1

Solutions

In this chapter we introduce notion of an ordinary differential equation and its solution. We prove that, essentially, there is only one which needs to be studied. This equation is a first order system. For a single first order equation we give its geometrical interpretation. This chapter ends with some rough statements of existence and uniqueness theorems for the initial value problems.

1.1 Introduction

We want to tranlate the feeling of what should be or what is an Ordinary Differential Equation (ODE) into mathematical terms. Defining some object like ODE, for which we have some rough feeling, in English words is not really useful unless we know how to put it mathematically. As such we can start by saying "Let us look at the following differential equation ...". But since many books give this definition, let us also have one such. The reader is referred to Remark 1.2 for an example of an "ODE" that we really do not want to be an ODE.

Let us start with

Hypothesis

Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain and $\mathbb{I} \subseteq \mathbb{R}$ be an interval. Let $F : \mathbb{I} \times \Omega \to \mathbb{R}$ be a function defined by $(x, z_1, \dots z_{n+1}) \mapsto F(x, z_1, \dots z_{n+1})$ such that F is not a constant function in the variable z_{n+1}.

With this notation and hypothesis on F we define the basic object in our study, namely, an Ordinary differential equation.

Definition 1.1. *[ODE] Assume the above hypothesis. An ordinary differential equation of order n is defined by the relation*

$$F\left(x, y, y^{(1)}, y^{(2)}, \dots y^{(n)}\right) = 0, \tag{1.1}$$

where $y^{(n)}$ stands for n^{th} derivative of unknown function $x \mapsto y(x)$ with respect to the independent variable x.

Remark 1.2.

1. As we are going to deal with only one independent variable through out this course, we use the terminology 'differential equation' in place of 'ordinary differential equation' at times. Also we use the acronym ODE which stands for Ordinary Differential Equation(s). Wherever convenient, we use the notation prime $'$ to denote a derivative w.r.t. independent variable x; for example, y' is used to denote $y^{(1)}$).

2. Note that the highest order of derivative of unknown function y appearing in the relation (1.1) is called the order of the ordinary differential equation. Look at the carefully framed hypothesis above that makes sure the appearance of n^{th} derivative of y in (1.1).
3. (Arnold) If we define an ODE as a relation between an unknown function and its derivates, then the following equation will also be an ODE but not really!
$$\frac{dy}{dx}(x) = y \circ y(x). \tag{1.2}$$
However, note that our Defintion 1.1 does not admit (1.2) as an ODE.

Having defined an ODE, we are interested in its solutions. This brings us to the question of existence of solutions and finding out all the solutions. We make clear what we mean by a solution of an ODE.

Definition 1.3. *[Solution of an ODE] A real valued function ϕ is said to be a solution of ODE (1.1) if $\phi \in \mathcal{C}^n(\mathbb{I})$ and*

$$F\left(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \dots \phi^{(n)}(x)\right) = 0, \qquad \forall x \in \mathbb{I}. \tag{1.3}$$

Remark 1.4.

1. There is no guarantee that an equation such as (1.1) will have a solution. In fact, the equation defined by $F(x, y, y') = (y')^2 + y^2 + 1 = 0$ has no solution. Thus we cannot hope to have a general theory for equations of type (1.1).
2. To convince ourselves that we do not expect every ODE to have a solution, let us recall the situation with other types of equations involving Polynomials, Systems of linear equations, Implicit functions. In each of these cases, existence of solutions was proved under some conditions. Some of those results also characterised equations that have solution(s), for example, for systems of linear equations the characterisation was in terms of ranks of matrix defining the linear system and the corresponding augmented matrix.
3. In the context of ODE, there are two basic existence theorems that hold for equations in a special form called **normal** form. We state them in Section 3.1.

As observed in the last remark, we need to work with a less general class of ODE if we expect them to have solutions. One such class is called ODE in normal form and is defined below.

Hypothesis (H)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\mathbb{I} \subseteq \mathbb{R}$ be an interval. Let $f : \mathbb{I} \times \Omega \to \mathbb{R}$ be a continuous function defined by $(x, z_1, \dots z_n) \mapsto f(x, z_1, \dots z_n)$.

Definition 1.5. *[ODE in Normal form] Assume Hypothesis (H) on f. An ordinary differential equation of order n is said to be in normal form if*

$$y^{(n)} = f\left(x, y, y^{(1)}, y^{(2)}, \dots y^{(n-1)}\right). \tag{1.4}$$

Definition 1.6. *[Solution of ODE in Normal form] A function $\phi \in C^n(\mathbb{I}_0)$ where $\mathbb{I}_0 \subseteq \mathbb{I}$ is a subinterval is called a solution of ODE (1.4) if for every $x \in \mathbb{I}_0$, the n-tuple $\left(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \dots \phi^{(n-1)}(x)\right) \in \mathbb{I} \times \Omega$ and*

$$\phi^{(n)} = f\left(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \dots \phi^{(n-1)}(x)\right), \quad \forall x \in \mathbb{I}_0. \tag{1.5}$$

Remark 1.7.

1. Observe that we want equation (1.5) to be satisfied "for all $x \in \mathbb{I}_0$" instead of "for all $x \in \mathbb{I}$". Compare now with defintion of solution given before in Definition 1.3 which is more stringent. We modified the concept of solution, by not requiring that the equation be satisfied by the solution on entire interval $\mathbb{I}$, due to various examples of ODEs that we shall see later which have solutions only on a subinterval of $\mathbb{I}$. We dont want to miss them!!
2. Compare Definition 1.5 with Definition 1.1. See the item (ii) of Remark 1.2, observe that we did not need any special effort in formulating Hypothesis (H) to ensure that n^{th} derivative makes an appearance in the equation (1.4).

Convention From now onwards an ODE in normal form will simply be called ODE for brevity.

Hypothesis ($\mathbf{H_S}$)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\mathbb{I} \subseteq \mathbb{R}$ be an interval. Let $\mathbf{f} : \mathbb{I} \times \Omega \to \mathbb{R}^n$ be a continuous function defined by $(x, \mathbf{z}) \mapsto \mathbf{f}(x, \mathbf{z})$ where $\mathbf{z} = (z_1, \ldots z_n)$.

Definition 1.8. *[System of ODEs] Assume Hypothesis* $(\mathrm{H_S})$ *on* $\mathbf{f}$. *A first order system of n ordinary differential equations is given by*

$$\mathbf{y}' = \mathbf{f}\,(x, \mathbf{y})\,. \tag{1.6}$$

The notion of solution for above system is defined analogous to Definition 1.5. A result due to D'Alembert enables us to restrict a general study of any ODE in normal form to that of a first order system.

Lemma 1.9 (D'Alembert). *An n^{th} order ODE (1.4) is equivalent to a system of n first order ODEs.*

Proof. Introducing a transformation $\mathbf{z} = (z_1, z_2, \ldots, z_n) := \left(y, y^{(1)}, y^{(2)}, \ldots y^{(n-1)}\right)$, we see that $\mathbf{z}$ satisfies the linear system

$$\mathbf{z}' = (z_2, \ldots, z_n, f(x, \mathbf{z})) \tag{1.7}$$

Equivalence of (1.4) and (1.7) means starting from a solution of either of these ODE we can produce a solution of the other. This is a simple calculation and is left as an exercise. □

Note that the first order system for $\mathbf{z}$ consists of n equations. This n is the order of (1.4).

Exercise 1.10. Define higher order systems of ordinary differential equations and define corresponding notion of its solution. Reduce the higher order system to a first order system.

1.2 Geometric interpretation of a first order ODE and its solution

We now define some terminology that we use while giving a geometric meaning of an ODE given by

$$\frac{dy}{dx} = f\,(x, y)\,. \tag{1.8}$$

We recall that f is defined on a domain D in $\mathbb{R}^2$.

Definition 1.11 (Line element). *A line element associated to a point $(x, y) \in D$ is a line passing through the point (x, y) with slope p. We use the triple (x, y, p) to denote a line element.*

Definition 1.12 (Direction field/ Vector field). *A direction field (sometimes called vector field) associated to the ODE (1.8) is collection of all line elements in the domain D where slope of the line element associated to the point (x, y) has slope equal to $f(x, y)$. In other words, a direction field is the collection*

$$\{ (x, y, f(x, y)) : (x, y) \in D \}. \tag{1.9}$$

Remark 1.13 (Interpretations).

1. The ODE (1.8) can be thought of prescribing line elements in the domain D.
2. Solving an ODE can be geometrically interpreted as finding curves in D that fit the direction field prescribed by the ODE. A solution (say ϕ) of the ODE passing through a point $(x_0, y_0) \in D$ (*i.e.,* $\phi(x_0) = y_0$) must satisfy $\phi'(x_0) = f(x_0, y_0)$. In other words,
$$(x_0, y_0, \phi'(x_0)) = (x_0, y_0, f(x_0, y_0)). \tag{1.10}$$
3. That is, the ODE prescribes the slope of the tangent to the graph of any solution (which is equal to $\phi'(x_0)$). This can be seen by looking at the graph of a solution.
4. Drawing direction field corresponding to a given ODE and fitting some curve to it will end up in finding a solution, at least, graphically. However note that it may be possible to fit more than one curve passing through some points in D, which is the case where there are more than one solution to ODE around those points. Thus this activity (of drawing and fitting curves) helps to get a rough idea of nature of solutions of ODE.
5. A big challenge is to draw direction field for a given ODE. One good starting point is to identify all the points in domain D at which line element has the same slope and it is easy to draw all these lines. These are called *isoclines*; the word means "leaning equally".

Exercise 1.14. Draw the direction field prescribed by ODEs where $f(x, y) = 1$, $f(x, y) = x$, $f(x, y) = y$, $f(x, y) = y^2$ and fit solution curves to them.

Finding a solution of an ODE passing through a point in D is known as Initial value problem. We address this in the next section.

1.3 Initial Value Problems

We consider an Initial Value Problem (also called Cauchy problem) for an ODE (1.4). It consists of solving (1.4) subject to what are called *Initial conditions*. The two basic theorems we are going to present are concerning an IVP for a first order ODE.

Definition 1.15 (Initial Value Problem for an ODE). *Let $x_0 \in \mathbb{I}$ and $(y_1, y_2, \ldots, y_n) \in \Omega$ be given. An Initial Value Problem (IVP) for an ODE in normal form is a relation satisfied by an unknown function y given by*

$$y^{(n)} = f\left(x, y, y^{(1)}, y^{(2)}, \ldots y^{(n-1)}\right), \qquad y^{(i)}(x_0) = y_i,\ i = 0, \ldots, (n-1). \tag{1.11}$$

Definition 1.16. *[Solution of an IVP for an ODE] A solution ϕ of ODE (1.4) (see Definition 1.6) is said to be a solution of IVP if $x_0 \in \mathbb{I}_0$ and*

$$\phi^{(i)}(x_0) = y_i,\ i = 0, \ldots, (n-1). \tag{1.12}$$

This solution is denoted by $\phi(.; x_0, y_0, y_1, \ldots, y_{n-1})$ to remember the IVP solved by ϕ.

Definition 1.17 (Local and Global solutions of an IVP). *Let ϕ be a solution of an IVP according to Definition 1.16.*

1. *If $\mathbb{I}_0 \subset \mathbb{I}$, then ϕ is called a local solution of IVP.*
2. *If $\mathbb{I}_0 = \mathbb{I}$, then ϕ is called a global solution of IVP.*

Remark 1.18.

1. Note that in all our definitions of solutions, a solution always comes with its domain of definition. Sometimes it may be possible to extend the given solution to a bigger domain. We address this issue in Chapter 2.
2. Geometrically speaking, graph of solution of an IVP is a curve passing through the point $\big(x_0, y_0, y_1, \ldots, y_{(n-1)}\big)$.

Exercise 1.19. Define an IVP for a first order system. Reduce an IVP for an n^{th} order ODE to that of an equivalent first order system.

1.3.1 A rough statement of existence and uniqueness theorems

We have shown that any arbitrary system of ODEs in normal is equivalent to that of a first order system. Thus we concentrate on first order systems henceforth and prove existence and uniqueness theorems for them.

Assume Hypothesis ($\mathrm{H_S}$) on $\mathbf{f}$. Let us consider an IVP for the system

$$\mathbf{y}' = \mathbf{f}\,(x, \mathbf{y})\,. \tag{1.13}$$

1. The IVP has *at least* one local solution.
2. If the function $\mathbf{y} \mapsto \mathbf{f}\,(x, \mathbf{y})$ is Frechet differentiable for every $x \in \mathbb{I}$, then the IVP has *at most* one local solution.
3. There are also results asserting the existence of a global solution for the IVP under some conditions on $\mathbf{f}$. We will discuss them in Chapter 2.
4. The conditions on $\mathbf{f}$ are only sufficient conditions but not necessary. Thus these results tell you when existence and uniqueness results hold and do not say a word if conditions on $\mathbf{f}$ are not met.

We will state and prove the existence and uniqueness results carefully later.