2

First order equations

In this chapter we focus our attention on first order equations. We do detailed study of the classical variable-separable method and linear equation. We discuss some existence and uniqueness results concerning IVP of a first order ODE and postpone the proofs to a later stage. We end this chapter with a discussion on obstructions in extending local solutions to global ones.

2.1 Variable-Separable method

Hypothesis ($\mathbf{H_{VS}}$)

Let $\mathbb{I} \subseteq \mathbb{R}$ and $\mathbb{J} \subseteq \mathbb{R}$ be intervals. Let $g : \mathbb{I} \to \mathbb{R}$ and $h : \mathbb{J} \to \mathbb{R} \setminus \{0\}$ be continuous functions.

If domains of functions are not specified, then they are assumed to be their “natural domains”.

We consider a first order ODE in **Variable-Separable form** given by

$$\frac{dy}{dx} = g(x)h(y) \tag{2.1}$$

We gain an understanding of the equation (2.1) and its solution in three simple steps.

2.1.1 Direction field independent of y

The equation (2.1) takes the form

$$\frac{dy}{dx} = g(x) \tag{2.2}$$

Geometric understanding: Observe that slope of line element associated to a point depends only on its x coordinate. Thus the direction field is defined on the strip $\mathbb{I} \times \mathbb{R}$ and is invariant under translation in the direction of $\mathbf{Y}$ axis. Therefore it is enough to draw line elements for points in the set $\mathbb{I} \times \{0\}$. This suggests that if we know a solution curve then its translates in the $\mathbf{Y}$ axis direction gives all solution curves.

Let $x_0 \in \mathbb{I}$ be fixed and let us define a primitive of g on $\mathbb{I}$ by

$$G(x) := \int_{x_0}^{x} g(s)\, ds. \tag{2.3}$$

The function G is a solution of ODE (2.2) by fundamental theorem of integral calculus, since $G'(x) = g(x)$ on $\mathbb{I}$. Moreover, $G(x_0) = 0$.

A solution curve passing through an arbitrary point $(\xi, \eta) \in \mathbb{I} \times \mathbb{R}$ can be obtained from G and is given by

$$y(x; \xi, \eta) = G(x) + (\eta - G(\xi)), \tag{2.4}$$

which is of the form $G(x)+C$ where $C = \eta - G(\xi)$. Thus all solutions of ODE (2.2) are determined. Moreover, all the solutions are *global*.

Example 2.1. Solve the ODE (2.2) with g given by (i). $x^2 + \sin x$ (ii). $x\sqrt{1-x^2}$ (iii). $\sin x^2$ (iv) Have you run out of patience? An explicit formula for the solution does not mean having it in a closed form!

2.1.2 Direction field independent of x

The equation (2.1) takes the form

$$\frac{dy}{dx} = h(y). \tag{2.5}$$

This type of equations are called "autonomous" since the RHS of the equation does not depend on the independent variable x.

Geometric understanding: Observe that slope of line element associated to a point depends only on its y coordinate. Thus the direction field is defined on the strip $\mathbb{R} \times \mathbb{J}$ and is invariant under translation in the direction of $\mathbf{X}$ axis. Therefore it is enough to draw line elements for points in the set $\{0\} \times \mathbb{I}$. This suggests that if we know a solution curve then its translates in the $\mathbf{X}$ axis direction gives solution curves. Note that this is one of the main features of autonomous equations.

Exercise 2.2. Formulate the geometric observation "if we know a solution curve then its translates in the $\mathbf{X}$ axis direction gives solution curves" as a mathematical statement and prove it. Compare and contrast with a similar statement made in Step 1.

This case is considerably different from that of Step 1 where solutions are defined on the entire interval $\mathbb{I}$. To expect the difficulties awaiting us in the present case, it is advisable to solve the following exercise.

Exercise 2.3. Verify that $\phi(x) = \frac{1}{(x+c)}$ are solutions of $y' = -y^2$ on certain intervals. Graph the solutions for $c = 0, \pm 1, \pm 2$. Verify that $\phi(x) \equiv 0$ is also a solution on $\mathbb{R}$.

Note from the last exercise that it has a constant solution (also called, rest/equilibrium point since it does not move!) defined globally on $\mathbb{R}$ and also non-constant solutions defined on only a subinterval of the real line that varies with c. There is no way we can get a non-zero solution from zero solution by translation. Thus it is a good time to review Exercise 2.2 and find out if there is any contradition.

Observation: If $\xi \in \mathbb{J}$ satisfies $h(\xi) = 0$, then ξ is a rest point and $\phi(x) = \xi$ is a solution of (2.5) for $x \in \mathbb{R}$.

Since the function h is continuous, it does not change sign. Therefore we may assume, without loss of generality (WLOG), that $h(y) > 0$ for $y \in \mathbb{J}$. There is no loss of generality because the other case, namely $h(y) < 0$ for $y \in \mathbb{J}$, can be disposed off in a similar manner.

Formally speaking, ODE (2.5) 'may be' written as

$$\frac{dx}{dy} = \frac{1}{h(y)}. \tag{2.6}$$

Thus all the conclusions of Step 1 can be translated to the current problem. But 'inversion of roles of x and y' needs to be justified!

Let $y_0 \in \mathbb{J}$ be fixed and let us define a primitive of $1/h$ on $\mathbb{J}$ by

$$H(y) := \int_{y_0}^{y} \frac{1}{h(s)}\, ds. \tag{2.7}$$

We record some properties of H below.

1. The function $H : \mathbb{J} \to \mathbb{R}$ is differentiable (follows from fundamental theorem of integral calculus).
2. Since $h > 0$ on $\mathbb{J}$, the function H is strictly monotonically increasing. Hence H is one-one.
3. The function $H : \mathbb{J} \to H(\mathbb{J})$ is invertible. By definition, $H^{-1} : H(\mathbb{J}) \to \mathbb{J}$ is onto, and is also differentiable.
4. A finer observation yields $H(\mathbb{J})$ is an interval containing 0.

We write

$$H \circ H^{-1}(x) = x \tag{2.8}$$

A particular solution

The function H gives rise to an implicit expression for a solution y of ODE (2.5) given by

$$H(y) = x \tag{2.9}$$

This assertion follows from the simple calculation

$$\frac{d}{dx}H(y) = \frac{dH}{dy}\frac{dy}{dx} = \frac{1}{h(y)}\frac{dy}{dx} = 1. \tag{2.10}$$

Note that $H(y_0) = 0$ and hence $x = 0$. This means that solution $y(x) := H^{-1}(x)$ passes through the point $(0, y_0)$. Note that this solution is defined on the interval $H(\mathbb{J})$ which may not be equal to $\mathbb{R}$.

Some more solutions

Note that the function $H^{-1}(x - c)$ is also a solution on the interval $c + H(\mathbb{J})$ (verify). Moreover, *these are all* the solutions of ODE (2.5).

All solutions

Let z be any solution of ODE (2.5) defined on some interval $\mathbb{I}_z$. Then we have

$$\frac{dz}{dx} = z'(x) = h(z(x))$$

Let $x_0 \in \mathbb{I}_z$ and let $z(x_0) = z_0$. Since $h \neq 0$ on $\mathbb{J}$, integrating both sides from x_0 to x yields

$$\int_{x_0}^{x} \frac{z'(x)}{h(z(x))}\,dx = \int_{x_0}^{x} 1\,dx$$

which reduces to

$$\int_{x_0}^{x} \frac{z'(x)}{h(z(x))}\,dx = x - x_0$$

The last equality, in terms of H defined by equation (2.7), reduces to

$$H(z) - \int_{y_0}^{z_0} \frac{1}{h(s)}\,ds = x - x_0$$

$$i.e., \qquad H(z) - H(z_0) = x - x_0 \tag{2.11}$$

From the equation (2.11), we conclude that $x - x_0 + H(z_0) \in H(\mathbb{J})$.
Thus, for x belonging to the interval $x_0 - H(z_0) + H(\mathbb{J})$, we can write the solution as

$$z(x) = H^{-1}\big(x - x_0 + H(z_0)\big). \tag{2.12}$$

Note that a solution of ODE (2.5) are not global in general.

Exercise 2.4. Give a precise statement of what we proved about solutions of ODE (2.5).

Exercise 2.5. Prove that a solution curve passes through every point of the domain $\mathbb{R} \times \mathbb{J}$.

Exercise 2.6. In view of what we have proved under the assumption of non-vanishing of h, revisit Exercise 2.3, Exercise 2.2 and comment.

Exercise 2.7. Solve the ODE (2.5) with h given by (i). ky (ii). $\sqrt{|y|}$ (iii). $1 + y^2$ (iv). $3y^{2/3}$ (v). $y^4 + 4$ (vi). $\frac{1}{y^4+4}$

2.1.3 General case

We justify the formal calculation used to obtain solutions of ODE (2.1) which we recall here for convenience.

$$\frac{dy}{dx} = g(x)h(y) \tag{2.13}$$

The formal calculation to solve (2.13) is

$$\int \frac{dy}{h(y)} = \int g(x)\,dx + C \tag{2.14}$$

We justify this formal calculation on a rigorous footing. We introduce two auxiliary ODEs:

$$\frac{dy}{du} = h(y) \quad \text{and} \quad \frac{du}{dx} = g(x). \tag{2.15}$$

Notation We refer to the first and second equations of (2.15) as equations (2.15a) and (2.15b) respectively. We did all the hard work in Steps 1-2 and we can easily deduce results concerning ODE (2.13) using the auxiliary equations (2.15) introduced above.

Claim Let $(x_1, y_1) \in \mathbb{I} \times \mathbb{J}$ be an arbitrary point. A solution y satisfying $y(x_1) = y_1$ is given by

$$y(x) = H^{-1}\big(G(x) - G(x_1) + H(y_1)\big). \tag{2.16}$$

Proof. Recalling from Step 2 that the function H^{-1} is onto $\mathbb{J}$, there exists a unique u_1 such that $H^{-1}(u_1) = y_1$. Now solution of (2.15a) satisfying $y(u_1) = y_1$ is given by (see, formula (2.12))

$$y(u) = H^{-1}\big(u - u_1 + H(y_1)\big) \tag{2.17}$$

and is defined on the interval $u_1 - H(y_1) + H(\mathbb{J})$.

Solution of (2.15b), defined on $\mathbb{I}$, satisfying $u(x_1) = u_1$ is given by (see, formula (2.4))

$$u(x; x_1, u_1) = G(x) + (u_1 - G(x_1)).G(x) \in G(x_1) - H(y_1) + H(\mathbb{J}) \tag{2.18}$$

Combining the two formulae (2.17)-(2.17), we get (2.16). This formula makes sense for $x \in \mathbb{I}$ such that $G(x) + (u_1 - G(x_1)) \in u_1 - H(y_1) + H(\mathbb{J})$. That is, $G(x) \in G(x_1) - H(y_1) + H(\mathbb{J})$.
Thus a general solution of (2.1) is given by

$$y(x) = H^{-1}\big(G(x) - c\big). \tag{2.19}$$

This ends the analysis of variable-separable equations modulo the following exercise. $\square$

Exercise 2.8. Prove that the set $\big\{x \in \mathbb{I} : G(x) \in G(x_1) - H(y_1) + H(\mathbb{J})\big\}$ is non-empty and contains a non-trivial interval.

Exercise 2.9. Prove: Initial value problems for variable-separable equations under Hypothesis $(\mathbf{H_{VS}})$ have unique solutions.

2.2 The Linear Equation

Hypothesis $(\mathbf{H_{LNH}})$

Let $\mathbb{I} \subseteq \mathbb{R}$ be an interval. Let $a, b : \mathbb{I} \to \mathbb{R}$ be continuous functions.

Definition 2.10. *Assume Hypothesis* $(\mathbf{H_{LNH}})$.

1. *A first order* **Linear homogeneous** *(LH) ODE is given by*

$$\frac{dy}{dx} = a(x)y. \tag{2.20}$$

2. *A first order* **Linear non-homogeneous** *(LNH) ODE is given by*

$$\frac{dy}{dx} = a(x)y + b(x). \tag{2.21}$$

Remark 2.11.

1. Equation (2.20) is called linear because the RHS of (2.20) is linear in the y variable. The adjective *homogeneous* appears because *If y_1 is a solution, then cy_1 is also a solution for every* $c \in \mathbb{R}$. In other words, we are looking for solutions of $Ly = 0$ where is L is a linear operator.
2. Equation (2.21) is called non-homogeneous because we are looking for solutions of $Ly = b$ where L is a linear operator. This is analogous to the problem of solving systems of linear equations.

Before we start finding solutions of equations (2.20) and (2.21), we can say something about structure of their solutions.

1. Solutions of equation (2.20) form a subspace of $C^1(\mathbb{I})$.
2. Any two solutions of equation (2.21) differ by a solution of (2.20). Thus, if z is one solution of (2.20) then any other solution is given by $z+w$ for some solution w of (2.20). This is analogous to the problem of solving systems of linear equations.

Solutions of (LH)

Let $x_0 \in \mathbb{I}$ be an arbitrary but fixed point. It is an easy exercise to solve (LH) by variable-separable method and find that all solutions are included in the family ($c \in \mathbb{R}$)

$$y(x;c) = c \exp\left(\int_{x_0}^{x} a(s)\,ds\right) \tag{2.22}$$

Note that these solutions are defined on the entire interval $\mathbb{I}$. That is solutions of (LH) are global.

Exercise 2.12. Prove the above assertion.

Exercise 2.13. Show that solutions of an IVP for (LH) are unique. From the family of solutions (2.22), find a solution that satisfies the initial condition $y(\xi) = \eta$.

Solutions of (LNH)

Solutions of (LNH)

We find the solution of (LNH) by the celebrated **Variation of parameters** (VoP) method. This method consists of proposing an ansatz, obtained by assuming the parameter c to be a function of x in the expression for general solutions of (LH) (2.22), for solution of (LNH) given by

$$z(x) = c(x) \quad \exp\left(\int_{x_0}^{x} a(s)\,ds\right) \tag{2.23}$$

Substituting ansatz (2.23) into ODE (2.21), we get

$$c'(x) \exp\left(\int_{x_0}^{x} a(s)\,ds\right) + c(x)\,a(x) \exp\left(\int_{x_0}^{x} a(s)\,ds\right) = a(x)c(x) \exp\left(\int_{x_0}^{x} a(s)\,ds\right) + b(x) \tag{2.24}$$

After simplification we see that the function c satisfies

$$c'(x) = \exp\left(-\int_{x_0}^{x} a(s)\,ds\right)\ b(x) \tag{2.25}$$

Integrating (2.25) from x_0 to x yields

$$c(x) = c(x_0) + \int_{x_0}^{x} \exp\left(-\int_{x_0}^{t} a(s)\,ds\right)\ b(t)\,dt \tag{2.26}$$

Substituting for c in the ansatz (2.23) we obtain

$$z(x) = \left[c(x_0) + \int\limits_{x_0}^{x} \exp\left(- \int\limits_{x_0}^{t} a(s)\, ds \right) b(t)\, dt \right] \quad \exp\left(\int\limits_{x_0}^{x} a(s)\, ds \right) \tag{2.27}$$

Note from the above equation, by evaluating $x = x_0$, that $z(x_0) = c(x_0)$. Thus we have

$$z(x) = \left[z(x_0) + \int\limits_{x_0}^{x} \exp\left(- \int\limits_{x_0}^{t} a(s)\, ds \right) b(t)\, dt \right] \quad \exp\left(\int\limits_{x_0}^{x} a(s)\, ds \right) \tag{2.28}$$

Note that the first term on RHS of (2.28) is a solution of (LH). Remark that these solutions (2.28) are defined on the entire interval $\mathbb{I}$. That is solutions of (LNH) are global. We will see later on that (VoP) helps us to solve (LNH) for systems and we have the same formula (2.28) with obvious modifications.

To understand the complexity of VoP computations, solve the following exercise.

Exercise 2.14. Solve the IVP: $y' = 2y + 3x^2 - x + 4$, $y(0) = 4$.

Constant coefficient case: Method of undetermined coefficients (MUC) for solving (LNH)

The formula (2.28) looks great but there are simpler methods to solve (LNH) if the function a is constant. These methods are useful for higher order equations as well but they work only for specific types of the function b, namely polynomials, exponentials, (Co)Sine functions.

The method itself will be given in detail later when we are dealing with higher order constant coefficient equations. For now, we prescribe the magic formula to solve Exercise (2.14). Try $y_{muc}(x) = Ax^2 + Bx + C$ to find a particular solution. To understand why this works, have a look at formula obtained from VoP.

The general prescription for (MUC) is

1. If b is a polynomial, then try a polynomial for y_{muc}
2. If b is $a \sin px$ or $b \cos px$, then try y_{muc} in the form $A \sin px + B \cos px$. We need to include both sin and cos in y_{muc} because one is the derivate of the other and we are dealing with ODE.
3. If b is e^{kx}, then try $y_{muc} = Ae^{kx}$ if $k \neq 2$. If $k = 2$ then e^{2x} is a solution of (LH), then try $y_{muc} = Axe^{kx}$.

Remark 2.15. Note the simplicity of (MUC) when compared to (VoP) while solving ODEs as in Exercise (2.14)

Exercise 2.16. Solve all first order linear non-homogeneous equations. This is really an exercise in integral calculus.