On the bipartite distance matrix and the bipartite Laplacian matrix

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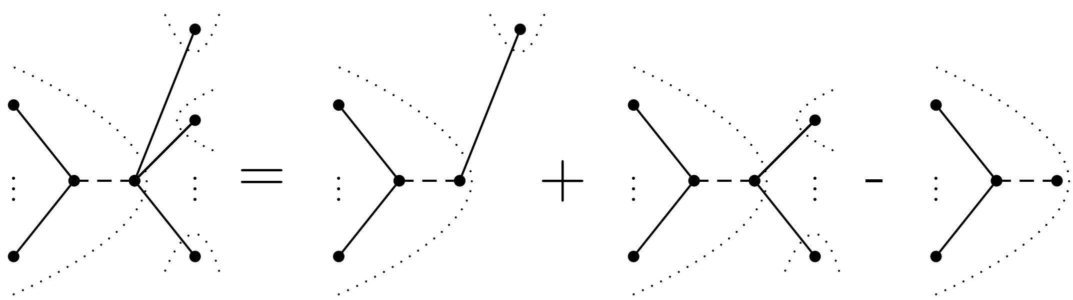
Abstract

All graphs considered here are simple and finite. The study of the properties of graph via matrices is widely studied subject that ties together two seemingly unrelated branches of mathematics; graph theory and linear algebra. For a bipartite graph, we do not require the complete adjacency matrix to display the adjacency information in the graph. Godsil in (*Inverses of trees*, Combinatorica, 5(1):33-39, 1985), used a smaller size matrix to display this information. Later on this matrix was named the bipartite adjacency matrix. The bipartite adjacency matrix (J. A. Bondy and U. S. R. Murty. *Graph theory*, Springer-Verlag London, New York, 2008) has appeared in many places in the literature and it is used widely in the study the inverse of the bipartite graph with unique perfect matching.

Similar to the bipartite adjacency matrix, we define the *bipartite distance matrix* of a bipartite graph with unique perfect matching. That is, the *bipartite distance matrix* $\mathfrak{B}(G)$ of a bipartite graph G with a unique perfect matching on $2p$ vertices is a $p \times p$ matrix whose (i, j)th entry is the distance between vertices l_i and r_j, where $L := \{l_1, \dots, l_p\}$, $R := \{r_1, \dots, r_p\}$ is a vertex bipartition of G. Although the size of the bipartite distance matrix is half of the size of the graph, but we observe that it still provides much information about the underlying graph. We observe that $\det \mathfrak{B}(G)$ is always a multiple of 2^{p-1}. Based on this observation, we define the *bipartite distance index* of G as $\mathtt{bd}(G) := \det \mathfrak{B}(G)/(-2)^{p-1}$.

We use the word 'nonsingular tree' to mean a tree with a (unique) perfect matching. We show that the bipartite distance index of a nonsingular tree T satisfies an interesting inclusion-exclusion type of principle at any matching edge of the tree which is explained below (the shaded line is the matching edge under consideration). This gives us a recursive formula to compute $\mathtt{bd}(T)$.



$\mathtt{bd}(T) = \mathtt{bd}(T_1) + \mathtt{bd}(T_2) - \mathtt{bd}(T_1 \cap T_2)$

Even more interestingly, we show that the bipartite distance index of a nonsingular tree T can be completely characterized by the structure of T via what we call the f-alternating sums. By the *f-alternating sum* $f_S(T)$ of a nonsingular tree T, we define the following sum

$$f_S(T) := \sum_{\substack{P \in \mathcal{A}_T \\ P=[u,\cdots,v]}} [d(u)-2][d(v)-2] S\Big(\frac{|P|}{2}\Big),$$

where $S : \mathbb{N} \to \mathbb{R}$ is a sequence, $\mathcal{A}_T$ is the set of all alternating path (a path which has the starting edge, each alternate edge thereafter and the last edge from the matching) in T. We show that the bipartite distance index of a nonsingular tree is actually the f-alternating sum corresponding to the sequence $S = (1, 1, 3, 3, 5, 5, \ldots)$. We also show that, in general, for any sequence S, the f-alternating sum of a nonsingular tree satisfies the above described inclusion-exclusion type of principle.

A well known result by Graham, Hoffman and Hosoya (*On the distance matrix of a directed graph*, Journal of Graph Theory, 1(1):85–88, 1977) is that the determinant of the distance matrix of a graph only depends on the blocks and is independent of how they are assembled. In a similar way, we identify some basic elements and a merging operation and show that each of the trees that can be constructed from a given set of elements, sequentially using this operation, have the same bipartite distance index, independent of the order in which the sequence is followed. For the class of trees that can be obtained in this way, we give a surprisingly simple way to evaluate the determinant of the bipartite distance matrix.

We observe that the bipartite distance matrix of a nonsingular tree is always invertible and its determinant can be described using the structure of the tree. What about the inverse? Can the entries of the inverse be described combinatorially? We supply answer to these. However, the answer to these questions leads us to an unexpected generalization of the usual Laplacian matrix of a graph. We call it the bipartite Laplacian matrix. This generalized Laplacian matrix is usually not symmetric, but it still has many properties like the usual Laplacian matrix. It turns out that the usual Laplacian matrix of any tree is a very special case of the bipartite Laplacian matrix. We study some of the fundamental properties of the bipartite Laplacian matrix and compare them with those of the usual Laplacian matrix. Further, we provide a formula for the inverse of the bipartite distance matrix of a nonsingular tree by the help of its bipartite Laplacian matrix.

There are many studies of distance-like matrices available in the literature. We consider two of them. The first one is the q-distance matrix which is a generalization of the usual distance matrix. We show that the determinant of the q-bipartite distance matrix of a tree with a unique perfect matching can still be expressed as a combinatorial sum over the set of alternating paths with respect to a suitable sequence. The second distance-like matrix we study is the exponential distance matrix, which has a very simple expression for the determinant. However, as another surprise, we show that the determinant of the exponential bipartite distance matrix of a tree with a unique perfect matching is independent of the tree structure.