3.1.2 Cauchy-Lipschitz-Picard existence theorem

From real analysis, we know that continuity of a function at a point is a local concept (as it involves values of the function in a neighbourhood of the point at which continuity of function is in question). We talk about uniform continuity of a function with respect to a domain. Similarly we can define Lipschitz continuity at a point and on a domain of a function defined on subsets of $\mathbb{R}^n$. For ODE purposes we need functions of $(n+1)$ variables and Lipschitz continuity w.r.t. the last n variables. Thus we straight away define concept of Lipschitz continuity for such functions.

Let $R \subseteq \mathbb{I} \times \Omega$ be a rectangle centred at $(x_0, \mathbf{y}_0)$ defined by two positive real numbers a, b (see equation (3.11)).

Definition 3.9 (Lipschitz continuity) *A function* $\mathbf{f}$ *is said to be Lipschitz continuous on a rectangle* R *with respect to the variable* y *if there exists a* $K > 0$ *such that*

$$\|\mathbf{f}(x, \mathbf{y}_1) - \mathbf{f}(x, \mathbf{y}_1)\| \leq K\|\mathbf{y}_1 - \mathbf{y}_2\| \quad \forall (x, \mathbf{y}_1),\, (x, \mathbf{y}_2) \in R. \tag{3.17}$$

Exercise 3.10 *1. Let* $n = 1$ *and* f *be differentiable w.r.t. the variable* y *with a continuous derivative defined on* $\mathbb{I} \times \Omega$. *Show that* f *is Lipschitz continuous on any rectangle* $R \subset \mathbb{I} \times \Omega$.

2. *If* f *is Lipschitz continuous on every rectangle* $R \subset \mathbb{I} \times \Omega$, *is* f *differentiable w.r.t. the variable* y*?*

3. *Prove that the function* h *defined by* $h(y) = y^{2/3}$ *on* $[0, \infty)$ *is not Lipschitz continuous on any interval containing* 0.

4. *Prove that the function* $f(x, y) = y^2$ *defined on doamin* $\mathbb{R} \times \mathbb{R}$ *is not Lipschitz continuous. (this gives yet another reason to define Lipschitz continuity on rectangles)*

We now state the existence theorem and the method of proof is different from that of Peano theorem and yields a bilateral interval containing x_0 on which existence of a solution is asserted.

Theorem 3.11 (Cauchy-Lipschitz-Picard) *Assume Hypothesis* $(\mathrm{H_{IVPS}})$. *Let* $\mathbf{f}$ *be Lipschitz continuous with respect to the variable* $\mathbf{y}$ *on* R. *Then the IVP* (3.1) *has at least one solution on the interval* $\mathbb{J}$ $|x - x_0| \leq \delta$ *where* $\delta = \min\{a, \frac{b}{M}\}$.

PROOF :

Step 1: Equivalent integral equation We recall (3.5), which is equivalent to the given IVP below.

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}(s))\, ds \quad \forall\, x \in \mathbb{I}. \tag{3.18}$$

By the equivalence of above integral equation with the IVP, it is enough to prove that the integral equation has a solution. This proof is accomplished by constructing, what are known as Picard approximations, a sequence of functions that converges to a solution of the integral equation (3.18).

Step 2: Construction of Picard approximations

Define the first function $\mathbf{y}_0(x)$, for $x \in \mathbb{I}$, by

$$\mathbf{y}_0(x) := \mathbf{y}_0 \tag{3.19}$$

Define $\mathbf{y}_1(x)$, for $x \in \mathbb{I}$, by

$$\mathbf{y}_1(x) := \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_0(s))\, ds. \tag{3.20}$$

Note that the function $\mathbf{y}_1(x)$ is well-defined for $x \in \mathbb{I}$. However, when we try to define the next member of the sequence, $\mathbf{y}_2(x)$, for $x \in \mathbb{I}$, by

$$\mathbf{y}_2(x) := \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_1(s))\, ds, \tag{3.21}$$

caution needs to exercised. This is because, we do not know about the values that the function $\mathbf{y}_1(x)$ assumes for $x \in \mathbb{I}$, there is no reason that those values are inside Ω. However, it happens that for $x \in \mathbb{J}$, where the interval $\mathbb{J}$ is as in the statement of the theorem, the expression on RHS of (3.21) which defined function $\mathbf{y}_2(x)$ is meaningful, and hence the function $\mathbf{y}_2(x)$ is well-defined for $x \in \mathbb{J}$. By restricting to the interval $\mathbb{J}$, we can prove that the following sequence of functions is well-defined: Define for $k \geq 1$, for $x \in \mathbb{J}$,

$$\mathbf{y}_k(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_{k-1}(s))\, ds. \tag{3.22}$$

Proving the well-definedness of Picard approximations is left as an exercise, by induction. In fact, the graphs of each Picard approximant lies inside the rectangle R (see statement of our theorem). That is,

$$\|\mathbf{y}_k(x) - \mathbf{y}_0\| \leq b, \quad \forall\, x \in [x_0 - \delta,\, x_0 + \delta]. \tag{3.23}$$

The proof is immediate from

$$\mathbf{y}_k(x) - \mathbf{y}_0 = \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_{k-1}(s))\, ds, \quad \forall\, x \in \mathbb{J}. \tag{3.24}$$

$$\text{Therefore,} \quad \|\mathbf{y}_k(x) - \mathbf{y}_0\| \leq M|x - x_0|, \quad \forall\, x \in \mathbb{J}. \tag{3.25}$$

and for $x \in \mathbb{J}$, we have $|x - x_0| \leq \delta$.

Step 3: Convergence of Picard approximations

We prove the uniform convergence of sequence of Picard approximations $\mathbf{y}_k$ on the interval $\mathbb{J}$, by proving that this sequence corresponds to the partial sums of a uniformly convergent series of functions, and the series is given by

$$\mathbf{y}_0 + \sum_{l=0}^{\infty} \big[\mathbf{y}_{l+1}(x) - \mathbf{y}_l(x)\big], \tag{3.26}$$

Note that the sequence $\mathbf{y}_{k+1}$ corresponds to partial sums of series (3.26). That is,

$$\mathbf{y}_{k+1}(x) = \mathbf{y}_0 + \sum_{l=0}^{k} \big[\mathbf{y}_{l+1}(x) - \mathbf{y}_l(x)\big]. \tag{3.27}$$

Step 3A: Unifrom convergence of series (3.26) on $x \in \mathbb{J}$

We are going to compare series (3.26) with a convergence series of real numbers, uniformly in $x \in \mathbb{J}$, and thereby proving uniform convergence of the series. From the expression

$$\mathbf{y}_{l+1}(x) - \mathbf{y}_l(x) = \int_{x_0}^{x} \big\{\mathbf{f}(s, \mathbf{y}_l(s)) - \mathbf{f}(s, \mathbf{y}_{l-1}(s))\big\}\, ds, \quad \forall\, x \in \mathbb{J}, \tag{3.28}$$

we can prove by induction the estimate (this is left an exercise):

$$\|\mathbf{y}_{l+1}(x) - \mathbf{y}_l(x)\| \leq ML^l \frac{|x - x_0|^{l+1}}{(l+1)!} \leq \frac{M}{L} L^{l+1} \frac{\delta^{l+1}}{(l+1)!} \tag{3.29}$$

We conclude that the series (3.26), and hence the sequence of Picard iterates, converge uniformly on $\mathbb{J}$. This is because the above estimate (3.29) says that general term of series (3.26) is uniformly smaller than that of a convergent series, namely, for the function $e^{\delta L}$ times a constant.

Let $\mathbf{y}(x)$ denote the uniform limit of the sequence of Picard iterates $\mathbf{y}_k(x)$ on $\mathbb{J}$.

Step 4: The limit function $\mathbf{y}(x)$ solves integral equation (3.18)

We want to take limit as $k \to \infty$ in

$$\mathbf{y}_k(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_{k-1}(s))\, ds. \tag{3.30}$$

Taking limit on LHS of (3.30) is trivial. Therefore, for $x \in \mathbb{J}$, if we prove that

$$\int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_{k-1}(s))\, ds \longrightarrow \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}(s))\, ds, \tag{3.31}$$

then we would obtain, for $x \in \mathbb{J}$,

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}(s))\, ds, \tag{3.32}$$

and this finishes the proof. Therefore, it remains to prove (3.31). Let us estimate, for $x \in \mathbb{J}$, the quantity

$$\int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}_{k-1}(s))\, ds - \int_{x_0}^{x} \mathbf{f}(s, \mathbf{y}(s))\, ds = \int_{x_0}^{x} \big\{\mathbf{f}(s, \mathbf{y}_{k-1}(s)) - \mathbf{f}(s, \mathbf{y}(s))\big\}\, ds \tag{3.33}$$

Since the graphs of $\mathbf{y}_k$ lie inside rectangle R, so does the graph of $\mathbf{y}$. This is because rectangle R is closed. Now we use that the vector field $\mathbf{f}$ is Lipschitz continuous (with Lipschitz constant L) in the variable $\mathbf{y}$ on R, we get

$$\left\| \int_{x_0}^{x} \big\{\mathbf{f}(s, \mathbf{y}_{k-1}(s)) - \mathbf{f}(s, \mathbf{y}(s))\big\}\, ds \right\| \leq \left| \int_{x_0}^{x} \left\| \big\{\mathbf{f}(s, \mathbf{y}_{k-1}(s)) - \mathbf{f}(s, \mathbf{y}(s))\big\} \right\| \, ds \right| \tag{3.34}$$

$$\leq L \left| \int_{x_0}^{x} \| \mathbf{y}_{k-1}(s) - \mathbf{y}(s) \| \, ds \right| \leq L|x - x_0| \sup_{\mathbb{J}} \| \mathbf{y}_{k-1}(x) - \mathbf{y}(x) \| \tag{3.35}$$

$$\leq L\delta \sup_{\mathbb{J}} \| \mathbf{y}_{k-1}(x) - \mathbf{y}(x) \| . \tag{3.36}$$

This estimate finishes the proof of (3.31), since $\mathbf{y}(x)$ is the uniform limit of the sequence of Picard iterates $\mathbf{y}_k(x)$ on $\mathbb{J}$, and hence for sufficiently large k, the quantity $\sup_{\mathbb{J}} \| \mathbf{y}_{k-1}(x) - \mathbf{y}(x) \|$ can be made arbitrarily small. ∎

Some comments on existence theorems

Remark 3.12 (i) *Note that the interval of existence depends only on the bound M and not on the specific function.*

(ii) *If $\mathbf{g}$ is any Lipschitz continuous function (with respect to the variable $\mathbf{y}$ on R) in a α neighbourhood of $\mathbf{f}$ and ζ is any vector in an β neighbourhood of $\mathbf{y}_0$, then solution to IVP with data $(\mathbf{g}, \zeta)$ exists on the interval $\delta = \min\{a, \frac{b-\beta}{M+\alpha}\}$. Note that this interval depends on data $(\mathbf{g}, \zeta)$ only in terms of its distance to $(\mathbf{f}, \mathbf{y}_0)$.*

Exercise 3.13 *Prove that Picard's iterates need not converge if the vector field does not satisfy Lipschitz condition as in the existence theorem. Compute successive approximations for the IVP*

$$y' = 2x - x\sqrt{y_+}, \quad y(0) = 0, \quad \textit{with} \quad y_+ = \max\{y,\, 0\},$$

and show that they do not converge, and also show that IVP has a unique solution. (Hint: $y_{2n} = 0$, $y_{2n+1} = x^2$, $n \in \mathbb{N}$).

Warning: We have only stated Peano theorem and hence you may ignore the details on its proof.