Chapter 5

Boundary Value Problems

A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point.

Let $\mathbb{I} = (a,\, b) \subseteq \mathbb{R}$ be an interval. Let $p, q, r : (a,\, b) \to \mathbb{R}$ be continuous functions.
Throughout this chapter we consider the linear second order equation given by

$$y'' + p(x)y' + q(x)y = r(x), \quad a < x < b. \tag{5.1}$$

Corresponding to ODE (5.1), there are four important kinds of (linear) boundary conditions. They are given by

$$\begin{aligned}
\text{Dirichlet or First kind} &: \quad y(a) = \eta_1, \quad y(b) = \eta_2,\\
\text{Neumann or Second kind} &: \quad y'(a) = \eta_1, \quad y'(b) = \eta_2,\\
\text{Robin or Third or Mixed kind} &: \quad \alpha_1 y(a) + \alpha_2 y'(a) = \eta_1, \quad \beta_1 y(b) + \beta_2 y'(b) = \eta_2,\\
\text{Periodic} &: \quad y(a) = y(b), \quad y'(a) = y'(b).
\end{aligned}$$

Remark 5.1 (On periodic boundary condition) *If the coefficients of ODE* (5.1) *are periodic functions with period $l = b - a$ and if ϕ is a solution of ODE* (5.1) *(note that this solution exists on $\mathbb{R}$), then ψ defined by $\psi(x) = \phi(x + l)$ is also a solution. If ϕ satisfies the periodic boundary conditions, then $\psi(a) = \phi(a)$ and $\psi'(a) = \phi'(a)$. Since solutions to IVP are unique in the present case, it must be that $\psi \equiv \phi$. In other words, ϕ is a periodic function of period l.*

Boundary Value Problems do not behave as nicely as Initial value problems. For, there are BVPs for which solutions do not exist; and even if a solution exists there might be many more. Thus existence and uniqueness generally fail for BVPs. The following example illustrate all the three possibilities.

Example 5.2 *Consider the equation*

$$y'' + y = 0 \tag{5.2}$$

(i) *The BVP for equation* (5.2) *with boundary conditions $y(0) = 1$, $y(\frac{\pi}{2}) = 1$ has a unique solution. This solution is given by $\sin x + \cos x$.*

(ii) *The BVP for equation* (5.2) *with boundary conditions $y(0) = 1$, $y(\pi) = 1$ has no solutions.*

(iii) *The BVP for equation* (5.2) *with boundary conditions $y(0) = 1$, $y(2\pi) = 1$ has an infinite number of solutions.*

5.1 Adjoint forms, Lagrange identity

In mathematical physics there are many important boundary value problems corresponding to second order equations. In the studies of vibrations of a membrane, vibrations of a structure one has to solve a homogeneous boundary value problem for real frequencies (eigen values). As is well-known in the case of symmetric matrices that there are only real eigen values and corresponding eigen vectors form a basis for the underlying vector space and thereby all symmetric matrices are diagonalisable. *Self-adjoint* problems can be thought of as corresponding ODE versions of symmetric matrices, and they play an important role in mathematical physics.

Let us consider the equation

$$\mathcal{L}[y] \equiv l(x)y'' + p(x)y' + q(x)y = 0, \quad a < x < b. \tag{5.3}$$

Integrating $z\mathcal{L}[y]$ by parts from a to x, we have

$$\int_a^x z\mathcal{L}[y]\,dx = \big[(zl)y' - (zl)'y + (zp)y\big]_a^x + \int_a^x \big[(zl)'' - (zp)' + (zq)\big]\,y\,dx. \tag{5.4}$$

If we define the second order operator $\mathcal{L}^*$ by

$$\mathcal{L}^*[z] \equiv (zl)'' - (zp)' + (zq) = l(x)z'' + (2l' - p)(x)z' + (l'' - p' + q)(x)z = 0, \tag{5.5}$$

then the equation (5.4) becomes

$$\int_a^x \big(z\,\mathcal{L}[y] - y\,\mathcal{L}^*[z]\big)\,dx = \big[l(y'z - yz') + (p - l')\,y\,z\big]_a^x. \tag{5.6}$$

The operator $\mathcal{L}^*$ is called the **adjoint operator** corresponding to the operator $\mathcal{L}$. It can be easily verified that adjoint of $\mathcal{L}^*$ is $\mathcal{L}$ itself. If $\mathcal{L}$ and $\mathcal{L}^*$ are the same, then $\mathcal{L}$ is said to be **self-adjoint** .

Thus, the necessary and sufficient condition for $\mathcal{L}$ to be self-adjoint is that

$$p = 2l' - p, \quad \text{and}\, q = l'' - p' + q, \tag{5.7}$$

which is satisfied if

$$p = l'. \tag{5.8}$$

Thus if $\mathcal{L}$ is self-adjoint, we have

$$\mathcal{L}[y] \equiv l(x)y'' + p(x)y' + q(x)y = \big(l(x)y'\big)' + q(x)y. \tag{5.9}$$

A general operator $\mathcal{L}$ may not be self-adjoint but it can always be converted into a self-adjoint by suitably multiplying $\mathcal{L}$ with a function.

Lemma 5.3 (self-adjointisation) *The operator $h(x)\mathcal{L}[y]$ is self-adjoint, where h is given by*

$$h(x) = \frac{1}{l(x)} \exp\left\{\int^x \frac{p(t)}{l(t)}\,dt\right\}. \tag{5.10}$$

In fact $h(x)\mathcal{L}[y]$ is given by

$$\frac{d}{dx}\left[\alpha(x)\frac{dy}{dx}\right] + \beta(x)\,y = 0, \tag{5.11}$$

where

$$\alpha(x) = \exp\left\{\int^x \frac{p(t)}{l(t)}\,dt\right\}, \quad \beta(x) = \frac{q(x)}{l(x)} \exp\left\{\int^x \frac{p(t)}{l(t)}\,dt\right\}. \blacksquare \tag{5.12}$$

Exercise 5.4 (i) *Prove that the self-adjoint form of the Legendre equation*

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{5.13}$$

$$\text{is} \quad \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)\,y = 0. \tag{5.14}$$

(ii) *Prove that the self-adjoint form of the Bessel equation*

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \tag{5.15}$$

$$\text{is} \quad \frac{d}{dx}\left[x\frac{dy}{dx}\right] + \left(x - \frac{\nu^2}{x}\right)y = 0. \tag{5.16}$$

Lagrange identity, Green's identity

Differentiating both sides of equation (5.6), we get **Lagrange identity**:

$$\big(z\,\mathcal{L}[y] - y\,\mathcal{L}^*[z]\big) = \frac{d}{dx}\big[l(y'z - yz') + (p-l')\,y\,z\big]. \tag{5.17}$$

If we take $x = b$ in the equation (5.6), we get **Green's identity**:

$$\int_a^b \big(z\,\mathcal{L}[y] - y\,\mathcal{L}^*[z]\big)\,dx = \big[l(y'z - yz') + (p-l')\,y\,z\big]_a^b. \tag{5.18}$$

If $\mathcal{L}$ is self-adjoint, then Green's identity (5.18) reduces to

$$\int_a^b \big(z\,\mathcal{L}[y] - y\,\mathcal{L}[z]\big)\,dx = \big[l(y'z - yz')\big]_a^b, \tag{5.19}$$

and Lagrange identity becomes

$$\big(z\,\mathcal{L}[y] - y\,\mathcal{L}[z]\big) = \frac{d}{dx}\big[l(y'z - yz')\big]. \tag{5.20}$$

5.2 Two-point boundary value problem

In this section we are going to set up the notations that we are going to use through out our discussion of BVPs.

We consider the linear nonhomogeneous second order in the self-adjoint form described below.

Hypothesis ($\mathbf{H_{BVP}}$)

Let f, q be continuous function on the interval $[a, b]$. Let p be a continuously differentiable and does not vanish on the interval $[a, b]$. Further assume that $p > 0$ on $[a, b]$. The linear nonhomogeneous BVP in self-adjoint form then conists of solving the ODE

$$\mathcal{L}[y] \equiv \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y = f(x), \tag{5.21}$$

along with two boundary conditions prescribed at a and b given by

$$\mathcal{U}_1[y] \equiv a_1y(a) + a_2y'(a) = \eta_1, \quad \mathcal{U}_2[y] \equiv b_1y(b) + b_2y'(b) = \eta_2, \tag{5.22}$$

where η_1 and η_2 are given constants; and the constants a_1, a_2, b_1, b_2 satisfy $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$.

Note that the boundary conditions are in the most general form, and they include the first three conditions given at the beginning of our discussion on BVPs as special cases.

Let us introduce some nomenclature here.

Definition 5.5 *Assume hypothesis* ($\mathbf{H_{BVP}}$)*. A* **nonhomogeneous boundary value problem** *consists of solving*

$$\mathcal{L}[y] = f, \quad \mathcal{U}_1[y] = \eta_1, \quad \mathcal{U}_2[y] = \eta_2, \tag{5.23}$$

for given constants η_1 *and* η_2*, and a given continuous function* f *on the interval* $[a, b]$*.*

Definition 5.6 *The associated* **homogeneous boundary value problem** *is then given by*

$$\mathcal{L}[y] = 0, \quad \mathcal{U}_1[y] = 0, \quad \mathcal{U}_2[y] = 0. \tag{5.24}$$

Let us list some properties of the solutions for BVP that are consequences of the linearity of the differential operator $\mathcal{L}$.

Lemma 5.7 (i) *A linear combination of solutions of the homogeneous BVP* (5.24) *is also a solution of the homogeneous BVP* (5.24)*.*

(ii) *If* u, v *are two solutions of the nonhomogeneous BVP* (5.23)*, then their difference* $u - v$ *is a solution of the homogeneous BVP* (5.24)*.*

(iii) *If* y *solves the nonhomogeneous BVP* (5.23) *and* z *solves the homogeneous BVP* (5.24)*, then the function* $y + z$ *nonhomogeneous BVP* (5.23)*.*

(iv) *Let* u *be a (fixed) solution of the nonhomogeneous BVP* (5.23)*. Then any solution* y *of the nonhomogeneous BVP* (5.23) *is given by* $y = u + z$ *for some function* z *that solves the homogeneous BVP* (5.24)*.* ∎

Given a fundamental pair of solutions to the ODE $\mathcal{L}[y] = 0$, it is possible to say whether (and when) the homogeneous BVP has only trivial solution and is characterised in terms of the given fundamental pair. Recall that a fundamental pair of solutions to $\mathcal{L}[y] = 0$ always exists.

Lemma 5.8 *Let* ϕ_1, ϕ_2 *be a fundamental pair of solutions to the ODE* $\mathcal{L}[y] = 0$*. Then the following are equivalent.*

(1) *The nonhomogeneous boundary value problem has a unique solution for any given constants* η_1 *and* η_2*, and a given continuous function* f *on the interval* $[a, b]$*.*

(2) *The associated homogeneous boundary value problem has only trivial solution.*

(3) *The determinant*

$$\begin{vmatrix} \mathcal{U}_1[\phi_1] & \mathcal{U}_1[\phi_2] \\ \mathcal{U}_2[\phi_1] & \mathcal{U}_2[\phi_2] \end{vmatrix} \neq 0. \tag{5.25}$$

Before we give a proof of this result, let us make an observation concerning the condition (5.25).

Remark 5.9 *Though (3) above depends on a given fundamental pair, due to its equivalence with the (1) and (2), it is indeed the case that the condition* (5.25) *is independent of the choice of fundamental pair.*

Also note that there is a subtle difference between equivalence of (1) and (2), and the statement(s) of Lemma 5.7. What is it? In effect, equivalence of (1) and (2) does not really follow from Lemma 5.7. Therefore we will have to prove the equivalence of (1) and (2).

Proof :

Our strategy for the proof is to prove (i). (1)$\Longleftrightarrow$(3) (ii). (2)$\Longleftrightarrow$(3). In fact, it is enough to prove (1)$\Longleftrightarrow$(3), since (2)$\Longleftrightarrow$(3) follows by taking $f = 0$, $\eta_1 = \eta_2 = 0$.

We already illustrated how to find solutions of $\mathcal{L}[y] = f$ starting from a fundamental pair of solutions to the ODE $\mathcal{L}[y] = 0$ and we gave an expression for a general solution of $\mathcal{L}[y] = f$. Let us pick any one of such solutions, let us denote it by z.

Since ϕ_1, ϕ_2 is a fundamental pair of solutions to the ODE $\mathcal{L}[y] = 0$, general soution y of $\mathcal{L}[y] = f$ is given by

$$y = z + c_1\phi_1 + c_2\phi_2, \quad c_1, c_2 \in \mathbb{R}. \tag{5.26}$$

Now y is a solution of nonhomogeneous BVP (5.23) if and only if we can solve for c_1 and c_2 from the algebraic equations

$$\mathcal{U}_1[y] = \mathcal{U}_1[z] + c_1\mathcal{U}_1[\phi_1] + c_2\mathcal{U}_1[\phi_2] = \eta_1, \quad \mathcal{U}_2[y] = \mathcal{U}_2[z] + c_1\mathcal{U}_2[\phi_1] + c_2\mathcal{U}_2[\phi_2] = \eta_2, \tag{5.27}$$

The above system (5.27) has a solution for every η_1 and η_2 if and only if (5.25) holds. ■

Exercise 5.10 *Consider the ODE*

$$y'' + y = f(x), \quad 0 \le x \le \pi. \tag{5.28}$$

Determine if the following BVPs for the ODE (5.28) *have unique solution for every* f, η_1, η_2 *by applying the above result.*

(i) $\mathcal{U}_1[y] \equiv y(0) + y'(0) = \eta_1, \quad \mathcal{U}_2[y] \equiv y(\pi) = \eta_2.$

(ii) $\mathcal{U}_1[y] \equiv y(0) = \eta_1, \quad \mathcal{U}_2[y] \equiv y(\pi) = \eta_2.$

Exercise 5.11 *Prove that the nonhomogeneous BVP (posed on* $[0, \pi]$*)*

$$y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 1 \tag{5.29}$$

has no solution. Comment on the solutions of the associated homogeneous BVP.

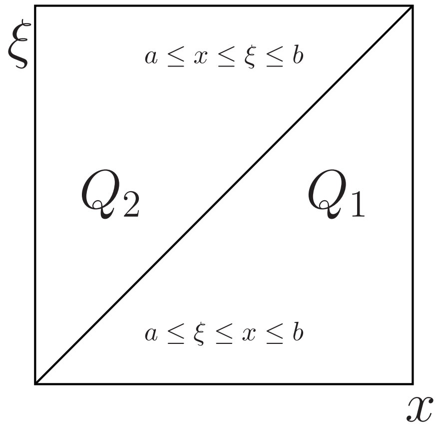
5.3 Fundamental solutions, Green's functions

Fundamental solution of an ODE gives rise to a representation (integral) formula for solution of the nonhomogeneous equation. When we want to take care of boundary conditions, we impose boundary conditions on fundamental solutions and get Green's functions. Thus Green's functions give rise to a representation formula for solution of the nonhomogeneous BVP. The concepts of a fundamental solution as well as a Green's function are defined in terms of the homogeneous BVPvassociated to the nonhomogeneous BVP.

Let Q denote the square $Q := [a, b] \times [a, b]$ in the $x\xi$-plane. Let us partition Q by the line $x = \xi$ and call the two resulting triangles Q_1 and Q_2. Let

$$\begin{aligned} Q_1 &= \{(x, \xi) \ : \ a \le \xi \le x \le b\}, \\ Q_2 &= \{(x, \xi) \ : \ a \le x \le \xi \le b\}. \end{aligned}$$

Note that the diagonal $x = \xi$ belongs to both the triangles.



Definition 5.12 (Fundamental solution) *A function $\gamma(x, \xi)$ defined in Q is called a fundamental solution of the homogeneous differential equation $\mathcal{L}[y] = 0$ if it has the following properties:*

(i) *The function $\gamma(x, \xi)$ is continuous in Q.*

(ii) *The first and second order partial derivatives w.r.t. variable x of the function $\gamma(x, \xi)$ exist and continuous up to the boundary on Q_1 and Q_2.*

(iii) *Let $\xi \in [a, b]$ be fixed. Then $\gamma(x, \xi)$, considered as a function of x, satisfies $\mathcal{L}[\gamma(., \xi)] = 0$ at every point of the interval $[a, b]$, except at ξ.*

(iv) *The first derivate has a jump across the diagonal $x = \xi$, of magnitude $1/p$, i.e.,*

$$\left[\frac{\partial \gamma}{\partial x}\right]_{x=\xi} := \frac{\partial \gamma}{\partial x}(\xi+, \xi) - \frac{\partial \gamma}{\partial x}(\xi-, \xi) = \frac{1}{p(\xi)}, \quad a < \xi < b, \tag{5.30}$$

where $\frac{\partial \gamma}{\partial x}(\xi+, , \xi)$ is defined as the limit of $\frac{\partial \gamma}{\partial x}(x, , \xi)$ as $x \to \xi+$, i.e., $(x, , \xi) \in Q_1$; and $\frac{\partial \gamma}{\partial x}(\xi-, , \xi)$ is defined as the limit of $\frac{\partial \gamma}{\partial x}(x, , \xi)$ as $x \to \xi-$, i.e., $(x, , \xi) \in Q_2$. Note that plus and minus signs indicate that limits are taken from right and left sides of the diagonal $x = \xi$.

A function is continuous up to the boundary of Q_i means that function can be extended continuously to the boundary of Q_i.

Exercise 5.13 *Prove that the condition* (5.30) *is equivalent to the condition*

$$\left[\frac{\partial \gamma}{\partial x}\right]_{x=\xi} := \frac{\partial \gamma}{\partial x}(x, x-) - \frac{\partial \gamma}{\partial x}(x, x+) = \frac{1}{p(x)}, \quad a < x < b. \tag{5.31}$$

Lemma 5.14 *Assume hypothesis* $(\mathbf{H_{BVP}})$. *A fundamental solution exists but is not unique.*

Proof :
For each $a \leq \xi \leq b$, let $y(x; \xi)$ be the solution of the IVP

$$\mathcal{L}[y] = 0, \quad y(\xi) = 0, \quad y'(\xi) = \frac{1}{p(\xi)}. \tag{5.32}$$

Then

$$\gamma(x, \xi) := \begin{cases} 0 & \text{for} \quad (x, \xi) \in Q_2 \\ y(x; \xi) & \text{for} \quad (x, \xi) \in Q_1 \end{cases} \tag{5.33}$$

is a fundamental solution, and can be verified easily.
Let z be a solution of $\mathcal{L}[z] = 0$, and let $h \in C[a, b]$. Then the function

$$\gamma_1(x, \xi) := \gamma(x, \xi) + z(x)h(\xi) \tag{5.34}$$

is also a fundamental solution. ■

Example 5.15 *With the definition* $a_+ := \max(0, a)$,
(i) a fundamental solution for $\mathcal{L}[y] \equiv y'' = 0$ *is* $(x-\xi)_+$.
(ii) a fundamental solution for $\mathcal{L}[y] \equiv y'' + \lambda^2 y = 0$ *is* $\frac{1}{\lambda}\sin\lambda(x-\xi)_+$.

Given a fundamental solution for the homogeneous equation $\mathcal{L}[y] = 0$, one can easily construct a solution of nonhomogeneous equation $\mathcal{L}[y] = f$, as asserted by the following result.

Theorem 5.16 *Assume hypothesis* ($\mathbf{H_{BVP}}$). *Let* $\gamma(x, \xi)$ *be a fundamental solution. Then the function* v *defined by*

$$v(x) = \int_a^b \gamma(x, \xi)\, f(\xi)\, d\xi \tag{5.35}$$

is twice continuously differentiable and is a solution of $\mathcal{L}[v] = f$ *on the interval* $[a, b]$.

Proof :

We split the integral in (5.35) at the point x and write

$$v(x) = \int_a^x \gamma(x, \xi)\, f(\xi)\, d\xi + \int_x^b \gamma(x, \xi)\, f(\xi)\, d\xi. \tag{5.36}$$

Differentiating (5.36) w.r.t. x yields

$$v'(x) = \gamma(x, x)\, f(x) + \int_a^x \frac{\partial\gamma}{\partial x}(x, \xi)\, f(\xi)\, d\xi - \gamma(x, x)\, f(x) + \int_x^b \frac{\partial\gamma}{\partial x}(x, \xi)\, f(\xi)\, d\xi. \tag{5.37}$$

Since $\gamma(x, \xi)$ is continuous, the above equation becomes

$$v'(x) = \int_a^x \frac{\partial\gamma}{\partial x}(x, \xi)\, f(\xi)\, d\xi + \int_x^b \frac{\partial\gamma}{\partial x}(x, \xi)\, f(\xi)\, d\xi. \tag{5.38}$$

Differentiating (5.38) w.r.t. x yields

$$v''(x) = \frac{\partial\gamma}{\partial x}(x, x-)\, f(x) + \int_a^x \frac{\partial^2\gamma}{\partial x^2}(x, \xi)\, f(\xi)\, d\xi - \frac{\partial\gamma}{\partial x}(x, x+)\, f(x) + \int_x^b \frac{\partial^2\gamma}{\partial x^2}(x, \xi)\, f(\xi)\, d\xi. \tag{5.39}$$

By Exercise 5.13, above equation (5.39) reduces to

$$v''(x) = \frac{1}{p(x)}\, f(x) + \int_a^x \frac{\partial^2\gamma}{\partial x^2}(x, \xi)\, f(\xi)\, d\xi + \int_x^b \frac{\partial^2\gamma}{\partial x^2}(x, \xi)\, f(\xi)\, d\xi. \tag{5.40}$$

Since $\mathcal{L}[\gamma(., \xi)] = 0$ (by definition of a fundamental solution), thanks to the above calculations, we get

$$\mathcal{L}[v] = pv'' + p'v' + qv = \int_a^b \mathcal{L}[\gamma(., \xi)](x, \xi)\, f(\xi)\, d\xi + f(x) = f(x). \blacksquare \tag{5.41}$$

Definition 5.17 (Green's function) *A Green's function* $G(x, \xi)$ *for the homogeneous boundary value problem* (5.24)

$$\mathcal{L}[y] = 0, \quad \mathcal{U}_1[y] = 0, \quad \mathcal{U}_2[y] = 0. \tag{5.42}$$

is defined by the following two properties:

(i) $G(x, \xi)$ *is a fundamental solution of* $\mathcal{L}[y] = 0$.

(ii) $\mathcal{U}_1[G(., \xi)] = 0, \quad \mathcal{U}_2[G(., \xi)] = 0$ *for each* $a < \xi < b$.

5.3.1 Construction of Green's functions

In this paragraph we are going to construct Green's functions under the assumption that the homogeneous BVP

$$\mathcal{L}[y] = 0, \quad \mathcal{U}_1[y] = 0, \quad \mathcal{U}_2[y] = 0. \tag{5.43}$$

has only trivial solution.

Let $(\lambda_1, \lambda_2) \neq (0,0)$ be such that $a_1\lambda_1 + a_2\lambda_2 = 0$ and let ϕ_1 be solution of $\mathcal{L}[y] = 0$ satisfying $\phi_1(a) = \lambda_1$ and $\phi_1'(a) = \lambda_2$. Choose another solution ϕ_2 of $\mathcal{L}[y] = 0$ similarly. This way of choosing ϕ_1 and ϕ_2 make sure that both are non-trivial solutions.

Note that ϕ_1 and ϕ_2 form a fundamental pair of solutions of $\mathcal{L}[y] = 0$, since we assumed that homogeneous BVP has only trivial solutions.
By Lagrange's identity (5.20), we get $\frac{d}{dx}\left[p(\phi_1'\phi_2 - \phi_1\phi_2')\right] = 0$. This implies

$$p(\phi_1'\phi_2 - \phi_1\phi_2') \equiv c, \text{ a constant and non-zero,} \tag{5.44}$$

as a consequence of $(\phi_1'\phi_2 - \phi_1\phi_2')$ being the wronskian corresponding to a fundamental pair of solutions.

Green's function is then given by

$$G(x,\,\xi) := \frac{1}{c}\begin{cases} \phi_1(x)\phi_2(\xi) & \text{for} \quad (x,\,\xi) \in Q_2 \\ \phi_1(\xi)\phi_2(x) & \text{for} \quad (x,\,\xi) \in Q_1. \end{cases} \tag{5.45}$$

Verifying that $G(x,\,\xi)$ has the required properties for it to be a Green's function is left as an exercise.

Exercise 5.18 *Verify that $G(x,\,\xi)$ is indeed a Green's function.*

Theorem 5.19 *Assume hypothesis* ($\mathbf{H_{BVP}}$). *Assume that the homogeneous BVP* (5.43) *has only the trivial solution. Then*

(i) *Green's function for* (5.43) *exists and is unique.*

(ii) *Green's function is explicitly given by* (5.45).

(iii) *Green's function is symmetric, i.e.,*

$$G(x,\,\xi) = G(\xi,\,x). \tag{5.46}$$

(iv) *The solution (which is unique by Lemma 5.8) of nonhomogeneous equation with homogogeneous boundary conditions, i.e.,*

$$\mathcal{L}[y] = f, \quad \mathcal{U}_1[y] = 0, \quad \mathcal{U}_2[y] = 0, \tag{5.47}$$

is given by

$$v(x) = \int_a^b G(x,\,\xi)\, f(\xi)\, d\xi. \tag{5.48}$$

PROOF :
Proof of (iv) is already done in the context of fundamental solutions; and thus it remains to check the boundary conditions. Recall the computations we did in that proof of v' and v''. Since $G(x,\,\xi)$ satisfies boundary conditions, we have for $i = 1, 2$,

$$\mathcal{U}_i[v] = \mathcal{U}_i\left[\int_a^b G(x,\,\xi)\, f(\xi)\, d\xi\right] = \int_a^b \mathcal{U}_i[G(.,\,\xi)]\, f(\xi)\, d\xi = 0. \tag{5.49}$$

Existence of Green's function follows from its construction, given before.

Let us turn to the proof of uniqueness of Green's function. If possible, let Γ be another Green's function. Let

$$v(x) = \int_a^b G(x,\,\xi)\, f(\xi)\, d\xi, \quad w(x) = \int_a^b \Gamma(x,\,\xi)\, g(\xi)\, d\xi \tag{5.50}$$

with continuous functions f, g. Since $\mathcal{U}_i[v] = \mathcal{U}_i[w] = 0$, from Green's identity for self-adjoint operators (5.19), and from (5.44), we get

$$\int_a^b \big(v\,\mathcal{L}[w] - w\,\mathcal{L}[v]\big)\, dx = \big[l(w'v - wv')\big]_a^b = 0. \tag{5.51}$$

Substituting the values of v and w in the above equation, and observing that $\mathcal{L}[v] = f$ and $\mathcal{L}[w] = g$, we get

$$\int_a^b \int_a^b g(x)G(x,\,\xi)f(\xi)\, dx\, d\xi = \int_a^b \int_a^b f(x)\Gamma(x,\,\xi)g(\xi)\, dx\, d\xi \tag{5.52}$$

By changing the order of integration, we get

$$\int_a^b \int_a^b (G(x,\,\xi) - \Gamma(\xi,\,x))\, f(\xi)g(x)\, d\xi\, dx = 0. \tag{5.53}$$

Since the relation holds for arbitrary f and g, we must have

$$G(x,\,\xi) = \Gamma(\xi,\,x). \tag{5.54}$$

Setting $\Gamma = G$, we get symmetry of Green's function G; and now once again we get $G = \Gamma$ and thus establishing uniqueness of Green's function. ■

Example 5.20 *Green's function for*

$$y'' + y = 0 \text{ in } [0,\,1], \quad y(0) = y(1) = 0 \tag{5.55}$$

is given by

$$G(x,\,\xi) := \begin{cases} \xi(x-1) & \text{for } \; 0 \le \xi \le x \le 1, \\ x(\xi - 1) & \text{for } \; 0 \le x \le \xi \le 1. \end{cases} \tag{5.56}$$

Note that the homogeneous BVP in this case, has only trivial solution.

Exercise 5.21 *How to solve general nonhomogeneous BVP? It is necessary to ask this question since Green's functions give rise to solutions of nonhomogeneous equations but with homogeneous boundary conditions. (Hint: Use linearity of the differential operator) Solve, for instance, the non-homogeneous BVP*

$$y'' = f(x) \text{ in } [0,\,1], \quad y(0) = 0, \quad y(1) + y'(1) = 2. \tag{5.57}$$

Exercise 5.22 *Solve the nonhomogeneous BVP*

$$y'' + y = e^x \text{ in } [0,\,1], \quad y(0) = y(1) = 0 \tag{5.58}$$

by (i) using a fundamental pair of solutions and a special solution of the nonhomogeneous differential equation; (ii) using Green's function.

Exercise 5.23 *Find Green's function for*

$$y'' = 0 \text{ in } [0,\,1], \quad y'(0) = y(1) = 0 \tag{5.59}$$

Exercise 5.24 *Find Green's function for*

$$y'' + \frac{1}{4x^2} = 0 \text{ in } [1,\,2], \quad y(1) = y(2) = 0 \tag{5.60}$$

(Hint: Try a change of variable $x = e^t$)

5.4 Generalised Green's function

In Section 5.3 we constructed Green's function for the homogeneous BVP when the latter problem had only trivial solutions. Now we investigate the case where the homogeneous BVP has non-trivial solutions.

Let us look at what happens in the case of linear system of equations of size k in k variables. Let A be a $k \times k$ matrix, and $\mathbf{b} \in \mathbb{R}^k$. Suppose that $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions. In this case we know that $A\mathbf{x} = \mathbf{b}$ does not have solution for every $\mathbf{b} \in \mathbb{R}^k$. We also know that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ belongs to orthogonal complement of a certain subspace of $\mathbb{R}^k$, *i.e.,* $\mathbf{b}$ must satisfy a compatibility condition. Note that, in this case, if $A\mathbf{x} = \mathbf{b}$ has one solution, then there are infinitely many solutions.

We are going to see that there exists an analogue of Green's function, called *Generalised Green's function*. We also prove that a situation similar to that of matricial analogy given above occurs here too; and the compatibility condition for the existence of a solution to nonhomogeneous BVP is a kind of Fredholm alternative.

Let us recall the notations Q_1 and Q_2.

$$\begin{aligned} Q_1 &= \{(x,\,\xi)\ :\ a \leq \xi \leq x \leq b\}, \\ Q_2 &= \{(x,\,\xi)\ :\ a \leq x \leq \xi \leq b\}. \end{aligned}$$

Definition 5.25 (Generalised Green's function) *Let ϕ_0 be a solution of $\mathcal{L}[y] = 0$ with homogeneous boundary conditions, such that $\int_a^b \phi_0^2(x)\,dx = 1$. A function $\Gamma(x,\,\xi)$ defined in Q is called a generalised Green's function if Γ has the following properties:*

(i) *The function $\Gamma(x,\,\xi)$ is continuous in Q.*

(ii) *The first and second order partial derivatives w.r.t. variable x of the function $\Gamma(x,\,\xi)$ exist and continuous up to the boundary on Q_1 and Q_2.*

(iii) *Let $\xi \in [a,\,b]$ be fixed. Then $\gamma(x,\,\xi)$, considered as a function of x, satisfies*

$$\mathcal{L}[\Gamma(.,\,\xi)] = \phi_0(x)\phi_0(\xi) \tag{5.61}$$

at every point of the interval $[a,\,b]$, except at ξ, satisfying the homogeneous boundary conditions $\mathcal{U}_1[y] = 0$ and $\mathcal{U}_2[y] = 0$.

(iv) *The first derivate has a jump across the diagonal $x = \xi$, of magnitude $1/p$, i.e.,*

$$\left[\frac{\partial \Gamma}{\partial x}\right]_{x=\xi} := \frac{\partial \Gamma}{\partial x}(\xi+,\,\xi) - \frac{\partial \Gamma}{\partial x}(\xi-,\,\xi) = \frac{1}{p(\xi)}, \quad a < \xi < b, \tag{5.62}$$

where $\frac{\partial \Gamma}{\partial x}(\xi+,,\xi)$ is defined as the limit of $\frac{\partial \Gamma}{\partial x}(x,,\xi)$ as $x \to \xi+$, i.e., $(x,,\xi) \in Q_1$; and $\frac{\partial \Gamma}{\partial x}(\xi-,,\xi)$ is defined as the limit of $\frac{\partial \Gamma}{\partial x}(x,,\xi)$ as $x \to \xi-$, i.e., $(x,,\xi) \in Q_2$. Note that plus and minus signs indicate that limits are taken from right and left sides of the diagonal $x = \xi$.

(v) *The function $\Gamma(x,\,\xi)$ satisfies the condition*

$$\int_a^b \Gamma(x,\,\xi)\phi_0(x)\,dx = 0 \tag{5.63}$$

The following result characterises the class of functions f, for which the nonhomogeneous equation $\mathcal{L}[y] = f$ has a solution satisfying homogeneous boundary conditions.

Theorem 5.26 *The non-homogenous BVP*

$$\mathcal{L}[y] = f, \quad \mathcal{U}_1[y] = 0, \quad \mathcal{U}_2[y] = 0 \tag{5.64}$$

has a solution

$$\phi(x) = \int_a^b \Gamma(x,\,\xi)\,f(\xi)\,d\xi, \tag{5.65}$$

where $\Gamma(x,\,\xi)$ is a generalised Green's function, if and only if

$$\int_a^b f(x)\phi_0(x)\,dx = 0 \tag{5.66}$$

where ϕ_0 be a solution of the associated homogeneous BVP, such that $\int_a^b \phi_0^2(x)\,dx = 1$. ■

Instead of proving the above result (proof is lengthy but simple) we construct explicitly in an example, a generalised Green's function.

Example 5.27 *Consider the BVP*

$$y'' = f(x) \text{ in } [0,\,1], \quad y(0) = 0, \quad y(1) - y'(1) = 0. \tag{5.67}$$

Step 1: Find the normalised solution ϕ_0 of the associated homogeneous problem. *In this example,*

$$\phi_0(x) = \sqrt{3}x. \tag{5.68}$$

Step 2: Checking the compatibility condition (5.66) for existence of solution to (5.67). *The compatibility condition is given by*

$$\int_0^1 xf(x)\,dx = 0. \tag{5.69}$$

Step 3: Construction of Generalised Green's function. *Once compatibility condition is satisfied, by above result there exists a generalised Green's function Γ. Let us construct it.*
Step 3A: *Γ must be a solution of*

$$\mathcal{L}[\Gamma(.,\,\xi)] = \phi_0(x)\phi_0(\xi) \tag{5.70}$$

at every point of the interval $[a,\,b]$, except at ξ. The above equation, in our case, becomes

$$\Gamma''(x,\,\xi) = 3x\xi. \tag{5.71}$$

Solve this equation.
Let us denote Γ in Q_i by Γ_i, for $i = 1, 2$. Thus we have

$$\Gamma_1(x,\,\xi) = \frac{\xi x^3}{2} + a_1(\xi)x + b_1(\xi) \tag{5.72}$$

$$\Gamma_2(x,\,\xi) = \frac{\xi x^3}{2} + a_2(\xi)x + b_2(\xi). \tag{5.73}$$

Γ_1 and Γ_2 must satisfy the first and second boundary conditions respectively. That is,

$$\Gamma_2(0,\,\xi) = 0 \tag{5.74}$$

$$\Gamma_1(1,\,\xi) - \Gamma_1'(1,\,\xi) = 0. \tag{5.75}$$

This yields

$$\Gamma_1(x,\,\xi) = \frac{\xi x^3}{2} + a_1(\xi)x + \xi \tag{5.76}$$

$$\Gamma_2(x,\,\xi) = \frac{\xi x^3}{2} + a_2(\xi)x. \tag{5.77}$$

Step 3B: Γ must be continuous.

$$\Gamma_1(\xi,\, \xi) = \Gamma_2(\xi,\, \xi). \tag{5.78}$$

This gives

$$a_2(\xi) = a_1(\xi) + 1. \tag{5.79}$$

Step 3C: Γ must satisfy (5.63). *In our case, this condition takes the form*

$$\int_0^1 \Gamma(x,\, \xi) x\, dx = 0 \tag{5.80}$$

This gives us

$$a_1(\xi) = \frac{\xi^3}{2} - \frac{9}{5}\xi. \tag{5.81}$$

Last Step : Formula for Generalised Green's function. *Generalised Green's function is given by*

$$\Gamma(x,\, \xi) := \begin{cases} \Gamma_1(x,\, \xi) & \textit{for} \quad 0 \le x \le \xi \le 1, \\ \Gamma_2(x,\, \xi) & \textit{for} \quad 0 \le \xi \le x \le 1. \end{cases} \tag{5.82}$$

Expression for solution (5.65).

$$\phi(x) = \int_0^1 \Gamma(x,\, \xi)\, f(\xi)\, d\xi = \int_0^x \Gamma_2(x,\, \xi)\, f(\xi)\, d\xi + \int_x^1 \Gamma_1(x,\, \xi)\, f(\xi)\, d\xi \blacksquare \tag{5.83}$$

Note *that we did not check the jump condition and it is satisfied automatically since ϕ_0 is normalised solution. This finishes our discussion on Green's functions.* ■