A walk through integrally closed domains.

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Let R be an integrally closed domain with quotient field K and θ be an element of an integral domain containing R with θ integral over R and $F(x)$ be the minimal polynomial of θ over K. Kummer proved that if $R[\theta]$ is an integrally closed domain then the maximal ideals of $R[\theta]$ which lie over $\mathfrak{p}$ can be explicitly determined from the irreducible factors of $F(x)$ modulo $\mathfrak{p}$. We shall discuss a necessary and sufficient criterion to be satisfied by $F(x)$ so that $R[\theta]$ is integrally closed when R is a valuation ring. We shall also give some applications of this criterion for algebraic number fields and derive necessary and sufficient conditions involving only the primes dividing a, b, m, n for $\mathbb{Z}[\theta]$ to be integrally closed when θ is a root of an irreducible trinomial $x^n + ax^m + b$ with integer coefficients. For any pair of algebraic number fields K_1, K_2 linearly disjoint over $K_1 \cap K_2$, we shall show that the relative discriminants of K_1/K and K_2/K to be coprime if and only if the composite ring $A_{K_1}A_{K_2}$ is integrally closed, A_{K_i} being the ring of algebraic integers of K_i. This provides converse of a well known result in algebraic number theory and will be discussed in a more general setting.