Examples

3.3 Continuation

We answer the following two questions in this section.

1. When can a given solution be continued?

2. If a solution can not be continued to a bigger interval, what prevents a continuation?

The existential results of Section 3.1 provide us with an interval containing the initial point on which solution for IVP exists and the length of the interval depends on the data of the problem as can be seen from expression for its length. However this does not rule out solutions of IVP defined on bigger intervals if not whole of $\mathbb{I}$. In this section we address the issue of extending a given solution to a bigger interval and the difficulties that arise in extending.

Intuitive idea for extension Take a local solution $\mathbf{u}$ defined on an interval $\mathbb{I}_0 = (x_1\, x_2)$ containing the point x_0. An intuitive idea to extend $\mathbf{u}$ is to take the value of $\mathbf{u}$ at the point $x = x_2$ and consider a new IVP for the same ODE by posing the initial condition prescribed at the point $x = x_2$ to be equal to $\mathbf{u}(x_2)$. Consider this IVP on a rectangle containing the point $(x_2, \mathbf{u}(x_2))$. Now apply any of the existence theorems of Section 3.1 and conclude the existence of solution on a bigger interval. Repete the previous steps and obtain a solution on the entire interval $\mathbb{I}$.

However note that this intuitive idea may fail for any one of the following reasons. They are (i). limit of function $\mathbf{u}$ does not exist as $x \to x_2$ (ii). limit in (i) may exist but there may not be any rectangle around the point $(x_2, \mathbf{u}(x_2))$ as required by existence theorems. In fact these are the principle difficulties in extending a solution to a bigger interval. In any case we show that for any solution there is a biggest interval beyond which it can not be extended as a solution of IVP and is called *maximal interval of existence* via Zorn's lemma.

We start with a few definitions.

Definition 3.20 (Continuation, Saturated solutions) *Let the function* $\mathbf{u}$ *defined on an interval* $\mathbb{I}_0$ *be a solution of IVP* (3.1)*. Then*

1. *The solution* $\mathbf{u}$ *is called continuable at the right if there exists a solution of IVP* $\mathbf{w}$ *defined on interval* $\mathbb{J}_0$ *satisfying* $\sup \mathbb{I}_0 < \sup \mathbb{J}_0$ *and the equality* $\mathbf{u}(x) = \mathbf{w}(x)$ *holds for* $x \in \mathbb{I}_0 \cap \mathbb{J}_0$*. Any such* $\mathbf{w}$ *is called an extension of* $\mathbf{u}$*. The solution* $\mathbf{u}$ *is called saturated at the right if it is not continuable at the right.*

2. *The solution* $\mathbf{u}$ *is called continuable at the left if there exists a solution of IVP* $\mathbf{z}$ *defined on interval* $\mathbb{K}_0$ *satisfying* $\inf \mathbb{K}_0 < \inf \mathbb{I}_0$ *and the equality* $\mathbf{u}(x) = \mathbf{z}(x)$ *holds for* $x \in \mathbb{I}_0 \cap \mathbb{K}_0$*. The solution* $\mathbf{u}$ *is called saturated at the left if it is not continuable at the left.*

3. *The solution* $\mathbf{u}$ *is called global at the right if* $\mathbb{I}_0 \supseteq \{x \in \mathbb{I} : x \geq x_0\}$*. Similarly,* $\mathbf{u}$ *is called global at the left if* $\mathbb{I}_0 \supseteq \{x \in \mathbb{I} : x \leq x_0\}$*.*

Remark 3.21 *If the solution* $\mathbf{u}$ *is continuable at the right, then by concactenating the solutions* $\mathbf{u}$ *and* $\mathbf{w}$ *we obtain a solution of IVP on a bigger interval* $(\inf \mathbb{I}_0, \sup \mathbb{J}_0)$*, where* $\mathbf{w}$ *defined on* $\mathbb{J}_0$ *is a right extension of* $\mathbf{u}$*. A similar statement holds if* $\mathbf{u}$ *is continuable at the left.*

Let us define the notion of a right (left) solution to the IVP (3.1).

Definition 3.22 *A function* $\mathbf{u}$ *defined on an interval* $[x_0, b)$ *(respectively, on* $(a, x_0]$*) is said to be a right solution (respectively, a left solution) if* $\mathbf{u}$ *is a solution of ODE* $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ *on* (x_0, b) *(respectively, on* (a, x_0)*) and* $\mathbf{u}(x_0) = \mathbf{y}_0$*.*

Note that if $\mathbf{u}$ is a solution of IVP (3.1) defined on an interval $(a,\, b)$, then $\mathbf{u}$ restricted to $[x_0,\, b)$ (respectively, to $(a,\, x_0]$) is a right solution (respectively, a left solution) to IVP.

For right and left solutions, the notions of continuation and saturated solutions become

Definition 3.23 (Continuation, Saturated solutions) *Let the function* $\mathbf{u}$ *defined on an interval* $[x_0,\, b)$ *be a right solution of IVP and let* $\mathbf{v}$ *defined on an interval* $(a,\, x_0]$ *be a left solution of IVP* (3.1)*. Then*

1. *The solution* $\mathbf{u}$ *is called continuable at the right if there exists a right solution of IVP* $\mathbf{w}$ *defined on interval* $[x_0,\, d)$ *satisfying* $b < d$ *and the equality* $\mathbf{u}(x) = \mathbf{w}(x)$ *holds for* $x \in [x_0,\, b)$*. Any such* $\mathbf{w}$ *is called a right extension of* $\mathbf{u}$*. The right solution* $\mathbf{u}$ *is called saturated at the right if it is not continuable at the right.*

2. *The solution* $\mathbf{v}$ *is called continuable at the left if there exists a solution of IVP* $\mathbf{z}$ *defined on interval* $(c,\, x_0]$ *satisfying* $c < a$ *and the equality* $\mathbf{u}(x) = \mathbf{z}(x)$ *holds for* $x \in (a,\, x_0]$*. The solution* $\mathbf{u}$ *is called saturated at the left if it is not continuable at the left.*

3. *The solution* $\mathbf{u}$ *is called global at the right if* $[x_0,\, b) = \{x \in \mathbb{I} \,:\, x \geq x_0\}$*. Similarly,* $\mathbf{u}$ *is called global at the left if* $(a,\, x_0] = \{x \in \mathbb{I} \,:\, x \leq x_0\}$*.*

The rest of the discussion in this section is devoted to analysing "continuability at the right", "saturated at the right" as the analysis for the corresponding notions "at the left" is similar. We drop suffixing "at the right" from now on to save space to notions of continuability and saturation of a solution.

3.3.1 Characterisation of continuable solutions

Lemma 3.24 *Assume Hypothesis* ($\mathrm{H}_{\mathrm{IVPS}}$)*. Let* $\mathbf{u} : \mathbb{I}_0 \to \mathbb{R}^n$ *be a right solution of IVP* (3.1) *defined on the interval* $[x_0,\, d)$*. Then the following statements are equivalent.*

(1) *The solution* $\mathbf{u}$ *is continuable.*

(2) (i) $d < \sup \mathbb{I}$ *and there exists*

(ii) $\mathbf{y}^* = \lim\limits_{x \to d_-} \mathbf{y}(x)$ *and* $\mathbf{y}^* \in \Omega$*.*

(3) *The graph of* $\mathbf{u}$ *i.e.,*

$$\mathit{graph}\, \mathbf{u} = \big\{(x,\, \mathbf{u}(x)) \,:\, x \in [x_0,\, d)\big\} \tag{3.27}$$

is contained in a compact subset of $\mathbb{I} \times \Omega$*.*

Proof :

We prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). The implication (1) $\Longrightarrow$ (2) is obvious.

Proof of (2) $\Longrightarrow$ (3)

In view of (2), we can extend the function $\mathbf{u}$ to the interval $[x_0,\, d]$ and let us call this extended function $\tilde{\mathbf{u}}$. Note that the function $x \longmapsto (x,\, \tilde{\mathbf{u}}(x)$ is continuous on the interval $[x_0,\, d]$ and the image of $[x_0,\, d]$ under this map is graph of $\tilde{\mathbf{u}}$, denoted by $\mathit{graph}\, \tilde{\mathbf{u}}$, is compact. But $\mathit{graph}\, \mathbf{u} \subset \mathit{graph}\, \tilde{\mathbf{u}} \subset \mathbb{I} \times \Omega$. Thus (3) is proved.

Proof of (3) $\Longrightarrow$ (2)

Assume that $\mathit{graph}\, \mathbf{u}$ is contained in a compact subset of $\mathbb{I} \times \Omega$. As a consequence, owing to continuity of the function $\mathbf{f}$ on $\mathbb{I} \times \Omega$, there exists $M > 0$ such that $\|\mathbf{f}(x,\, \mathbf{u}(x))\| < M$ for all $x \in [x_0,\, d)$. Also, since $\mathbb{I}$ is an open interval, necessarily $d < \sup \mathbb{I}$. We will now prove that the limit in (2)(ii) exists.

Since $\mathbf{u}$ is solution of IVP (3.1), by Lemma 3.4, we have

$$\mathbf{u}(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{u}(s))\, ds \quad \forall\, x \in [x_0,\, d). \tag{3.28}$$

Thus for $\xi, \eta \in [x_0,\, d)$, we get

$$\|\mathbf{u}(\xi) - \mathbf{u}(\eta)\| \leq \int_{\xi}^{\eta} \|\mathbf{f}(s, \mathbf{u}(s))\|\, ds \leq M|\xi - \eta|. \tag{3.29}$$

Thus $\mathbf{u}$ satisfies the hypothesis of Cauchy test on the existence of finite limit at d. Indeed, the inequality (3.29) says that $\mathbf{u}$ is uniformly continuous on $[x_0,\, d)$ and hence limit of $\mathbf{u}(x)$ is finite as $x \to d$. This follows from a property of uniformly continuous functions, namely they map Cauchy sequences to Cauchy sequences. Let us denote the limit by y^*. In principle being a limit, $y^* \in \overline{\Omega}$. To complete the proof we need to show that $y^* \in \Omega$. This is a consequence of the hypothesis that graph of $\mathbf{u}$ is contained in a compact subset of $\mathbb{I} \times \Omega$ and the fact that $\mathbb{I}$ and Ω are open sets.

Proof of (2) $\Longrightarrow$ (1)

As we shall see, the implication $2 \Longrightarrow 1$ is a consequence of existence theorem for IVP (Theorem 3.8) and concactenation lemma (Lemma 3.5).

Let $\mathbf{w}$ be a solution to IVP corresponding to the initial data $(d, y^*) \in \mathbb{I} \times \Omega$ defined on an interval $(e,\, f)$ containing the point d. Let $\mathbf{w}|_{[d,\, f)}$ be the restriction of $\mathbf{w}$ to the interval $[d,\, f) \subseteq \mathbb{I}$. Let $\tilde{\mathbf{u}}$ be defined as the continuous extension of $\mathbf{u}$ to the interval $(c,\, d])$ which makes sense due to the existence of the limit in (2)(ii). Concactenating $\tilde{\mathbf{u}}$ and $\mathbf{w}|_{[d,\, f)}$ yields a solution of the original IVP (3.1) that is defined on the interval $(c,\, f)$ and $d < f$. ∎

Remark 3.25 *The important message of the above result is that a solution can be extended to a bigger interval provided the solution curve remains "well within" the domain $\mathbb{I} \times \Omega$ and its right end-point lies in Ω.*

3.3.2 Existence and Classification of saturated solutions

The following result is concerning the existence of saturted solutions for an IVP. Once again we study "saturated at the right" and corresponding results for "saturated at the left" can be obtained by similar arguments. Thus for this discussion we always consider a soluton as defined on interval of the form $[x_0, d)$

Theorem 3.26 (Existence of saturated solutions) *If $\mathbf{u}$ defined on an interval $[x_0, d)$ is a right solution of IVP* (3.1)*, then either $\mathbf{u}$ is saturated, or $\mathbf{u}$ can be continued up to a saturated one.*

Proof :

If $\mathbf{u}$ is saturated, then there is nothing to prove. Therefore, we assume $\mathbf{u}$ is not saturated. By definition of saturatedness of a solution, $\mathbf{u}$ is continuable. Thus the set $\mathcal{S}$, defined below, is non-empty.

$$\mathcal{S} = \text{Set of all solutions of IVP (3.1) which extend } \mathbf{u}. \tag{3.30}$$

We define a relation $\preceq$ on the set $\mathcal{S}$, called *partial order* as follows. For $\mathbf{w}, \mathbf{z} \in \mathcal{S}$ defined on intervals $[x_0,\, d_{\mathbf{w}})$ and $[x_0,\, d_{\mathbf{z}})$ respectively, we say that $\mathbf{w} \preceq \mathbf{z}$ if $\mathbf{z}$ is a continuation of $\mathbf{w}$.
Roughly speaking, if we take the largest (w.r.t. order $\preceq$) element of $\mathcal{S}$ then by it can not be further continued. To implement this idea, we need to apply *Zorn's lemma.* Zorn's lemma is equivalent to axiom of choice (see the book on Topology by JL Kelley for more) and helps in asserting existence of "maximal elements" provided the totally ordered subsets of $\mathcal{S}$ have an upper bound (upper bound for a subset $\mathcal{T} \subseteq \mathcal{S}$ is an element $\mathbf{h} \in \mathcal{S}$ such that $\mathbf{w} \preceq \mathbf{h}$ for all $\mathbf{w} \in \mathcal{S}$).

Exercise 3.27 *Show that the relation $\preceq$ defines a partial order on the set $\mathcal{S}$. Prove that each totally ordered subset of $\mathcal{S}$ has an upper bound.*

By Zorn's lemma, there exists a maximal element $\mathbf{q}$ in $\mathcal{S}$. Note that this maximal solution is saturated in view of the definition of $\preceq$ and maximality $\mathbf{q}$. ∎

Remark 3.28 *Under the hypothesis of previous theorem, if a solution* $\mathbf{u}$ *of IVP* (3.1) *is continuable, then there may be more than one saturated solution extending* $\mathbf{u}$. *This possibility is due to non-uniqueness of solutions to IVP* (3.1). *The following exercise is concerned with this phenomenon. Further note that if solution* $\mathbf{u}$ *to IVP* (3.1) *is unique, then there will be a unique saturated solution extending it.*

Exercise 3.29 *Let* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x, y) = 3\,y^{2/3}$. *Show that the solution* $y : [-1,\, 0] \to \mathbb{R}$ *defined by* $y(x) = 0$ *for all* $x \in [-1,\, 0]$ *of IVP satisfying the initial condition* $y(-1) = 0$ *has at least two saturated solutions extending it.*

Theorem 3.30 (Classification of saturated solutions) *Let* $\mathbf{u}$ *be a saturated right solution of IVP* (3.1), *and its domain of definition be the interval* $[x_0, d)$. *Then one of the following alternatives holds.*

(1) *The function* $\mathbf{u}$ *is unbounded on the interval* $[x_0, d)$.

(2) *The function* $\mathbf{u}$ *is bounded on the interval* $[x_0, d)$, *and* $\mathbf{u}$ *is global i.e.,* $d = \sup \mathbb{I}$.

(3) *The function* $\mathbf{u}$ *is bounded on the interval* $[x_0, d)$, *and* $\mathbf{u}$ *is not global i.e.,* $d < \sup \mathbb{I}$ *and each limit point of* $\mathbf{u}$ *as* $x \to d_-$ *lies on the boundary of* Ω.

Proof :

If (1) is not true, then definitely (2) or (3) will hold. Therefore we assume that both (1) and (2) do not hold. Thus we assume that $\mathbf{u}$ is bounded on the interval $[x_0, d)$ and $d < \sup \mathbb{I}$. We need to show that each limit point of $\mathbf{u}$ as $x \to d_-$ lies on the boundary of Ω.

Our proof is by method of contradiction. We assume that there exists a limit point $\mathbf{u}^*$ of $\mathbf{u}$ as $x \to d_-$ in Ω. We are going to prove that $\lim\limits_{x \to d_-} \mathbf{u}(x)$ exists. Note that, once the limit exists it must be equal to $\mathbf{u}^*$ which is one of its limit points. Now applying Lemma 3.24, we infer that the solution $\mathbf{u}$ is continuable and thus contradicting the hypothesis that $\mathbf{u}$ is a saturated solution.

Thus it remains to prove that $\lim\limits_{x \to d_-} \mathbf{u}(x) = \mathbf{u}^*$ *i.e.,* $\|\mathbf{u}(x) - \mathbf{u}^*\|$ can be made arbitrarily small for x near $x = d$.

Since Ω is an open set and $\mathbf{u}^* \in \Omega$, there exists $r > 0$ such that $B[\mathbf{u}^*, r] \subset \Omega$. As a consequence, $B[\mathbf{u}^*, \epsilon] \subset \Omega$ for every $\epsilon < r$. Thus on the rectangle $R \subset \mathbb{I} \times \Omega$ defined by

$$R = [x_0,\, d] \times \{\mathbf{y} \,:\, \|\mathbf{y} - \mathbf{u}^*\| \le r\}, \tag{3.31}$$

$\|\mathbf{f}(x, \mathbf{y})\| \le M$ for some $M > 0$ since R is a compact set and $\mathbf{f}$ is continuous.

Since $\mathbf{u}^*$ is a limit point of $\mathbf{u}$ as $x \to d_-$, there exists a sequence (x_m) in $[x_0,\, d)$ such that $x_m \to d$ and $\mathbf{u}(x_m) \to \mathbf{u}^*$. As a consequence of definition of limit, we can find a $k \in \mathbb{N}$ such that

$$|x_k - x| < \min\left\{\frac{\epsilon}{2M},\, \frac{\epsilon}{2}\right\} \quad \text{and} \quad \|\mathbf{u}(x_k) - \mathbf{u}^*\| < \min\left\{\frac{\epsilon}{2M},\, \frac{\epsilon}{2}\right\}. \tag{3.32}$$

Claim: $\{(x,\, \mathbf{u}(x)) \,:\, x \in [x_k,\, d)\} \subset \mathbb{I} \times B[\mathbf{u}^*, \epsilon]$.

Proof of claim: If the inclusion in the claim were false, then there would exist a point on the graph of $\mathbf{u}$ (on the interval $[x_k,\, d)$) lying outside the set $\mathbb{I} \times B[\mathbf{u}^*, \epsilon]$. Owing to the continuity of $\mathbf{u}$, the graph must meet the boundary of $B[\mathbf{u}^*, \epsilon]$. Let $x^* > x_k$ be the first instance at which the trajectory touches the boundary of $B[\mathbf{u}^*, \epsilon]$. That is, $\epsilon = \|\mathbf{u}(x^*) - \mathbf{u}^*\|$ and $\|\mathbf{u}(x) - \mathbf{u}^*\| < \epsilon$ for $x_k \le x < x^*$. Thus

$$\epsilon = \|\mathbf{u}(x^*) - \mathbf{u}^*\| \;\le\; \|\mathbf{u}(x^*) - \mathbf{u}(x_k)\| + \|\mathbf{u}(x_k) - \mathbf{u}^*\| < \int_{x_k}^{x^*} \|\mathbf{f}(s, \mathbf{u}(s))\|\, ds + \frac{\epsilon}{2} \tag{3.33}$$

$$< \; M(x^* - x_k) + \frac{\epsilon}{2} \;<\; M(d - x_k) + \frac{\epsilon}{2} \;<\; \epsilon \tag{3.34}$$

This contradiction finishes the proof of Claim.

Therefore, limit of $\mathbf{u}(x)$ as $x \to d$ exists. As noted at the beginning of this proof, it follows that $\mathbf{u}$ is continuable. This finishes the proof of the theorem. ∎