictions

$$\gamma_x : \bigvee_{i=1}^m S_i^1 \times \{x\} \to Y \text{ and } \gamma_i : S_i^1 \times D^2 \to Y$$

are smooth embeddings for each $x \in D^2$ and for each $i,\ 1 \le i \le m$. Let $\hat{\gamma}_x$ denote the family of holonomy maps

$$\hat{\gamma}_x : A_Y \to \underbrace{SU(2) \times \cdots \times SU(2)}_{m \text{ times}},\ x \in D^2.$$

The holonomy is conjugated under the action of the group of gauge transformations and we continue to denote by $\hat{\gamma}_x$ the induced map on the quotient $\mathcal{B}_Y = \mathcal{A}_Y/\mathcal{G}$. Let $\mathcal{F}_m$ denote the set of smooth functions

$$h : \underbrace{SU(2) \times \cdots \times SU(2)}_{m \text{ times}} \to \mathbb{R}$$

which are invariant under the adjoint action of $SU(2)$. Floer's set of perturbations Π is defined as

$$\Pi := \bigcup_{m \in \mathbb{N}} \Gamma_m \times \mathcal{F}_m.$$

Floer proves that for each $(\gamma, h) \in \Pi$ the function

$$h_\gamma : \mathcal{B}_Y \to \mathbb{R} \text{ defined by } h_\gamma(\alpha) = \int_{D^2} h(\hat{\gamma}_x(\alpha))$$

is a smooth function and that for a dense subset $\mathcal{P} \subset \mathcal{RM}(Y) \times \Pi$ the critical points of the perturbed function

$$f_{(\gamma,h)} := f_{CS} + h_\gamma$$

are non-degenarate and the corresponding moduli space decomposes into smooth, oriented manifolds of regular trajectories of the gradient flow of the function $f_{(\gamma,h)}$ with respect to a generic metric $\sigma \in \mathcal{RM}(Y)$. Furthermore, the homology groups of the perturbed chain complex are independent of the choice of perturbation in $\mathcal{P}$. We shall assume that this has been done. Let α, β be two critical points of the function f_{CS}. Considering the spectral flow (denoted by sf) from α to β we obtain the moduli space $\mathcal{M}(\alpha, \beta)$ as the moduli space of self-dual connections on $Y \times \mathbb{R}$ which are asymptotic to α and β (as $t \to \pm\infty$). Let $\mathcal{M}^j(\alpha, \beta)$ denote the component of dimension j in $\mathcal{M}(\alpha, \beta)$. There is a natural action of $\mathbb{R}$ on $\mathcal{M}(\alpha, \beta)$. Let $\hat{\mathcal{M}}^j(\alpha, \beta)$ denote the component of dimension $j-1$ in $\mathcal{M}(\alpha, \beta)/\mathbb{R}$. Let $\#\hat{\mathcal{M}}^1(\alpha, \beta)$ denote the signed sum of the number of points in $\hat{\mathcal{M}}^1(\alpha, \beta)$. Floer defines the Morse index of α by considering the spectral flow from α to the trivial connection θ. It can be shown that the spectral flow and hence the Morse index are defined modulo 8. Now define the chain groups by

$$\mathcal{R}_n(Y) = \mathbb{Z}\{\alpha \in \mathcal{R}^*(Y) \mid sf(\alpha) = n\}, \; n \in \mathbb{Z}_8$$

and define the boundary operator ∂

$$\partial : \mathcal{R}_n(Y) \to \mathcal{R}_{n-1}(Y)$$

by

$$\partial\alpha = \sum_{\beta \in \mathcal{R}_{n-1}(Y)} \#\hat{\mathcal{M}}^1(\alpha, \beta)\beta. \tag{34}$$

It can be shown that $\partial^2 = 0$ and hence $(\mathcal{R}(Y), \partial)$ is a complex. This complex can be thought of as an infinite dimensional generalization [24] of Witten's instanton tunnelling and we will call it the **Floer-Witten Complex** of the pair $(Y, SU(2))$. Since the spectral flow and hence the dimensions of the components of $\mathcal{M}(\alpha, \beta)$ are congruent modulo 8, this complex defines the Floer homology groups $FH_j(Y),\ j \in \mathbb{Z}_8$, where j is the spectral flow of α to θ modulo 8. If r_j denotes the rank of the Floer homology group $FH_j(Y),\ j \in \mathbb{Z}_8$, then we can define the corresponding Euler characteristic $\chi_F(Y)$ by

$$\chi_F(Y) := \sum_{j \in \mathbb{Z}_8} (-1)^j r_j.$$

Combining this with Taubes' interpretation of the Casson invariant $c(Y)$ we get

$$c(Y) = \chi_F(Y) = \sum_{j \in \mathbb{Z}_8} (-1)^j r_j. \tag{35}$$

An important feature of Floer's instanton homology is that it can be regarded as a functor from the category of homology 3-spheres with morphisms given by oriented cobordism, to the category of graded abelian groups. Let M be a smooth, oriented cobordism from Y_1 to Y_2 so that $\partial M = Y_2 - Y_1$. By a careful analysis of instantons on M, Floer showed [23] that M induces a graded homomorphism

$$M_j : FH_j(Y_1) \to FH_{j+b(M)}(Y_2),\ j \in \mathbb{Z}_8, \tag{36}$$

where

$$b(M) = 3(b_1(M) - b_2(M)). \tag{37}$$

Then the homomorphisms induced by cobordism has the following functorial properties.

$$(Y \times \mathbb{R})_j = id, \tag{38}$$

$$(MN)_j = M_{j+b(N)} N_j. \tag{39}$$

An algorithm for computing the Floer homology groups for Seifert-fibered homology 3-spheres with three exceptional fibers (or orbits) has been discussed in [22].

In addition to these invariants of 3-manifolds and the linking number, there are several other invariants of knots and links in 3-manifolds. We introduce them in the next section and study their field theory interpretations in the later sections.

7 Knot Polynomials

In the second half of the nineteenth century, a systematic study of knots in $\mathbb{R}^3$ was made by Tait. He was motivated by Kelvin's theory of atoms modelled on knotted vortex tubes of ether. Tait classified the knots in terms of the crossing number of a plane projection and made a number of observations about some general properties of knots which have come to be known as the "Tait conjectures". Recall that a knot κ in S^3 is an embedding of the circle S^1 and that a link is a disjoint union of knots. A **link diagram** of κ is a plane projection with crossings marked as over or under. By changing a link diagram at one crossing we can obtain three diagrams corresponding to links κ_+, κ_- and κ_0.

In the 1920s, Alexander gave an algorithm for computing a polynomial invariant $A_\kappa(q)$ of a knot κ, called the **Alexander polynomial**, by using its projection on a plane. He also gave its topological interpretation as an annihilator of a certain cohomology module associated to the knot κ. In the 1960s, Conway defined his polynomial invariant and gave its relation to the Alexander polynomial. This polynomial is called the **Alexander-Conway polynomial** or simply the Conway polynomial. The Alexander-Conway polynomial of an oriented link L is denoted by $\nabla_L(z)$ or simply by $\nabla(z)$ when L is fixed. We denote the corresponding polynomials of L_+, L_- and L_0 by ∇_+, ∇_- and ∇_0 respectively. The Alexander-Conway polynomial is uniquely determined by the following simple set of axioms.

AC1. Let L and L' be two oriented links which are ambient isotopic. Then

$$\nabla_{L'}(z) = \nabla_L(z) \tag{40}$$

AC2. Let S^1 be the standard unknotted circle embeded in S^3. It is usually referred to as the **unknot** and is denoted by $\mathcal{O}$. Then

$$\nabla_{\mathcal{O}}(z) = 1. \tag{41}$$

AC3. The polynomial satisfies the following **skein relation**

$$\nabla_+(z) - \nabla_-(z) = z\nabla_0(z). \tag{42}$$

We note that the original Alexander polynomial Δ_L is related to the Alexander-Conway polynomial by the relation

$$\Delta_L(t) = \nabla_L(t^{1/2} - t^{-1/2}).$$

Despite these and other major advances in knot theory, the Tait conjectures remained unsettled for more than a century after their formulation. Then in the 1980s, Jones discovered his polynomial invariant $V_\kappa(q)$, called the **Jones polynomial**, while studying Von Neumann algebras and gave its interpretation in terms of statistical mechanics. These new polynomial invariants have led to the proofs of most of the Tait conjectures. As with the earlier invariants, Jones' definition of his poynomial invariants is algebraic and combinatorial in nature and was based on representations of the braid groups and related Hecke algebras. The Jones polynomial $V_\kappa(t)$ of κ is a Laurent polynomial in t (polynomial in t and t^{-1}) which is uniquely determined by a simple set of properties similar to the axioms for the Alexander-Conway polynomial. More generally, the Jones polynomial can be defined for any oriented link L as a Laurent polynomial in $t^{1/2}$. Reversing the orientation of all components of L leaves V_L unchanged. In particular, V_κ does not depend on the orientation of the knot κ. For a fixed link, we denote the Jones polynomial simply by V. Recall that there are 3 standard ways to change a link diagram at a crossing point. The Jones polynomials of the corresponding links are denoted by V_+, V_- and V_0 respectively. Then the Jones polynomial is characterized by the following properties:
JO1. Let κ and κ' be two oriented links which are ambient isotopic. Then

$$V_{\kappa'}(t) = V_\kappa(t) \tag{43}$$

JO2. Let $\mathcal{O}$ denote the unknot. Then

$$V_{\mathcal{O}}(t) = 1. \tag{44}$$

JO3. The polynomial satisfies the following skein relation

$$t^{-1}V_+ - tV_- = (t^{1/2} - t^{-1/2})V_0. \tag{45}$$

An important property of the Jones polynomial that is not shared by the Alexander-Conway polynomial is its ability to distinguish between a knot and its mirror image. Let κ_m be the mirror image of the knot κ. Then

$$V_{\kappa_m}(t) = V_\kappa(t^{-1}) \neq V_\kappa(t) \tag{46}$$

Since the Jones polynomial is not symmetric in t and t^{-1}. Soon after Jones' discovery a two variable polynomial generalizing V was found by several mathematicians. It is called the **HOMFLY polynomial** and is denoted by P. The HOMFLY polynomial $P(\alpha, z)$ satisfies the following skein relation

$$\alpha P_+ - \alpha^{-1} P_- = z P_0. \tag{47}$$

If we put $\alpha = t^{-1}$ and $z = (t^{1/2} - t^{-1/2})$ in equation (47) we get the skein relation for the original Jones polynomial V. If we put $\alpha = 1$ we get the skein relation for the Alexander-Conway polynomial.

Knots and links in $\mathbb{R}^3$ can also be obtained by using braids. A **braid** on n strands (or with n strings or simply an n-braid) can be thought of as a set of n pairwise disjoint strings joining n distinct points in one plane with n distinct points in a parallel plane in $\mathbb{R}^3$. The set of equivalence classes of n-braids is denoted by $\mathcal{B}_n$. A braid is called elementary if only two neighboring strings cross. We denote by σ_i the elementary braid where the i-th string crosses over the $(i+1)$-th string.
Theorem (M. Artin): The set $\mathcal{B}_n$ with multiplication operation induced by concatenation of braids is a group generated by the elementary braids $\sigma_i, 1 \leq i \leq n-1$ subject to the braid relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ , \ 1 \leq i \leq n-2. \tag{48}$$

and the far commutativity relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \ , \ 1 \leq i, j \leq n-1 \text{ and } |i-j| > 1. \tag{49}$$

The closure of a braid b obtained by gluing the endpoints is a link denoted by $c(b)$. A classical theorem of Alexander shows that the closure map from the set of braids to the set of links is surjective, i.e. any link (and, in particular, knot) is the closure of some braid. Moreover, if braids b and b' are equivalent, then the links $c(b)$ and $c(b')$ are equivalent. There are several descriptions of the braid group leading to various approaches to the study of its representations and invariants of links. For example, $\mathcal{B}_n$ is isomorphic to the fundamental group of the configuration space of n distinct points in the plane. The action of $\mathcal{B}_n$ on the homology of the configuration space is related to the representations of certain Hecke algebras leading to invariants of links such as the Jones polynomial that we have discussed earlier. The group $\mathcal{B}_n$ is also isomorphic to the mapping class group of the n-punctured disc. This definition was recently used by Krammer and Bigelow in showing the linearity of $\mathcal{B}_n$ over the ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials in two variables.

8 Categorification of Knot Polynomials

We begin by recalling that a categorification of an invariant I is the construction of a suitable (co)homology H^* such that its Euler characteristic $\chi(H^*)$ (the alternating sum of the ranks of (co)homology groups) equals I. Historically, the Euler charactristic was defined and understood well before the advent of algebraic topology. Theorema egregium of Gauss and the closely related Gauss-Bonnet theorem and its generalization by Chern give a geometric interpretation of the Euler characteristic $\chi(M)$ of a manifold M. They can be regarded as precursors of Chern-Weil theory as well as index theory. Categorification $\chi(H^*(M))$ of this Euler characteristic $\chi(M)$ by various (co)homolgy theories $H^*(M)$ came much later. A well known recent example that we have discussed is the categorification of the Casson invariant by the Fukaya-Floer homology. Categorification of quantum invariants such as Knot Polynomials requires the use of quantum Euler characteristic and multi-graded knot homologies.

Recently Khovanov [34] has obtained a categorification of the Jones polynomial $V_\kappa(q)$ by constructing a bi-graded $sl(2)$-homology $H_{i,j}$ determined by the knot κ. It is called the **Khovanov homology** of the knot κ and is denoted by $KH(\kappa)$. The **Khovanov polynomial** $Kh_\kappa(t,q)$ is defined by

$$Kh_\kappa(t,q) = \sum_{i,j} t^j q^i dim\ H_{i,j}\ .$$

It can be thought of as a two variable generalization of the Poincarè polynomial. The quantum or graded Euler characteristic of the Khovanov homology equals the Jones polynomial. i.e.

$$V_\kappa(q) = \chi_q(KH(\kappa)) = \sum_{i,j}(-1)^j q^i dim\ H_{i,j}\ .$$

Khovanov's construction folllows Kauffman's state-sum model of the link L and his alternative definition of the Jones polynomial. Let $\hat{L}$ be a regular projection of L with $n = n_+ + n_-$ labelled crossings. At each crossing we can define two resolutions or states, the vertical or 1-state and horizontal or 0-state. Thus there are 2^n total resolutions of $\hat{L}$ which can be put into one to one correspondence with the vertices of an n-dimensional unit cube. For each vertex x let $|x|$ be the sum of its coordinates and let $c(x)$ be the number of disjoint circles in the resolution $\hat{L}_x$ of $\hat{L}$ determined by x. Kauffman's state-sum expression for the non-normalized Jones polynomial $\hat{V}(L)$ can be written as follows:

$$\hat{V}(L) = (-1)^{n_-} q^{(n_+ - 2n_-)} \sum (-q)^{|x|} (q + q^{-1})^{c(x)} \ . \tag{50}$$

Dividing this by the unknot value $(q+q^{-1})$ gives the usual normalized Jones polynomial $V(L)$. The Khovanov complex is constructed as follows. Let V be a graded vector space over a fixed ground field K, generated by two basis vectors $v_\pm$ with respective degrees ± 1. To the manifold M_x at each vertex x we associate the graded vector space

$$V_x(L) := V^{\otimes c(x)}\{|x|\} \ , \tag{51}$$

where $\{k\}$ is the degree shift by k. We define the Frobenius structure on V as follows. Multiplication $m : V \otimes V \to V$ is defined by

$$m(v_+ \otimes v_+) = v_+, \qquad m(v_+ \otimes v_-) = v_-,$$

$$m(v_- \otimes v_+) = v_-, \qquad m(v_- \otimes v_-) = 0.$$

Co-multiplication $\Delta : V \to V \otimes V$ is defined by

$$\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+ \ , \qquad \Delta(v_-) = v_- \otimes v_-.$$

Thus v_+ is the unit. The co-unit $\delta \in V^*$ is defined by mapping v_+ to 0 and v_- to 1 in the base field. The r-th chain group $C_r(L)$ in the Khovanov complex is the direct sum of all vector spaces $V_x(L)$, where $|x| = r$, and the differential is defined by the Frobenius structure. Thus

$$C_r(L) := \oplus_{|x|=r} V_x(L). \tag{52}$$

Remark

The total resolution associates to each vertex x a one dimensional manifold M_x consisting of $c(x)$ disjoint circles. We can construct a $(1+1)$-dimensioal TQFT (along the lines of Atiyah-Segal axioms discussed in the next section) for each edge of the cube as follows. If xy is an edge of the cube we can get a pair of pants cobordism from M_x to M_y by noting that a circle at x can split into two at y or two circles at x can fuse into one at y. If a circle goes to a circle than the cylinder provides the cobordism. TQFT then corresponds to the Frobenius algebra structure on V defined above.

The r-th homology group of the Khovanov complex is denoted by KH_r. Khovanov has proved that the homology is independent of the various choices made in defining it. Thus we have

Theorem 2 *The homology groups KH_r are link invariants. In particular, the Khovanov polynomial*

$$Kh_L(t,q) = \sum_j t^j dim_q(KH_j)$$

is a link invariant that specializes to the non-normalized Jones polynomial. The Khovanov polynomial is strictly stronger than the Jones polynomial.

We note that the knots 9_{42} and 10_{125} are chiral. Their chirality is detected by the Khovanov polynomial but not by the Jones polynomial. Also there are several pairs of knots with the same Jones polynomials but different Khovanov polynomials. For example $(5_1, 10_{132})$ is such a pair.

8.1 Categorification of $V(3_1)$

Using equations (51) and (52) and the algebra structure on V the calculation of the Khovanov complex can be reduced to an algorithm. A computer program implementing such an algorithm is discussed in [5]. A table of Khovanov polynomials for knots and links upto 11 crossings is also given there. We now illustrate Khovanov's categorification of the Jones polynomial of the right handed trefoil knot 3_1. For the standard diagram of the trefoil, $n = n_+ = 3$ and $n_- = 0$. The quantum dimensions of the non-zero terms of the Khovanov complex with the shift factor included are given by

$$C_0 = (q+q^{-1})^2, C_1 = 3q(q+q^{-1}), C_2 = 3q^2(q+q^{-1})^2, C_3 = q^3(q+q^{-1})^3. \quad (53)$$

The non-normalized Jones polynomial can be obtained from (53) or directly from (50) giving

$$\hat{V}(L) = (q + q^3 + q^5 - q^9) \quad (54)$$

The normalized or standard Jones polynomial is then given by

$$V(q) = (q + q^3 + q^5 - q^9)/(q + q^{-1}) = q^2 + q^6 - q^8.$$

By direct computation or using the program in [5] we obtain the following formula for the Khovanov polynomial of the trefoil

$$Kh(t,q) = q + q^3 + t^2q^5 + t^3q^9, Kh(-1,q) = \chi_q = \hat{V}(L).$$

Based on computations using the program described in [5], Khovanov, Garoufalidis and Bar-Natan (BKG) have formulated some conjectures on the structure of Khovanov polynomials over different base fields. We now state these conjectures.

The BKG Conjectures: For any prime knot κ, there exists an even integer $s = s(\kappa)$ and a polynomial $Kh'_\kappa(t,q)$ with only non-negative coefficients such that

1. Over the base field $K = \mathbb{Q}$,

$$Kh_\kappa(t,q) = q^{s-1}[1 + q^2 + (1 + tq^4)Kh'_\kappa(t,q)]$$

2. Over the base field $K = \mathbb{Z}_2$,

$$Kh_\kappa(t,q) = q^{s-1}(1 + q^2)[1 + (1 + tq^2)Kh'_\kappa(t,q)]$$

3. Moreover, if the κ is alternating, then $s(\kappa)$ is the signature of the knot and $Kh'_\kappa(t,q)$ contains only powers of tq^2.

The conjectured results are in agreement with all the known values of the Khovanov polynomials.

If $S \subset \mathbb{R}^4$ is an oriented surface cobordism between links L_1 and L_2, then it induces a homomorphism of Khovanov homologies of links L_1 and L_2. These homomorphisms define a functor from the category of link cobordisms to the category of bigraded abelian groups. Khovanov homology extends to colored links (i.e. oriented links with components labelled by irreducible finite dimensional representations of $sl(2)$) to give a categorification of the colored Jones polynomial. Khovanov and Rozansky have defined an $sl(n)$-homology for links colored by either the defining representation or its dual. This gives categorification of the specialization of the HOMFLY polynomial $P(\alpha, q)$ with $a = q^n$. The sequence of such specializations for $n \in \mathbb{N}$ would categorify the two variable HOMFLY polynomial $P(\alpha, q)$. For $n = 0$ the theory coincides with the Heegaard Floer homology of Ozsváth and Szabo.

In the 1990s Reshetikhin, Turaev and other mathematicians obtained several quantum invariants of triples $(\mathbf{g}, L, M)$, where $\mathbf{g}$ is a simple Lie algebra, $L \subset M$ is an oriented, framed link with components labelled by irreducible representations of $\mathbf{g}$ and M is a 2-framed 3-manifold. In particular, there are polynomial invariants $< L >$ that take values in $\mathbb{Z}[q^{-1}, q]$. Khovanov has conjectured that at least for some classes of Lie algebras (e.g. simply-laced) there exists a bigraded homology theory of labelled links such that the polynomial invariant $< L >$ is the quantum Euler characteristic of this homology. It should define a functor from the category of framed link cobordisms to the category of bigraded abelian groups. In particular, the homology of the unknot labelled by an irreducible representation U of $\mathbf{g}$ should be a Frobenius algebra of dimension $dim(U)$.

9 Topological Quantum Field Theory

Quantization of classical fields is an area of fundamental importance in modern mathematical physics. Although there is no satisfactory mathematical theory of quantization of classical dynamical systems or fields, physicists have developed several methods of quantization that can be applied to specific problems. Most successful among these is QED (Quantum Electrodynamics), the theory of quantization of electromagnetic fields. The physical significance of electromagnetic fields is thus well understood at both the classical and the quantum level. Electromagnetic theory is the prototype of classical gauge theories. It is therefore, natural to try to extend the methods of QED to the quantization of other gauge field theories. The methods of quantization may be broadly classified as non-perturbative and perturbative. The literature pertaining to each of these areas is vast. See for example [7, 18, 71, 75, 76]. Our aim in this section is to discuss some aspects of a new area of research in quantum field theory, namely, topological quantum field theory (or TQFT for short). Ideas from TQFT have already led to new ways of looking at old topological invariants as well as to surprising new invariants.

9.1 Atiyah-Segal axioms for TQFT

In 2 and 3 dimensional geometric topology, Conformal Field Theory (CFT) methods have proved to be useful. An attempt to put the CFT on a firm mathematical foundation was begun by Segal in [73] by propossing a set of axioms for CFT. CFT is a two dimensional theory and it was necessary to modify and generalize these axioms to apply to topological field theory in any dimension. We now discuss briefly these TQFT axioms following Atiyah The Atiyah-Segal axioms for TQFT (see, for example, [1], [46]) arose from an attempt to give a mathematical formulation of the non-perturbative aspects of quantum field theory in general and to develop, in particular, computational tools for the Feynman path integrals that are fundamental in the Hamiltonian approach to Witten's topological QFT. The most spectacular application of the non-perturbative methods has been in the definition and calculation of the invariants of 3-manifolds with or without links and knots. In most physical applications however, it is the perturbative calculations that are predominantly used. Recently, perturbative aspects of the Chern-Simons theory in the context of TQFT have been considered in [4]. For other approaches to the invariants of 3-manifolds see [35, 37, 63, 79, 81]

Let $\mathcal{C}_n$ denote the category of compact, oriented, smooth n-dimensional manifolds with morphism given by oriented cobordism. Let $\mathcal{V}_{\mathbb{C}}$ denote the category of finite dimensional complex vector spaces. An $(n+1)$-dimensional TQFT is a functor $\mathcal{T}$ from the category $\mathcal{C}_n$ to the category $\mathcal{V}_{\mathbb{C}}$ which satisfies the following axioms.

A1. Let $-\Sigma$ denote the manifold Σ with the opposite orientation of Σ and let V^* be the dual vector space of $V \in \mathcal{V}_{\mathbb{C}}$. Then

$$\mathcal{T}(-\Sigma) = (\mathcal{T}(\Sigma))^*, \ \forall \Sigma \in \mathcal{C}_n.$$

A2. Let $\sqcup$ denote disjoint union. Then

$$\mathcal{T}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{T}(\Sigma_1) \otimes \mathcal{T}(\Sigma_2), \ \forall \Sigma_1, \Sigma_2 \in \mathcal{C}_n.$$

A3. Let $Y_i : \Sigma_i \to \Sigma_{i+1}, \ i = 1, 2$ be morphisms. Then

$$\mathcal{T}(Y_1 Y_2) = \mathcal{T}(Y_2)\mathcal{T}(Y_1) \in Hom(\mathcal{T}(\Sigma_1), \mathcal{T}(\Sigma_3)),$$

where $Y_1 Y_2$ denotes the morphism given by composite cobordism $Y_1 \cup_{\Sigma_2} Y_2$.

A4. Let $\emptyset_n$ be the empty n-dimensional manifold. Then

$$\mathcal{T}(\emptyset_n) = \mathbb{C}.$$

A5. For every $\Sigma \in \mathcal{C}_n$

$$\mathcal{T}(\Sigma \times [0, 1]) : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$$

is the identity endomorphism.

We note that if Y is a compact, oriented, smooth $(n+1)$-manifold with compact, oriented, smooth boundary Σ, then

$$\mathcal{T}(Y) : \mathcal{T}(\phi_n) \to \mathcal{T}(\Sigma)$$

is uniquely determined by the image of the basis vector $1 \in \mathbb{C} \equiv \mathcal{T}(\phi_n)$. In this case the vector $\mathcal{T}(Y) \cdot 1 \in \mathcal{T}(\Sigma)$ is often denoted simply by $\mathcal{T}(Y)$ also. In particular, if Y is closed, then

$$\mathcal{T}(Y) : \mathcal{T}(\phi_n) \to \mathcal{T}(\phi_n) \text{ and } \mathcal{T}(Y) \cdot 1 \in \mathcal{T}(\phi_n) \equiv \mathbb{C}$$

is a complex number which turns out to be an invariant of Y. Axiom A3 suggests a way of obtaining this invariant by a cut and paste operation on Y as follows. Let $Y = Y_1 \cup_\Sigma Y_2$ so that Y_1 (resp. Y_2) has boundary Σ (resp. $-\Sigma$). Then we have

$$\mathcal{T}(Y) \cdot 1 = < \mathcal{T}(Y_1) \cdot 1, \mathcal{T}(Y_2) \cdot 1 >, \tag{55}$$

where $< \, , \, >$ is the pairing between the dual vector spaces $\mathcal{T}(\Sigma)$ and $\mathcal{T}(-\Sigma) = (\mathcal{T}(\Sigma))^*$. Equation (55) is often referred to as a gluing formula. Such gluing formulas are characteristic of TQFT. They also arise in Fukaya-Floer homology theory of 3-manifolds, Floer-Donaldson theory of 4-manifold invariants as well as in 2-dimensional conformal field theory. For specific applications the Atiyah axioms need to be refined, supplemented and modified. For example, one may replace the category $\mathcal{V}_\mathbb{C}$ of complex vector spaces by the category of finite-dimensional Hilbert spaces. This is in fact, the situation of the $(2+1)$-dimensional Jones-Witten theory. In this case it is natural to require the following additional axiom.

A6. Let Y be a compact oriented 3-manifold with $\partial Y = \Sigma_1 \sqcup (-\Sigma_2)$. Then the linear transformations

$$\mathcal{T}(Y) : \mathcal{T}(\Sigma_1) \to \mathcal{T}(\Sigma_2) \text{ and } \mathcal{T}(-Y) : \mathcal{T}(\Sigma_2) \to \mathcal{T}(\Sigma_1)$$

are mutually adjoint.
For a closed 3-manifold Y the axiom A6 implies that

$$\mathcal{T}(-Y) = \overline{\mathcal{T}(Y)} \in \mathbb{C}.$$

It is this property that is at the heart of the result that in general, the Jones polynomials of a knot and its mirror image are different, i.e.

$$V_\kappa(t) \neq V_{\kappa_m}(t),$$

where κ_m is the mirror image of the knot κ.

An important example of a $(3+1)$-dimensional TQFT is provided by the Floer-Donaldson theory. The functor $\mathcal{T}$ goes from the category $\mathcal{C}$ of compact, oriented Homology 3-spheres to the category of $\mathbb{Z}_8$-graded abelian groups. It is defined by

$$\mathcal{T} : Y \to HF_*(Y),\ Y \in \mathcal{C}.$$

For a compact, oriented, 4-manifold M with $\partial M = Y$, $\mathcal{T}(M)$ is defined to be the vector $q(M, Y)$

$$q(M, Y) := (q_1(M, Y), q_2(M, Y), \ldots),$$

where the components $q_i(M, Y)$ are the relative polynomial invariants of Donaldson defined on the relative homology group $H_2(M, Y; \mathbb{Z})$.

The axioms also suggest algebraic apporaches to TQFT. The most widely studied of these approaches are based on quantum groups, operator algebras, modular tensor categories and Jones' theory of subfactors. See, for example, books [41, 38, 39, 80], and articles [79, 81, 82]. Turaev and Viro gave an algebraic construction of such a TQFT by using the quantum $6j$-symbols for the quantum group $U_q(sl_2)$ at roots of unity.. Ocneanu [64] starts with a special type of subfactor to generate the data which can be used with the Turaev and Viro construction.

The correspondence between geometric (topological) and algebraic structures has played a fundamental role in the development of modern mathematics. Its roots can be traced back to the classical work of Descartes. Recent developments in low dimensional geometric topology have raised this correppodence to a new level bringing in ever more exotic algebraic structures such as quantum groups, vertex algebras, monoidal and higher categories. This broad area is now often referred to as quantum topology. See, for example, [91, 49].

9.2 Quantum Observables

A **quantum field theory** may be considered as an assignment of the **quantum expectation** $< \Phi >_\mu$ to each gauge invariant function $\Phi : \mathcal{A}(M) \to \mathbb{C}$, where $\mathcal{A}(M)$ is the space of gauge potentials for a given gauge group G and the base manifold (space-time) M. Φ is called a **quantum observable** or simply an **observable** in quantum field theory. Note that the invariance of Φ under the group of gauge transformations $\mathcal{G}$ implies that Φ descends to a function on the moduli space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of gauge equaivalence classes of gauge potentials. In the Feynman path integral approach to quantization the quantum or vacuum expectation $< \Phi >_\mu$ of an observable is given by the following expression.

$$< \Phi >_\mu = \frac{\int_{\mathcal{B}(M)} e^{-S_\mu(\omega)} \Phi(\omega) \mathcal{D}\mathcal{B}}{\int_{\mathcal{B}(M)} e^{-S_\mu(\omega)} \mathcal{D}\mathcal{B}}, \tag{56}$$

where $e^{-S_\mu} \mathcal{D}\mathcal{B}$ is a suitably defined measure on $\mathcal{B}(M)$. It is customary to express the quantum expectation $< \Phi >_\mu$ in terms of the **partition function** Z_μ defined by

$$Z_\mu(\Phi) := \int_{\mathcal{B}(M)} e^{-S_\mu(\omega)} \Phi(\omega) \mathcal{D}\mathcal{B}. \tag{57}$$

Thus we can write

$$< \Phi >_\mu = \frac{Z_\mu(\Phi)}{Z_\mu(1)}. \tag{58}$$

In the above equations we have written the quantum expectation as $< \Phi >_\mu$ to indicate explicitly that, in fact, we have a one-parameter family of quantum expectations indexed by the coupling constant μ in the action. There are several examples of gauge invariant functions. For example, primary characteristic classes evaluated on suitable homology cycles give an important family of gauge invariant functions. The instanton number and the Yang-Mills action are also gauge invariant functions. Another important example is the Wilson loop functional well known in the physics literature.

Wilson loop functional: Let ρ denote a representation of G on V. Let $\alpha \in \Omega(M, x_0)$ denote a loop at $x_0 \in M$. Let $\pi : P(M, G) \to M$ be the canonical projection and let $p \in \pi^{-1}(x_0)$. If ω is a connection on the principal bundle $P(M, G)$, then the parallel translation along α maps the fiber $\pi^{-1}(x_0)$ into itself. Let $\hat{\alpha}_\omega : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ denote this map. Since G acts transitively on the fibers, $\exists g_\omega \in G$ such that $\hat{\alpha}_\omega(p) = p g_\omega$. Now define

$$\mathcal{W}_{\rho,\alpha}(\omega) := Tr[\rho(g_\omega)] \; \forall \omega \in \mathcal{A}. \tag{59}$$

We note that g_ω and hence $\rho(g_\omega)$, change by conjugation if, instead of p, we choose another point in the fiber $\pi^{-1}(x_0)$, but the trace remains unchanged. We call these $\mathcal{W}_{\rho,\alpha}$ the Wilson loop functionals associated to the representation ρ and the loop α. In the particular case when $\rho = Ad$ the adjoint representation of G on $\mathbf{g}$, our constructions reduce to those considered in physics. If $L = (\kappa_1, \ldots, \kappa_n)$ is an oriented link with component knots κ_i, $1 \le i \le n$ and if ρ_i is a representation of the gauge group asscociated to κ_i, then we can define the quantum observable $\mathcal{W}_{\rho,L}$ associated to the pair (L, ρ), where $\rho = (\rho_1, \ldots, \rho_n)$ by

$$\mathcal{W}_{\rho,L} = \prod_{i=1}^{n} \mathcal{W}_{\rho_i,\kappa_i} \ .$$

9.3 Link Invariants

In the 1980s, Jones discovered his polynomial invariant $V_\kappa(q)$, called the **Jones polynomial**, while studying Von Neumann algebras and gave its interpretation in terms of statistical mechanics. These new polynomial invariants have led to the proofs of most of the Tait conjectures. As with most of the earlier invariants, Jones' definition of his polynomial invariants is algebraic and combinatorial in nature and was based on representations of the braid groups and related Hecke algebras. The Jones polynomial $V_\kappa(t)$ of κ is a Laurent polynomial in t (polynomial in t and t^{-1}) which is uniquely determined by a simple set of properties similar to the well known axioms for the Alexander-Conway polynomial. More generally, the Jones polynomial can be defined for any oriented link L as a Laurent polynomial in $t^{1/2}$.

A geometrical interpretation of the Jones' polynomial invariant of links was provided by Witten by applying ideas from QFT to the Chern-Simons Lagrangian constructed from the Chern-Simons action

$$\mathcal{A}_{CS} = \frac{k}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where A is the gauge potential of the $SU(n)$ connection ω. Chern-Simons action is not gauge invariant. Under a gauge transformation g the action transforms as follows:

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A) + 2\pi k \mathcal{A}_{WZ}, \tag{60}$$

where $\mathcal{A}_{WZ}$ is the **Wess-Zumino action functional**. It can be shown that the Wess-Zumino functional is integer valued and hence, if the Chern-Simons coupling constant k is taken to be an integer, then the partition function In fact, Witten's model allows us to consider the knot and link invariants in any compact 3-manifold M. Z defined by

$$Z(\Phi) := \int_{\mathcal{B}(M)} e^{-i\mathcal{A}_{CS}(\omega)} \Phi(\omega) \mathcal{D}\mathcal{B}$$

is gauge invariant. We take for Φ the Wilson loop functional $\mathcal{W}_{\rho,L}$, where ρ is a representation of $SU(n)$ and L is the link under consideration.

We denote the Jones polynomial of L simply by V. Recall that there are 3 standard ways to change a link diagram at a crossing point. The Jones polynomials of the corresponding links are denoted by V_+, V_- and V_0 respectively. To verify the defining relations for the Jones' polynomial of a link L in S^3, Witten [85] starts by considering the Wilson loop functionals for the associated links L_+, L_-, L_0. For a framed link L, we denote by $< L >$ the expectation value of the corresponding Wilson loop functional for the Chern-Simons theory of level k and gauge group $SU(n)$ and with ρ_i the fundamental representation for all i. To verify the defining relations for the Jones' polynomial of a link L in S^3, Witten considers the expectation values of the Wilson loop functionals for the associated links L_+, L_-, L_0 and obtains the relation

$$\alpha < L_+ > + \beta < L_0 > + \gamma < L_- > = 0 \tag{61}$$

where the coefficients α, β, γ are given by the following expressions

$$\alpha = -exp(\frac{2\pi i}{n(n+k)}), \tag{62}$$

$$\beta = -exp(\frac{\pi i(2-n-n^2)}{n(n+k)}) + exp(\frac{\pi i(2+n-n^2)}{n(n+k)}), \tag{63}$$

$$\gamma = exp(\frac{2\pi i(1-n^2)}{n(n+k)}). \tag{64}$$

We note that the result makes essential use of 3-manifolds with boundary. The calculation of the coefficients α, β, γ is closely related to the Verlinde fusion rules [83] and $2d$ conformal field theories. Subistituting the values of α, β, γ into equation (61) and cancelling a common factor $exp(\frac{\pi i(2-n^2)}{n(n+k)})$, we get

$$-t^{n/2} < L_+ > + (t^{1/2} - t^{-1/2}) < L_0 > + t^{-n/2} < L_- > = 0, \tag{65}$$

where we have put

$$t = exp(\frac{2\pi i}{n+k}).$$

This is equivalent to the following skein relation for the polynomial invariant V of the link

$$t^{n/2} V_+ - t^{-n/2} V_- = (t^{1/2} - t^{-1/2}) V_0 \tag{66}$$

For $SU(2)$ Chern-Simons theory, equation (66) is the skein relation that defines a variant of the original Jones' polynomial. This variant also occurs in the work of Kirby and Melvin [36] where the invariants are studied by using representation theory of certain Hopf algebras and the topology of framed links. It is not equivalent to the Jones polynomial. In an earlier work [57] I had observed that under the transformation $\sqrt{t} \rightarrow -1/\sqrt{t}$, it goes over into the equation which is the skein relation characterizing the Jones polynomial. The Jones polynomial belongs to a different family that corresponds to the negative values of the level. Note that the coefficients in the skein relation (66) are defined for positive valuse of the level k. To extend them to negative values of the level we must also note that the shift in k by the dual Coxeter number would now change the level $-k$ to $-k-n$. If in equation (66) we now allow negative values of n and take t to be a formal variable, then the extended family includes both positive and negative levels.

Let $V^{(n)}$ denote the Jones-Witten polynomial corresponding to the skein relation (66), (with $n \in \mathbb{Z}$) then the family of polynomials $\{V^{(n)}\}$ can be shown to be equivalent to the two variable HOMFLY polynomial $P(\alpha, z)$ which satisfies the following skein relation

$$\alpha P_+ - \alpha^{-1} P_- = z P_0. \tag{67}$$

If we put $\alpha = t^{-1}$ and $z = (t^{1/2} - t^{-1/2})$ in equation (47) we get the skein relation for the original Jones polynomial V. If we put $\alpha = 1$ we get the skein relation for the Alexander-Conway polynomial.

To compare our results with those of Kirby and Melvin we note that they use q to denote our t and t to denote its fourth root. They construct a modular Hopf algebra U_t as a quotient of the Hopf algebra $U_q(sl(2,\mathbb{C}))$ which is the well known q-deformation of the universal envelopping algebra of the Lie algebra $sl(2,\mathbb{C})$. Jones polynomial and its extensions are obtained by studying the representaions of the algebras U_t and U_q.

9.4 WRT invariants

If $Z_k(1)$ exists, it provides a numerical invariant of M. For example, for $M = S^3$ and $G = SU(2)$, using the Chern-Simons action Witten obtains the following expression for this partition function as a function of the level k

$$Z_k(1) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right). \tag{68}$$

This partition function provides a new family of invariants for $M = S^3$, indexed by the level k. Such a partition function can be defined for a more general class of 3-manifolds and gauge groups. More precisely, let G be a compact, simply connected, simple Lie group and let $k \in \mathbb{Z}$. Let M be a 2-framed closed, oriented 3-manifold. We define the **Witten invariant** $\mathcal{T}_{G,k}(M)$ of the triple (M, G, k) by

$$\mathcal{T}_{G,k}(M) := Z(1) := \int_{\mathcal{B}(M)} e^{-i\mathcal{A}_{CS}} \mathcal{D}\mathcal{B}, \tag{69}$$

where $e^{-i\mathcal{A}_{CS}}\mathcal{D}\mathcal{B}$, is a suitable measure on $\mathcal{B}(M)$. We note that no precise definition of such a measure is available at this time and the definition is to be regarded as a formal expression. Indeed, one of the aims of TQFT is to make sense of such formal expressions. We define the **normalized Witten invariant** $\mathcal{W}_{G,k}(M)$ of a 2-framed, closed, oriented 3-manifold M by

$$\mathcal{W}_{G,k}(M) := \frac{\mathcal{T}_{G,k}(M)}{\mathcal{T}_{G,k}(S^3)}. \tag{70}$$

If G is a compact, simply connected, simple Lie group and M, N be two 2-framed, closed, oriented 3-manifolds. Then we have the following results:

$$\mathcal{T}_{G,k}(S^2 \times S^1) = 1 \tag{71}$$

$$\mathcal{T}_{SU(2),k}(S^3) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \tag{72}$$

$$\mathcal{W}_{G,k}(M \# N) = \mathcal{W}_{G,k}(M)\mathcal{W}_{G,k}(N) \tag{73}$$

In his work Kohno [40] defined a family of invariants $\Phi_k(M)$ of a 3-manifold M by its Heegaard decomposition along a Riemann surface Σ_g and representations of its mapping class group in the space of conformal blocks. Similar results were also obtained, independently, by Crane [16]. The agreement of these results (up to normalization) with those of Witten may be regarded as strong evidence for the usefulness of the ideas from TQFT and CFT in low dimensional geometric topology. We remark that a mathematically precise definition of the Witten invariants via solutions of the Yang-Baxter equations and representations of the corresponding quantum groups was given by Reshetikhin and Turaeev. For this reason, we now refer to them as Witten-Reshetikhin-Turaeev or WRT invariants. The invariant is well defined only at roots of unity. But in special cases it can be defined near roots of unity by a perturbative expansion in Chern-Simons theory. A similar situation occurs in the study of classical modular functions and Ramanujan's mock theta functions. Ramanujan had introduced his mock theta functions in a letter to Hardy in 1920 (the famous last letter) to describe some power series in variable $q = e^{2\pi iz}, z \in \mathbb{C}$. He also wrote down (without proof, as was usual in his work) a number of identities involving these series which were completely verified only in 1988 . Recently, Lawrence and Zagier [47] have obtained several different formulas for the Witten invariant $\mathcal{W}_{SU(2),k}(M)$ of the Poincaré homology sphere $M = \Sigma(2, 3, 5)$. They show how the Witten invariant can be extended from integral k to rational k and give its relation to the mock theta function. This extension is obtained by a mathematical procedure, Its physical meaning is not yet understood. For integral k they obtain the following fantastic formula, a la Ramanujan, for the Witten invariant of the Poincaré homology sphere

$$\mathcal{W} = 1 + \sum_{n=1}^{\infty} x^{-n^2}(1 + x)(1 + x^2) \ldots (1 + x^{n-1})$$

where $x = e^{\pi i/(k+2)}$. We note that the series on the right hand side of this formula terminates after $k + 2$ terms[1].

10 Chern-Simons and String Theory

The general question "what is the relationship between gauge theory and string theory?" is not meaningful at this time. So I will follow the strong admonition by Galileo against[2] "disputar lungamente delle massime questioni senza conseguir verità nissuna". However, interesting special cases where such relationship can be established are emerging. For example, Witten [87] has argued that Chern-Simons gauge theory on a 3-manifold M can be viewed as a string theory constructed by using a topological sigma model with target space T^*M. The perturbation theory of this string will coincide with Chern-Simons perturbation theory, in the form discussed by Axelrod and Singer [3]. The coefficient of k^{-r} in the perturbative expansion of $SU(n)$ theory in powers of $1/k$ comes from Feynman digrams with r loops. Witten shows how each diagram can be replaced by a Riemann surface Σ of

[1] I would like to thank Don Zagier for bringing this work to my attention

[2] lengthy discussions about the greatest questions that fail to lead to any truth whatever.

genus g with h holes (boundary components) with $g = (r - h + 1)/2$. Gauge theory would then give an invariant $\Gamma_{g,h}(M)$ for every topological type of Σ. Witten shows that this invariant would equal the corresponding string partition function $Z_{g,h}(M)$. We now give an example of gauge theory to string theory correspondence relating the non-perturbative WRT invariants in Chern-Simons theory with gauge group $SU(n)$ and topological string amplitudes which generalize the GW (Gromov-Witten) invariants of Calabi-Yau 3-folds. The passage from real 3 dimensional Chern-Simons theory to the 10 dimensional string theory and further onto the 11 dimensional M-theory can be schematically represented by the following:

$$
\begin{aligned}
3 + 3 &= 6 \text{ (real symplectic 6-manifold)} \\
&= 6 \text{ (conifold in } \mathbb{C}^4 \text{)} \\
&= 6 \text{ (Calabi-Yau manifold)} \\
&= 10 - 4 \text{ (string compactification)} \\
&= (11 - 1) - 4 \text{ (M-theory)}
\end{aligned}
$$

We now discuss the significance of the various terms of the above equation array. Recall that string amplitudes are computed on a 6-dimensional manifold which in the usual setting is a complex 3-dimensional Calaby-Yau manifold obtained by string compactification. This is the most extensively studied model of passing from the 10-dimensional space of supersymmetric string theory to the usual 4-dimensional space-time manifold. However, in our work we do allow these so called extra dimensions to form an open or a symplectic Calabi-Yau manifold. We call these the generalized Calabi-Yau manifolds. The first line suggests that we consider open topological strings on such a generalized Calabi-Yau manifold, namely, the cotangent bundle T^*S^3, with Dirichlet boundary conditions on the zero section S^3. We can compute the open topological string amplitudes from the $SU(n)$ Chern-Simons theory. Conifold transition [74] has the effect of closing up the holes in open strings to give closed strings on the Calabi-Yau manifold obtained by the usual string compactification from 10 dimensions. Thus we recover a topological gravity result starting from gauge theory. In fact, as we discussed earlier, Witten had anticipated such a gauge theory string theory correspondance almost ten years ago. Significance of the last line is based on the conjectured equivalence of M-theory compactified on S^1 to type IIA strings compactified on a Calabi-Yau threefold. We do not consider this aspect here. The crucial step that allows us to go from a real, non-compact, synplectic 6-manifold to a compact Calabi-Yau manifold is the conifold or geometric transition. Such a change of geometry and topology is expected to play an important role in other applications of string theory as well.

10.1 Conifold Transition

To understand the relation of the WRT invariant of S^3 for $SU(n)$ Chern-Simons theory with open and closed topological string amplitudes on "Calabi-Yau" manifolds we need to discuss the concept of conifold transition. From the geometrical point of view this corresponds to symplectic surgery in six dimensions. It replaces a vanishing Lagrangian 3-sphere by a symplectic S^2. The starting point of the construction is the observation that T^*S^3 minus its zero section is symplectomorphic to the cone $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ minus the origin in $\mathbb{C}^4$, where each manifold is taken with its standard symplectic structure. The complex singularity at the origin can be smoothed out by the manifold M_τ defined by $z_1^2 + z_2^2 + z_3^2 + z_4^2 = \tau$ producing a Lagrangian S^3 vanishing cycle. There are also two so called small resolutions $M^\pm$ of the singularity with exceptional set $\mathbb{C}P^1$.

They are defined by

$$M^\pm := \left\{ z \in \mathbb{C}^4 \mid \frac{z_1 + iz_2}{z_3 \pm iz_4} = \frac{-z_3 \pm iz_4}{z_1 - iz_2} \right\} .$$

Note that $M_0 \setminus \{0\}$ is symplectomorphic to each of $M^\pm \setminus \mathbb{C}P^1$. Blowing up the exceptional set $\mathbb{C}P^1 \subset M^\pm$ gives a resolution of the singularity which can be expressed as a fiber bundle F over $\mathbb{C}P^1$. Going from the fiber bundle T^*S^3 over S^3 to the fiber bundle F over $\mathbb{C}P^1$is referred to in the physics literature as the conifold transition. We note that the holomorphic automorphism of $\mathbb{C}^4$ given by $z_4 \mapsto -z_4$ switches the two small resolutions $M^\pm$ and changes the orientation of S^3. Conifold transition can also be viewed as an application of mirror symmetry to Calabi-Yau manifolds with singularities. Such an interpretation requires the notion of symplectic Calabi-Yau manifolds and the corresponding enumerative geometry. The geometric structures arising from the resolution of singularities in the conifold transition can also be interpreted in terms of the symplectic quotient construction of Marsden and Weinstein.

10.2 WRT Invariants and String Amplitudes

To find the relation between the large n limit of $SU(n)$ Chern-Simons theory on S^3 to a special topological string amplitude on a Calabi-yau manifold we begin by recalling the formula for the partition function (vacuum amplitude) of the theory $\mathcal{T}_{SU(n),k}(S^3)$ or simply $\mathcal{T}$. Upto a phase, it is given by

$$\mathcal{T} = \frac{1}{\sqrt{n(k+n)^{(n-1)}}} \prod_{j=1}^{n-1} \left[2 \sin \left(\frac{j\pi}{k+n} \right) \right]^{n-j} . \tag{74}$$

Let us denote by $F_{(g,h)}$ the amplitude of an open topological string theory on T^*S^3 of a Riemann surface of genus g with h holes. Then the generating function for the free energy can be expressed as

$$-\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \lambda^{2g-2+h} n^h F_{(g,h)} \tag{75}$$

This can be compared directly with the result from Chern-Simons theory by expanding the $\log \mathcal{T}$ as a double power series in λ and n.

Instead of that we use the conifold transition to get the topological amplitude for a closed string on a Calabi-Yau manifold. We want to obtain the large n expansion of this amplitude in terms of parameters λ and τ which are defined in terms of the Chern-Simons parameters by

$$\lambda = \frac{2\pi}{k+n}, \tau = n\lambda = \frac{2\pi n}{k+n}. \tag{76}$$

The parameter λ is the string coupling constant and τ is the 't Hooft coupling $n\lambda$ of the Chern-Simons theory. The parameter τ entering in the string amplitude expansion has the geometric interpretation as the Kähler modulus of a blown up S^2 in the resolved $M^{\pm}$. If $F_g(\tau)$ denotes the amplitude for a closed string at genus g then we have

$$F_g(\tau) = \sum_{h=1}^{\infty} \tau^h F_{(g,h)} \tag{77}$$

So summing over the holes amounts to filling them up to give the closed string amplitude.

The large n expansion of $\mathcal{T}$ in terms of parameters λ and τ is given by

$$\mathcal{T} = \exp\left[-\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(\tau)\right] \ , \tag{78}$$

where F_g defined in (77) can be interpreted on the string side as the contribution of closed genus g Riemann surfaces. For $g > 1$ the F_g can be expressed in terms of the Euler characteristic χ_g and the Chern class c_{g-1} of the Hodge bundle of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g as follows

$$F_g = \int_{\mathcal{M}_g} c_{g-1}^3 - \frac{\chi_g}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-n(\tau)} \ . \tag{79}$$

The integral appearing in the formula for F_g can be evaluated explicitly to give

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{(-1)^{(g-1)}}{(2\pi)^{(2g-2)}} 2\zeta(2g-2)\chi_g. \tag{80}$$

The Euler characteristic is given by the Harer-Zagier [30] formula

$$\chi_g = \frac{(-1)^{(g-1)}}{(2g)(2g-2)} B_{2g} \ , \tag{81}$$

where B_{2g} is the $(2g)$-th Bernoulli number. We omit the special formulas for the genus 0 and genus 1 cases. The formulas for F_g for $g \geq 0$ coincide with those of the g-loop topological string amplitude on a suitable Calabi-Yau manifold. The change in geometry that leads to this calculation can be thought of as the result of coupling to gravity. Such a situation occurs in the quantization of Chern-Simons theory. Here the classical Lagrangian does not depend on the metric, however, coupling to the gravitational Chern-Simons term is necessary to make it TQFT.

We have mentioned the following four approaches that lead to the WRT invariants.
1. Witten's QFT calculation of the Chern-Simons partition function
2. Quantum group (or Hopf algebraic) computations initiated by Reshetikhin and Turaev
3. Kohno's special functions corresponding to representations of mapping class groups in the space of conformal blocks and a similar approach by Crane

4. open or closed string amplitudes in suitable Calabi-Yau manifolds
These methods can also be applied to obtain invariants of links, such as the Jones polynomial. Indeed, this was the objective of Witten's original work. WRT invariants were a byproduct of this work. Their relation to topological strings came later.

The WRT to string theory correspondence has beeen extended by Gopakumar and Vafa (see, hep-th/9809187, 9812127) by using string theoretic arguments to show that the expectation value of the quantum observables defined by the Wilson loops in the Chern-Simons theory also has a similar interpretation in terms of a topological string amplitude. This leads them to conjecture a correspondence between certain knot invariants (such as the Jones polynomial) and Gromov-Witten type invariants of generalized Calabi-Yau manifolds. Gromov-Witten invariants of a Calabi-Yau 3-fold X are in general rational numbers, since one has to get the weighted count by dividing by the order of automorphism groups.. Using M-theory Gopakumar and Vafa have argued that the generating series F_X of Gromov-Witten invariants in all degrees and all genera is determined by a set of integers $n(g, \beta)$. They give the following remarkable formula for F_X

$$F_X(\lambda, q) = \sum \sum_{g \geq 0} \sum_{k \geq 1} \frac{1}{k} n(g, \beta)(2sin(k\lambda/2))^{2g-2} q^{k\beta},$$

where λ is the string coupling constant and the first sum is taken over all nonzero elements β in $H_2(X)$. We note that for a fixed genus there are only finitely many nonzero integers $n(g, \beta)$. A mathematical formulation of the Gopakumar-Vafa conjecture (GV conjecture) has been given in [66]. Special cases of the conjecture have been verified (see, for example [67] and references therein). In [48] a new geometric approach relating the Gromov-Witten invariants to equivariant index theory and 4-dimensional gauge theory has been used to compute the string partition functions of some local Calabi-Yau spaces and to verify the GV conjecture for them.

A knot should correspond to a Lagrangian D-brane on the string side and the knot invariant would then give a suitably defined count of compact holomorphic curves with boundary on the D-brane. To understand a propossed proof, recall first that a categorification of an invariant I is the construction of a suitable homology such that its Euler characteristic equals I. A well known example of this is Floer's categorification of the Casson invariant. We have already discussed earlier, Khovanov's categorification of the Jones polynomial $V_\kappa(q)$ by constructing a bi-graded $sl(2)$-homology $H_{i,j}$ determined by the knot κ. Its quantum or graded Euler characteristic equals the Jones polynomial. i.e.

$$V_\kappa(q) = \sum_{i,j} (-1)^j q^i \dim H_{i,j} \ .$$

Now let L_κ be the Lagrangian submanifold corresponding to the knot κ of a fixed Calabi-Yau space X. Let r be a fixed relative integral homology class of the pair (X, L_κ). Let $\mathcal{M}_{g,r}$ denote the moduli space of pairs (Σ_g, A), where Σ_g is a compact Riemann surface in the class r with boundary S^1 and A is a flat $U(1)$ connection on Σ_g. This data together with the cohomology groups $H^k(\mathcal{M}_{g,r})$ determines a tri-graded homology. It generalizes the Khovanov homology. Its Euler characteristic is a generating function for the BPS states' invariants in string theory and these can be used to obtain the Gromov-Witten invariants. Taubes has given a construction of the Lagrangians in the Gopakumar-Vafa conjecture. We note that counting holomorphic curves with boundary on a Lagrangian manifold was introduced by Floer in his work on the Arnold conjecture.

The tri-graded homology is expected to unify knot homologies of the Khovanov type as well as knot Floer homology constructed by Ozswáth and Szabó [65] which provides a categorification of the Alexander polynomial. Knot Floer homology is defined by counting pseudo-holomorphic curves and has no known combinatorial description. An explicit construction of a tri-graded homology for certain torus knots has been recently given by Dunfield, Gukov and Rasmussen [math.GT/0505662].

11 Yang-Mills, Gravity and Strings

Recall that in string theory, an elementary particle is identified with a vibrational mode of a string. Different particles correspond to different harmonics of vibration. The Feynman diagrams of the usual QFT are replaced by fat graphs or Riemann surfaces that are generated by moving strings splitting or joining together. The particle interactions described by these Feynman diagrams are built into the basic structure of string theory. The appearance of Riemann surfaces expalains the relation to conformal field theory. We have already discussed Witten's argument relating gauge and string theories. It now forms a small part of the program of relating quantum group invariants and topological string amplitudes. In general, the string states are identified with fields. The ground state of the closed string turns out to be a massless spin two field which may be interpreted as a graviton. In the large distance limit, (at least at the lower loop levels) string theory includes the vacuum equations of Einstein's general relativity theory. String theory avoids the ultraviolet divergences that appear in conventional attempts at quantizing gravity. In physically interesting string models one expects the string space to be a non-trivial bundle over a Lorentzian space-time M with compact or non-compact fibers.

11.1 Yang-Mills and Gravity

In 1954 Yang and Mills [90, 89] obtained the following now well-known non-abelian gauge field equations for the vector potential b_μ of isotopic spin in interaction with a field ψ of isotopic spin 1/2:

$$\partial f_{\mu\nu}/\partial x_\nu + 2\epsilon(b_\nu \times f_{\mu\nu}) + J_\mu = 0$$

where the quantities

$$f_{\mu\nu} = \partial b_\mu/\partial x_\nu - \partial b_\nu/\partial x_\mu - 2\epsilon b_\mu \times b_\nu,$$

are the components of an $SU(2)$-gauge field and J_μ is the current density of the source field ψ. There was no immediate physical application of these equations since they seemed to predict massless gauge particles as in Maxwell's theory. In Maxwell's theory the predicted massless particle is identified as photon, the massless carrier of electromagnetic field. No such identification could be made for the massless particles predicted by Yang-Mills theory. This difficulty is overcome by the introduction of the Higgs mechanism [31] which shows how spontaneous symmetry breaking can give rise to massive gauge vector bosons. This paved the way for a gauge theoretic formulation of the standard model of electroweak theory and subsequent development of the general framework for a unified treatment of strong, weak and electromagnetic interactions. The Yang-Mills equations may be thought of as a matrix valued generalization of the equations for the classical vector potential of Maxwell's theory. The gauge field that they obtained turns out to be the curvature of a connection in a principal fiber bundle with gauge group $SU(2)$. The general theory of such connections was developed in 1950 by Ehresman, following the fundamental work of Elie Cartan. However, physicists continued to use the classical theory of connections and curvature that was the cornerstone of Einstein's general relativity theory. In fact, using this classical theory, Ikeda and Miyachi[3] [32, 33] essentially linked the Yang-Mills theory with the theory of connections. However, this work does not seem to be well known in the mathematical physics community and it was not until the early seventies that the identification between curvature of a connection in a principal bundle and the gauge field was made in [88]. This identification unleashed a flurry of activity among both physicists and mathematicians and has already had great successes some of which we have discussed in this paper. The physical literature on gauge theory is vast and is, in general, aimed at applications to elementary-particle physics and quantum field theories. The most successful quantum field theory is quantum electrodynamics (QED) which deals with quantization of electromagnetic fields. Its predictions have been verified to a very high degree of accuracy. However, there is as yet no generally accepted mathematical theory of quantization of non-abelian gauge fields.

Relating the usual Einstein's equations with cosmological constant with the Yang-Mills equations requires the ten dimensional manifold $\Lambda^2(M)$ of differential forms of degree two. There are several differences between the Riemannian functionals used in theories of gravitation and the Yang-Mills functional used to study gauge field theories. The most important difference is that the Riemannian functionals are dependent on the bundle of frames of M or its reductions, while the Yang-Mills functional can be defined on any principal bundle over M. However, we have the following interesting theorem [6].

Theorem: Let (M, g) be a compact, 4-dimensional, Riemannian manifold. Let $\Lambda^2_+(M)$ denote the bundle of self-dual 2-forms on M with induced metric G_+. Then the Levi-Civita connection λ_g on M satisfies the Euclidean gravitational instanton equations if and only if the Levi-Civita connection λ_{G_+} on $\Lambda^2_+(M)$ satisfies the Yang-Mills instanton equations.

11.2 Gravitational Field Equations

There are several ways of deriving Einstein's gravitational field equations. For example, we can consider natural tensors satisfying the conditions that they contain derivatives of the fundamental (pseudo-metric) tensor up to

[3]I would like to thank Prof. Akira Asada for introducing me to Prof. Miyachi and his work.

order two and depend linearly on the second order derivatives. Then we obtain the tensor

$$c_1 R^{ij} + c_2 g^{ij} S + c_3 g^{ij},$$

where R^{ij} are the components of the Ricci tensor Ric and S is the scalar curvature. Requiring this tensor to be divergenceless and using the Bianchi identities leads to the relation $c_1 + 2c_2 = 0$ between the constants $c_1,\ c_2,\ c_3$. Choosing $c_1 = 1$ and $c_3 = 0$ we obtain Einstein's equations (without the cosmological constant) which may be expressed as

$$E = -T \tag{82}$$

where $E := Ric - \frac{1}{2}Sg$ is the **Einstein tensor** and T is an energy-momentum tensor on the space-time manifold which acts as the source term. Now the Bianchi identities satisfied by the curvature tensor imply that

$$div\ E := \nabla_i E^{ij} = 0.$$

Hence, if Einstein's equations (82) are satisfied, then for consistency we must have

$$div\ T = \nabla_i T^{ij} = 0. \tag{83}$$

Equation (83) is called the differential (or local) law of conservation of energy and momentum. However, integral (or global) conservation laws can be obtained by integrating equation (83) only if the space-time manifold admits Killing vectors. Thus equation (83) has no clear physical meaning, except in special cases. An interesting discussion of this point is given by Sachs and Wu [69]. Einstein was aware of the tentative nature of the right hand side of equation (82), but he believed strongly in the expression on the left hand side of (82). By taking the trace of both sides of equations (82) we are led to the condition

$$S = t \tag{84}$$

where t denotes the trace of the energy-momentum tensor. The physical meaning of this condition seems even more obscure than that of condition (83). If we modify equation (82) by adding the cosmological term Λg (Λ is called the **cosmological constant**) to the left hand side of equation (82), we obtain Einstein's equation with cosmological constant

$$E + \Lambda g = -T. \tag{85}$$

This equation also leads to the consistency condition (83), but condition (84) is changed to

$$S = t + 4\Lambda. \tag{86}$$

Using (86), equation (85) can be rewritten in the following form

$$K = -(T - \frac{1}{4}tg), \tag{87}$$

where

$$K = (Ric - \frac{1}{4}Sg) \tag{88}$$

is the trace-free part of the Ricci tensor of g. We call equations (87) **generalized field equations** of gravitation. We now show that these equations arise naturally in a geometric formulation of Einstein's equations. We begin by defining a tensor of curvature type.

Let C be a tensor of type $(4,0)$ on M. We can regard C as a quadrilinear mapping (pointwise) so that for each $x \in M$, C_x can be identified with a multilinear map

$$C_x : T_x^*(M) \times T_x^*(M) \times T_x^*(M) \times T_x^*(M) \to \mathbb{R}.$$

We say that the tensor C is of curvature type if C_x satisfies the following conditions for each $x \in M$ and for all $\alpha, \beta, \gamma, \delta \in T_x^*(M)$.

1. $C_x(\alpha, \beta, \gamma, \delta) = -C_x(\beta, \alpha, \gamma, \delta)$;
2. $C_x(\alpha, \beta, \gamma, \delta) = -C_x(\alpha, \beta, \delta, \gamma)$;
3. $C_x(\alpha, \beta, \gamma, \delta) + C_x(\alpha, \gamma, \delta, \beta) + C_x(\alpha, \delta, \gamma, \beta) = 0$.

From the above definition it follows that a tensor C of curvature type also satisfies the following condition:

$$C_x(\alpha, \beta, \gamma, \delta) = C_x(\gamma, \delta, \alpha, \beta), \ \forall x \in M.$$

We denote by $\mathcal{C}$ the space of all tensors of curvature type. The Riemann-Christoffel curvature tensor Rm is of curvature type. Indeed, the definition of tensors of curvature type is modelled after this fundamental example. Another important example of a tensor of curvature type is the tensor G defined by

$$G_x(\alpha, \beta, \gamma, \delta) = g_x(\alpha, \gamma)g_x(\beta, \delta) - g_x(\alpha, \delta)g_x(\beta, \gamma), \ \forall x \in M \tag{89}$$

where g is the fundamental or metric tensor of M.

We now define the curvature product of two symmetric tensors of type $(2,0)$ on M. It was introduced by the author in [50] and used in [54] to obtain a geometric formulation of Einstein's equations.

Let g and T be two symmetric tensors of type $(2,0)$ on M. The **curvature product** of g and T, denoted by $g \times_c T$, is a tensor of type $(4,0)$ defined by

$$\begin{aligned}(g \times_c T)_x(\alpha, \beta, \gamma, \delta) := \quad & \tfrac{1}{2}[g(\alpha,\gamma)T(\beta,\delta) + g(\beta,\delta)T(\alpha,\gamma) \\ & -g(\alpha,\delta)T(\beta,\gamma) - g(\beta,\gamma)T(\alpha,\delta)],\end{aligned}$$

for all $x \in M$ and $\alpha, \beta, \gamma, \delta \in T^*_x(M)$.

In the following proposition we collect together some important properties of the curvature product and tensors of curvature type.

Proposition 3 *) Let g and T be two symmetric tensors of type $(2,0)$ on M and let C be a tensor of curvature type on M. Then we have the following:*

1. *$g \times_c T = T \times_c g$.*
2. *$g \times_c T$ is a tensor of curvature type.*
3. *$g \times_c g = G$, where G is the tensor defined in (89).*
4. *G_x induces a pseudo-inner product on $\Lambda^2_x(M), \forall x \in M$.*
5. *C_x induces a symmetric, linear transformation of $\Lambda^2_x(M), \forall x \in M$.*

The orthogonal group $O(g)$ of the metric acts on the space $\mathcal{C}$ and splits it into three irreducible subspaces of dimensions 10, 9, and 1. Under this splitting the Riemann curvatureR Rm into three parts as follows:

$$Rm = W + c_1(K \times_c g) + c_2 S(g \times_c g).$$

The ten dimensional part W is the Weyl conformal curvature tensor. It splits further into its self-dual part W_+ and anti-dual part W_- under the action of $SO(g)$. The part involving the trace-free Ricci tensor K is 9 dimensional. All of these tensors occur in functionals on the space of metrics.

We denote the Hodge star operator on $\Lambda^2_x(M)$ by J_x. The fact that M is a Lorentz 4-manifold implies that J_x defines a complex structure on $\Lambda^2_x(M)$, $\forall x \in M$. Using this complex structure we can give a natural structure of a complex vector space to $\Lambda^2_x(M)$. Then we have the following proposition.

Proposition 4 *Let* $U : \Lambda^2_x(M) \to \Lambda^2_x(M)$ *be a real, linear transformation. Then the following are equivalent:*

1. *L commutes with* J_x*.*
2. *L is a complex linear transformation of the complex vector space* $\Lambda^2_x(M)$*.*
3. *The matrix of L with respect to a* G_x*-orthonormal basis of* $\Lambda^2_x(M)$ *is of the form*
$$\begin{pmatrix} A & B \\ \text{-}B & A \end{pmatrix} \tag{90}$$
where A, B *are real* 3×3 *matrices.*

We now define the gravitational tensor W_{gr}, of curvature type, which includes the source term. Let M be a space-time manifold with fundamental tensor g and let T be a symmetric tensor of type $(2, 0)$ on M. Then the **gravitational tensor** W_{gr} is defined by

$$W_{gr} := Rm + g \times_c T, \tag{91}$$

where Rm is the Riemann-Christoffel curvature tensor of type $(4, 0)$.

We are now in a position to give a geometric formulation of the generalized field equations of gravitation.

Theorem 5 *Let* W_{gr} *denote the gravitational tensor defined by (91) with source tensor* T*. We denote by* $\hat{W}_{gr}$ *the linear transformation of* $\Lambda^2_x(M)$ *induced by* W_{gr}*. Then the following are equivalent:*

1. *g satisfies the generalized field equations of gravitation (87);*
2. $\hat{W}_{gr}$ *commutes with* J_x*;*
3. $\hat{W}_{gr}$ *is a complex linear transformation of the complex vector space* $\Lambda^2_x(M)$*.*

We shall call the triple (M, g, T) a **generalized gravitational field** if any one of the conditions of Theorem 5 is satisfied. Generalized gravitational field equations were introduced by the author in [50]. Their mathematical properties have been studied in [56, 51, 62]. In local coordinates, the generalized gravitational field equations can be written as

$$R^{ij} - \frac{1}{4}Rg^{ij} = -(T^{ij} - \frac{1}{4}Tg^{ij}). \tag{92}$$

We observe that the equation (92) does not lead to any relation between the scalar curvature and the trace of the source tensor, since both sides of equation (92) are trace-free. Taking divergence of both sides of equation (92) and using the Bianchi identities we obtain the generalized conservation condition

$$\nabla_i T^{ij} - g^{ij}\Phi_i = 0, \tag{93}$$

where ∇_i is the covariant derivative with respect to the vector $\frac{\partial}{\partial x^i}$,

$$\Phi = \frac{1}{4}(T - R) \tag{94}$$

and $\Phi_i = \frac{\partial}{\partial x^i}\Phi$. Using the function Φ defined by equation (94), the field equations can be written as

$$R^{ij} - \frac{1}{2}Rg^{ij} - \Phi g^{ij} = -T^{ij}. \tag{95}$$

In this form the new field equations appear as Einstein's field equations with the cosmological constant replaced by the function Φ, which we may call the cosmological function. The cosmological function is intimately connected with the classical conservation condition expressing the divergence-free nature of the energy-momentum tensor as is shown by the following proposition.

Proposition 6 *The energy-momentum tensor satisfies the classical conservation condition*

$$\nabla_i T^{ij} = 0 \tag{96}$$

if and only if the cosmological function Φ is a constant. Moreover, in this case the generalized field equations reduce to Einstein's field equations with cosmological constant.

We note that, if the energy-momentum tensor is non-zero but is localized in the sense that it is negligible away from a given region, then the scalar curvature acts as a measure of the cosmological constant. By setting the energy-momentum tensor to zero in (92) we obtain various characterizations of the usual gravitational instanton. Solutions of the generalized gravitational field equations which are not solutions of Einstein's equations have been discussed in [12].

We note that the theorem (5) and the last condition in proposition (3) can be used to discuss the Petrov classification of gravitational fields (see Petrov

[68]). The tensor W_{gr} can be used in place of R in the usual definition of sectional curvature to define the gravitational sectional curvature on the Grassmann manifold of non-degenerate 2-planes over M and to give a further geometric characterization of gravitational field equations. We observe that the generalized field equations of gravitation contain Einstein's equations with or without the cosmological constant as special cases. Solutions of the source-free generalized field equations are called **gravitational instantons** If the base manifold is Riemannian, then gravitational instantons correspond to Einstein spaces. A detailed discussion of the structure of Einstein spaces and their moduli spaces may be found in [6]. Over a compact, 4-dimensional, Riemannian manifold (M, g), the gravitational instantons that are not solutions of the vacuum Einstein equations are critical points of the quadratic, Riemannian functional or action $\mathcal{A}_2(g)$ defined by

$$\mathcal{A}_2(g) = \int_M S^2 dv_g.$$

Furthermore, the standard **Hilbert-Einstein action**

$$\mathcal{A}_1(g) = \int_M S dv_g$$

also leads to the generalized field equations when the variation of the action is restrcted to metrics of volume 1.

11.3 Geometrization Conjecture and Gravity

The classification problem for low dimensional manifolds is a natural question after the success of the case of surfaces by the uniformization theorem . In 1905, Poincaré formulated his famous conjecture which states in the smooth case: A closed, simply-connected 3-manifold is diffeomorphic to S^3, the standard sphere. A great deal of work in three dimensional topology in the next 100 years was motivated by this. In the 1980s Thurston studied hyperbolic manifolds. This led him to his "Geometrization Conjecture" about the existence of homogeneous metrics on all 3-manifolds. It includes the Poincaré conjecture as a special case. In the case of 4-manifolds, there is at present no analogue of the geometrization conjecture. We discuss briefly the current state of these problems in the next two subsections.

11.3.1 The 3-manifold case

: The Ricci flow equations

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

for a Riemannian metric g were introduced by Hamilton in [28]. They form a system of nonlinear second order partial differential equations. Hamilton proved that this equation has a unique solution for a short time for any smooth metric on a closed manifold. The evolution equation for the metric leads to the evolution equations for the curvature and Ricci tensors and for the scalar curvature. By developing a maximum principle for tensors, Hamilton proved that the Ricci flow preserves the positivity of the Ricci tensor in dimension three and that of the curvature operator in dimension four [29]. In each of these cases he proved that the evolving metrics converge to metrics of constant positive curvature (modulo scaling). These and a series of further papers led him to conjecture that the Ricci flow with surgeries could be used to prove the Thruston geometrization conjecture. In a series of e-prints Perelman developed the essential framework for implementing the Hamilton program. We would like to add that the full Einstein equations with dilaton field as source play a fundamental role in Perelman's work (see, arXiv.math.DG/0211159, 0303109, 0307245) on the geometrization conjecture. A corrolary of this work is the proof of the long standing Poincaré conjecture. A complete proof of the geometrization conjecture by applying the Hamilton-Perelman theory of the Ricci flow has just appeared in [13] in a special issue dedicated to the memory of S.-S. Chern,[4] one of the greatest mathematicians of the twentieth century.

The Ricci flow is perturbed by a scalar field which corrosponds in string theory to the dilaton. It is suppossed to determine the overall strength of all interactions. The low energy effective action of the dilaton field coupled to gravity is given by the action functional

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dv \ .$$

[4] I first met Prof. Chern and his then newly arrived student S.-T. Yau in 1973 at the AMS summer workshop on differential geometry held at Stanford University. He was a gourmet and his conference dinners were always memorable. I attended the first one in 1973 and the last one in 2002 on the occassion of the ICM satelite conference at his institute in Tianjin. Inspite of his advanced age and poor health he participated in the entire program and then continued with his duties as President of the ICM in Beijing.

Note that when f is the constant function the action reduces to the classical Hilbert-Einstein action. The first variation can be written as

$$\delta\mathcal{F}(g,f) = \int_M [-\delta g^{ij}(R_{ij}+\nabla_i f\nabla_j f)+(\frac{1}{2}\delta g^{ij}(g_{ij}-\delta f)(2\Delta f-|\nabla f|^2+R)]dm\ ,$$

where $dm = e^{-f}dv$. If $m = \int_M e^{-f}dv$ is kept fixed, then the second term in the variation is zero and then the symmetric tensor $-(R_{ij}+\nabla_i f\nabla_j f)$ is the L^2 gradient flow of the action functional $\mathcal{F}^m = \int_M (R+|\nabla f|^2)dm$. The choice of m is similar to the choice of a gauge. All choices of m lead to the same flow, up to diffeomorphism, if the flow exists. We remark that in the quantum field theory of the two-dimensional nonlinear σ-model, Ricci flow can be considered as an approximation to the renormalization group flow. This suggests gradient-flow like behaviour for the Ricci flow, from the physical point of view. Perelman's calculations confirm this result. The functional $\mathcal{F}^m$ has also a geometric interpretation in terms of the classical Bochner-Lichnerowicz formulas with the metric measure replaced by the dilaton twisted measure dm.

The corresponding variational equations are

$$R_{ij} - \frac{1}{2}Rg_{ij} = -(\nabla_i\nabla_j f - \frac{1}{2}(\Delta f)g_{ij}).$$

These are the usual Einstein equations with the energy-momentum tensor of the dilaton field as source. They lead to the decoupled evolution equations

$$(g_{ij})_t = -2(R_{ij}+\nabla_i\nabla_j f), f_t = -R - \Delta f.$$

After applying a suitable diffeomorphism these equations lead to the gradient flow equations. This modified Ricci flow can be pushed through the singularities by surgery and rescaling. A detailed case by case analysis is then used to prove Thurston's geometrization conjecture. This includes as a special case the classical Poincaré conjecture.

11.3.2 The 4-manifold case

In the category of topological manifolds a complete classification of closed, 1-connected, oriented 4-manifolds was carried out by Freedman (see [25]) in 1981. To state his results we begin by recalling the general scheme of classification of symmetric, non-degenerate, unimodular, bilinear forms (referred to simply as forms in the rest of this section) on lattices (see Milnor and Husemoller [61]). The classification of forms has a long history and is an important area of classical mathematics with applications to algebra, number theory and more recently to topology and geometry. Each form has a set of invariants, its rank b_n, signature (b_n^+, b_n^-)(also defined as the integer $\sigma(M) := b_n^+ - b_n^-$) and type. Recall that a form ι on the lattice L is **even** or of **type II** if $\iota(a, a)$ is even for all $a \in L$. Otherwise we say that it is **odd** or of **type I**. It can be shown that for even (type II) forms the signature $\sigma(M)$ is divisible by 8. In particular, the rank of a positive definite even form is divisible by 8. The indefinite forms are completely classified by the rank, signature and type. The classification of definite forms is much more involved. The number $N(r)$ of equivalence classes of definite forms (which counts the inequivalent forms) grows very rapidly with the rank r of the form. For example, $N(8) = 1, N(16) = 2, N(24) = 24, N(32) > 10^7$.

Theorem 7 *Let $\mathcal{M}_{sp}$ (resp. $\mathcal{M}_{ns}$) denote the set of topological equivalence classes (i.e. homeomorphism classes) of closed,* 1*-connected, oriented, spin (resp. non-spin)* 4*-manifolds. Let $\mathcal{I}_{ev}$ (resp. $\mathcal{I}_{od}$) denote the set of equivalence classes of even (resp. odd) forms. Then we have the following:*

1. *the map $\iota : \mathcal{M}_{sp} \to \mathcal{I}_{ev}$ is bijective;*
2. *the map $\iota : \mathcal{M}_{ns} \to \mathcal{I}_{od}$ is surjective and is exactly two-to-one. The two classes in the preimage of a given form are distinguished by a cohomology class $\kappa(M) \in H^4(M; \mathbb{Z}_2)$ called the Kirby-Siebenmann invariant.*

We note that the **Kirby-Siebenmann invariant** represents the obstruction to the existence of a piecewise linear structure on a topological manifold of dimension at least four. Applying the above theorem to the empty rank zero form provides a proof of the 4-dimensional Poincaré Conjecture. Freedman's theorem is regarded as one of the fundamental results of modern topology.

In the smooth category the situation is much more complicated. It is well known that the map ι is not surjective in this case. In fact, we have the following theorem:

Theorem 8 *(Rochlin) Let M be a smooth, closed,* 1*-connected, oriented, spin manifold of dimension* 4*. Then $\sigma(M)$, the signature of M, is divisible by* 16*.*

Now as we observed earlier, the signature of an even form is always divisible by 8, but it need not be divisible by 16. Thus we can define the **Rochlin invariant** $\rho(\mu)$ of an even form μ by

$$\rho(\mu) := \frac{1}{8}\sigma(\mu) \pmod 2.$$

We note that the Rochlin invariant and the Kirby-Siebenmann invariant are equal in this case, but for non-spin manifolds the Kirby-Siebenmann invariant is not related to the intersection form and thus provides a further obstruction to smoothability. From Freedman's classification and Rochlin's theorem it follows that a topological manifold with non-zero Rochlin invariant is not smoothable. For example, the topological manifold corresponding to the Cartan matrix of the exceptional Lie group E_8 has signature 8 (Rochlin invariant 1) and hence is not smoothable.

Until recently very little progress was made beyond the result of the above theorem in the smooth category. Then in 1982 through his study of the topology and geometry of the moduli space of instantons on 4-manifolds Donaldson discovered the following unexpected result. Donaldson's theorem has led to a number of important new results including the existence of uncountably many exotic differentiable structures on the standard Euclidean topological space $\mathbb{R}^4$.

Theorem 9 *(Donaldson) Let M be a smooth, closed, 1-connected, oriented manifold of dimension 4 with positive definite intersection form ι_M. Then $\iota_M \cong b_2(1)$, the diagonal form of rank b_2, the second Betti number of M.*

Donaldson's work uses in an essential way the solution space of the Yang-Mills instanton equations for $SU(2)$ gauge group and has already had profound influence on the applications of physical theories to mathematical problems. It is reasonable to say that a new branch of mathematics which may be called "Physical Mathematics" has been created. As I remarked in the introduction, I have been using this name in my work for over ten years.

We now give a brief account of some implications of Donaldson's theorem for the classification of smooth 4-manifolds. In the first part of this section we have discussed the classification of a class of topological 4-manifolds. In addition to these there are two other categories of manifolds that topologists are interested in studying. These are the category PL of piecewise linear manifolds and the category $DIFF$ of smooth manifolds. We have the inclusions

$$DIFF \subset PL \subset TOP.$$

In general, these are strict inclusions, however, it is well known that every piecewise linear 4-manifold carries a unique smooth structure compatible with its piecewise linear structure. Thus in this case the Kirby-Siebenman invariant (and its ancestor, the Rochlin invariant) was the only known obstruction to smoothability of a topological 4-manifold. Theorem 9 combined with Freedman's classification of topological 4-manifolds gives many examples of non-smoothable 4-manifolds with zero Kirby-Siebenman invariant. In fact, we have the following classification of smooth 4-manifolds upto homeomorphism.

Theorem 10 *Let M be a smooth, closed, 1-connected, oriented 4-manifold. Let $\cong_h$ denote the relation of homeomorphism and let ι_M denote the intersection form of M. If $\iota_M = \emptyset$ then $M \cong_h S^4$ and if $\iota_M \neq \emptyset$ then we have the following cases:*

1. *M is non-spin with odd intersection form*

$$\iota_M \cong j(1) \oplus k(-1), \quad j, k \geq 0 \text{ and } M \cong_h j(\mathbb{C}P^2)\#k(\overline{\mathbb{C}P}^2).$$

i.e. M is homeomorphic to the connected sum of j copies of $\mathbb{C}P^2$ and k copies of $\overline{\mathbb{C}P}^2$ ($\mathbb{C}P^2$ with the opposite complex structure and orientation).

2. *M is spin with even intersection form*

$$\iota \cong m\sigma_1 \oplus pE_8 \quad m, p \geq 0 \text{ and } M \cong_h m(S^2 \times S^2)\#p(|E_8|).$$

i.e. M is homeomorphic to the connected sum of m copies of $S^2 \times S^2$ and p copies of $|E_8|$, the unique topological manifold corresponding to the intersection form E_8.

In [19] Donaldson used the moduli spaces of instantons to define a new set of invariants of M which can be regarded as polynomials on the second homology $H_2(M)$. In [43] Kronheimer and Mrowka obtained a structure theorem for the Donaldson invariants in terms of their basic classes and introduced a technical property called simple type for a closed, simply connected 4-manifold M. Then in 1993 the Seiberg-Witten (SW) equations were obtained as a by product of the study of super Yang-Mills equations. These

equations are defined using a $U(1)$ monopole gauge theory and the Dirac operator obtained by coupling to a $Spin^c$ structure. In [86], Witten used the moduli space of solutions of SW equations to define the SW invariants. The structure of this SW modli space is much simpler than the instanton moduli spaces used to define Donaldson's polynomial invariants. This has led to a number of new results that had met with insurmountable difficulties in the Donaldson theory (see, for example, [44, 78]). There is also a simple type condition and basic classes in SW theory. Witten used the idea of taking ultraviolate and infrared limits of $N = 2$ supersymmetric quantum Yang-Mills theory and metric independence of correlation functions to relate D and SW invariants. The precise form of Witten's conjecture can be expressed as follows:
A closed, simply connected 4-manifold M has KM-simple type if and only if it has SW-simple type. If M has simple type and if $D(\alpha)$ (resp. $SW(\alpha)$) denote the generating function series for the Donaldson (resp. Seiberg-Witten) invariants with $\alpha \in H_2(M; \mathbb{R})$, then we have

$$D(\alpha) = 2^c e^{\iota_M(\alpha)/2} SW(\alpha), \forall \alpha \in H_2(M; \mathbb{R}).$$

In the above formula ι_M is the intersection form of M and c is a constant given by

$$c = 2 + \frac{7\chi(M) + 11\sigma(M)}{4}.$$

A mathematical approach to a proof of Witten's conjecture was proposed by Pidstrigatch and Tyurin (see, dg-ga/9507004). In a series of papers, Feehan and Lenses (see, [21] and references therein) have used similar ideas but employ an $SO(3)$ monopole gauge theory which generalizes both the instanton and $U(1)$ monopole theories. The problem of relating this proof to Witten's TQFT argument remains open.

Inspite of these impressive new developments, there is at present no analogue of the geometrization conjecture in the case of 4-manifolds. Here geometric topologists are studying the variational problems on the space of metrics on a closed, oriented 4-manifold M for one of the classical curvature functionals such as the square of the L^2 norm of the Riemann curvature Rm, Weyl conformal curvature W and its self-dual and anti-dual parts W_+ and W_- respectively, and Ric, the Ricci curvature. We have already discussed earlier the role of the scalar curvature functionals in the study of gravitational field equations. Einstein metrics, i.e. metrics satisfying the equation

$$K := Ric - \frac{1}{4}Rg = 0$$

are critical points of all of the functionals listed above. Here K is the trace-free part of the Ricci tensor. In many cases, the Einstein metrics are minimizers but there are large classes of minimizers which are not Einstein. A well known obstruction to the existence of Einstein metrics is the Hitchin-Thorpe inequality $\chi(M) \geq \frac{3}{2}|\tau(M)|$ where $\chi(M)$ is the Euler characteristic and $\tau(M)$ is the signature of M. A number of new obstructions are now known. Some of these indicate that their existence may depend on the smooth structure of M as opposed to just the topological structure. These obstructions can be interpreted as implying a coupling of matter fields to gravity (see, [52, 53, 17]). The basic problem is to understand the existence and moduli spaces of these metrics on a given manifold and perhaps to find a geometric decomposition of M with respect to a special functional. One of the most important tools for developing such a theory is the Chern-Gauss-Bonnet theorem which states that

$$\chi(M) = \frac{1}{8\pi^2}\int(|Rm|^2 - |K|^2)dv = \frac{1}{8\pi^2}\int(|W|^2 - \frac{1}{2}|K|^2 + \frac{1}{24}R^2)dv$$

This result allows one to control the full Riemann curvature in terms of the Ricci curvature Ric. It is interesting to note that in [45], Lanczos had arrived at the same result while searching for Lagrangians to generalize Einstein's gravitational field equations. He noted the curious property of the Euler class that it contains no dynamics (or is an invariant). He had thus obtained the first topological gravity invariant (without realizing it). Chern's fundamental paper [15], appeared in the same journal seven years later. Chern-Weil theory and Hirzebruch's signature theorem give the following expression for the signature $\tau(M)$

$$\tau(M) = \frac{1}{12\pi^2}\int(|W_+|^2 - |W_-|^2)dv.$$

The Hitchin-Thorpe inequality follows from this result and the Chern-Gauss-Bonnet theorem. In dimension 4, all the classical functionals are conformally (or scale) invariant, so it is customery to work with the spce of unit volume metrics on M.

12 Concluding Remarks

We have seen that QFT calculations have their counterparts in string theory. One can speculate that this is a topological quantum gravity (TQG) interpretation of a result in TQFT, in the Euclidean version of the theories. If modes of vibration of a string are identified with fundamental particles, then their interactions are already built into the theory. Consistency with known physical theories requires string theory to include supersymmetry. While supersymmetry has had great success in mathematical applications, its physical verification is not yet available. However, there are indications that it may be the theory that unifies fundamental forces in the standard model at energies close to those at currently existing and planned accelarators. Perterbative supersymmetric string theory (at least up to lower loop levels) avoids the ultraviolet divergences that appear in conventional attempts at quantizing gravity. Recent work relating the Hartle-Hawking wave function to string partition function can be used to obtain a wave function for the metric fluctuations on S^3 embeded in a Calabi-Yau manifold. This may be a first step in a realistic quantum cosmology relating the entropy of certain black holes with the top[ological string wave function. While a string theory model unifying all fundamental forces is not yet available, a number of small results (some of which we have discussed in this paper) are emerging to suggest that supersymmetric string theory could play a fundamental role in constructing such a model. Developing a theory and phenomenology of 4-dimensional string vacua and relating them to experimental physics and cosmological data would be a major step in this direction. New mathematical ideas may be needed for the completion of this project.

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